On the Laguerre's Theorem Concerning the Separation of Real Roots.

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(1) Let \( F(x) = A_0 x^n + A_1 x^{n-1} + A_2 x^{n-2} + \ldots + A_n = 0 \) be an equation with real coefficients.

Suppose that \( x \geq 0 \) \( F(x) \neq 0 \), and put

\[
F_0(x) = A_n, \\
F_{n-1}(x) = A_n x - A_{n-1}, \\
F_{n-2}(x) = A_n x^2 + A_{n-2} x + A_{n-3}, \\
\ldots \\
F(x) = A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + \ldots + A_0
\]

then we have the theorem:

The number of real roots of the equation \( F(x) = 0 \), which are greater than \( x \), is either equal to the number of variations of signs in \( F_0(x), F_{n-1}(x), F_{n-2}(x), \ldots, F(x) \), or less than that number by an even integer. Terms which vanish are to be dropped out before counting the variations of signs. The multiplicity of the root is here taken account of roots.

This very useful theorem was obtained by Laguerre analytically by means of the consideration of power series (1), but it can be proved directly in an algebraic manner as follows.

(2) Lemma 1. Let \( \{x_n\} = x_0, x_1, x_2, \ldots, x_n \), be a sequence of numbers. Make of \( \{x_n\} \) a new sequence \( \{t_n\} = t_0, t_1, t_2, \ldots, t_{n+1} \), as follows,

\[
t_0 = x_0, \\
t_i = a_i t_{i-1} - b_{i-1} t_{i-1}, \quad 1 \leq i \leq n, \\
t_{n+1} = -l_{n+1} t_n
\]

\( a_i, b_i \) being positive numbers.

Then the number of variations of signs in \( \{t_n\} \) exceeds that number in \( \{x_n\} \) by an odd integer.

(1) Jour. d. Math. purs et appliquées. 3e série, t IX, 1889.

Theorem 1, p. 4
Because, if a variation of signs takes place from $s_{i-1}$ to $s_i$, $1 \leq i \leq n$, then $t_i = a_i s_i - b_{i-1} s_{i-1} \neq 0$ must be of the same sign as $s_i$. Let us suppose, for another case, that a variation of signs takes place from $s_i$ to $s_k$, and $s_i = 0$, $i < j < k$, then $t_k = a_k s_i - b_{k-1} s_{k-1} = a_k s_i \neq 0$ has the same sign as $s_i$. Now we observe the terms of $\{s_m\}$, to which the variations of signs take place, and then it is not difficult to see that the number of variations of signs in $\{t_m\}$ exceeds that number in $\{s_m\}$ by an odd integer.

Lemma 2. If we take $t_{i+1} = b_{i} s_i$ in the above sequence $\{t_m\}$, then the number of variations of signs in $\{t_m\}$ is either equal to that number in $\{s_m\}$, or exceeds it by an even integer.

Lemma 3. If we make of $\{s_m\}$ another sequence $\{t_m\}$ as follows,

\[
  t_n = u_{n}s_n \\
  t_i = u_{i}s_i + b_{i-1}s_{i-1}, \quad 1 \leq i \leq n \\
  t_{i+1} = b_{i} s_i
\]

then the number of variations of signs in $\{t_m\}$ does not exceed that number in $\{s_m\}$. If it is less, the difference is an even integer.

This can be proved similarly as Lemma 1, considering the terms in $\{s_m\}$, to which the permanences of signs take place instead of the variations.

(3) Put

\[
  \varphi(x) = (x - \alpha') F(x) \quad \alpha < \alpha'
\]

\[
  = a_1 x^{n+1} + (a_1 - \alpha' \Delta_1) x^n + (a_2 - \alpha' \Delta_1) x^{n-1} + \ldots - \alpha' \Delta_n
\]

and

\[
  G_{n+1}(\alpha) = a_n \\
  G_n(\alpha) = a_n \alpha + (a_1 - \alpha' \Delta_1) \\
  G_{n-1}(\alpha) = a_n \alpha^2 + (a_1 - \alpha' \Delta_1) \alpha + (a_2 - \alpha' \Delta_1) \\
  \vdots
\]

\[
  G(\alpha) = a_n \alpha^{n+1} + (a_1 - \alpha' \Delta_1) \alpha^n + (a_2 - \alpha' \Delta_1) \alpha^{n-1} + \ldots - \alpha' \Delta_n
\]

as before. Then we have

\[
  G_{n+1}(\alpha) = F_{n+1}(\alpha) \\
  G_i(\alpha) = F_{i-1}(\alpha) - \alpha' F_i(\alpha) \quad 1 \leq i \leq n \\
  G(\alpha) = \sum_{i=0}^{n} (-\alpha') F_i(\alpha)
\]

Let us denote the number of variations of signs in $F_i(\alpha), F_{i-1}(\alpha) \ldots F(\alpha)$, by $v_{\alpha}$, then according to lemma 1 $v_{\alpha} = v_{\alpha} + 2\alpha + 1$, $\alpha$ being a positive integer.

Suppose that

\[
  F(x) = \varphi(x)(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)
\]
\[ \alpha < \alpha_i \quad i = 1, 2, \ldots, n \]
and \( \varphi(x) = 0 \) has no greater root than \( \alpha \). Consequently \( \lambda_{m}^{\prime} \) is an even integer. Then
\[ \lambda_{m}^{\prime} = \lambda_{m}^{\prime} + \sum_{k=1}^{m} (2k+1) = 2k + \nu \]
Hence we obtain the Laguerre's theorem.

(4) We will now investigate further the Horner's scheme, which was also employed by Laguerre.

\[ \frac{\alpha}{b_0} \quad \frac{\alpha}{b_1} \quad \frac{\alpha}{b_2} \quad \ldots \quad \frac{\alpha}{b_{n-1}} \quad \frac{\alpha}{b_{n-2}} \]

\[ \frac{c_0}{c_1} \quad \frac{c_1}{c_2} \quad \ldots \quad \frac{c_{n-2}}{c_{n-1}} \]

\[ \frac{f_0}{f_1} \quad \frac{f_1}{f_2} \quad \ldots \quad \frac{f_{n-1}}{f_n} \]

\[ \frac{g_0}{g_1} \quad g_2 \quad g_3 \quad \ldots \quad g_{n} \]

Evidently we have
\[ u_0 = F_{n}(\alpha), \quad u_1 = F_{n-1}(\alpha), \ldots, \quad u_n = F(\alpha), \]
and we know that the terms \( a_n, b_n, c_{n-1}, \ldots \), which appear in the last diagonal are equal respectively to \( F(\alpha) F'(\alpha) F''(\alpha) \ldots \)

On this scheme, starting from a point of the first column, say \( g_0 \), we come to \( a_n \), counting the variations of signs at first horizontally and next diagonally. Let us denote this number of variations of signs by \( \lambda'(g_0, a_n) \). Then, \( \lambda'(g_0, a_n) \) does not exceed \( \lambda'(f_0, a_n) \), and if it is less, the difference is an even integer. Because, we have
\[ f_0 = g_0, \quad f_1 = g_1 - ag_1, \quad \text{(lemma 2)} \]
Hence we know the theorem:

About any such path, the number of variations of signs is either equal to that number in \( F'(\alpha), F''(\alpha), F'''(\alpha), \ldots, F^{(n)}(\alpha) \), or exceeds that number by an even integer. Moreover, the number of variations of signs in the above successive derivatives, consequently the number of variations of signs in any such path, is not less than the number of greater roots than \( \alpha \) of \( F(x) = 0 \), the difference being an even integer.
Truly, if we put  \( G(x) = (x - \alpha') F(x) \),
then we have  \( G^{n+1}(\alpha) = (n+1)F^{n}(\alpha) \)
\[
G^{n}(\alpha) = i F^{n-1}(\alpha', -(\alpha' - \alpha) F^{0}(\alpha) \quad 1 \leq i \leq n
\]
\[
F(\alpha) = -(\alpha' - \alpha) F(\alpha),
\]
therefore, if \( \alpha' > \alpha \), the number of variations of signs at \( \alpha \) in the successive derivatives of \( G(x) \) exceeds that number in those of \( F(x) \) by an odd integer (lemma 1). Thus the Fourier's theorem follows analogously as in (3).

(5) Let us observe the successive derivatives of \( G(x) = xF(x) \).
\[
G(\alpha) = \alpha F(\alpha)
\]
\[
G^{n}(\alpha) = \alpha F^{n}(\alpha) + i F^{n-1}(\alpha) \quad 1 \leq i \leq n
\]
\[
G^{n+1}(\alpha) = (n+1)F^{n}(\alpha)
\]

Therefore, we know that, by the prolongation of the scheme, we get an equal or less number of variations of signs in the last diagonal (lemma 3). As it can be easily known, this happens in any our path. The amount of loss is an even integer.

\[
\begin{array}{cccccccc}
\alpha & \alpha_1 & \alpha_2 & \ldots & \alpha_n \\
\alpha & a_0 & a_1 & a_2 & \ldots & a_n & a_{n+1} \\
b_0 & b_1 & b_2 & \ldots & b_{n-1} & b_n \\
c_0 & c_1 & c_2 & \ldots & c_{n-2} & c_{n-1} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
f_0 & f_1 & f_2 & \ldots & f_n & f_{n+1} \\
\end{array}
\]

ex.
\[
F(x) = 2x^3 - 5x^2 + 8x^2 - 3x - 4 = 0
\]
\[
\begin{array}{cccccc}
2 & -5 & 0 & 8 & -3 & -4 \\
1 & 2 & -3 & 5 & 2 & -2 \\
2 & -1 & -4 & 1 & 3 & 1 \\
2 & 1 & -3 & -2 & 1 & 2 \\
2 & 3 & 0 & -2 & -1 & 1 \\
2 & 5 & 5 & 3 & 2 & \ldots \\
\end{array}
\]
Hence we know that the equation has only one positive root in the interval (1, 2), and two negative roots one of which is equal -1, and the other lies in the interval (-1, 0).

Separating the real roots of $F'(x) = 0$, $F''(x) = 0$ by this method, we can easily draw the graph of $y = F(x)$.

$F'(x) = 0: \ (-1, \ 0), \ (0, \ 1)$
$F''(x) = 0: \ (-1, \ 0), \ (0, \ 1), \ (1, \ 2)$