On the Linear Displacements in the Generalized ManiJold.

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The theory of connections which was, at first, considered by Ricci, Levi-Civita, has been developed by Cartan, Eddington, Weyl, Schouten, Struik, Eisenhart, Veblen, Thomas, and several other authors.

Moreover, since P. Finsler published his well-known papers, "Ueber Kurven und Flächen in allgemeinen Räumen, Dissertation: Göttingen 1918. p. 121.", the theory of connections in the Finsler's manifold has been developed by E. Noether, L. Berwald, J. L. Synge, J. H. Taylor, J. Douglas, E. Cartan, T. Hosokawa and many other writers.

Recently Prof. A. Kawaguchi has published his excellent investigation with respect to the most generalized Finsler's manifold(1) and, at last he has published a theory of connections in the abstract space which contains all of these theories of connections.(2)

P. Finsler has introduced a line-element(3) (or "ausgezeichnete Linienelment" as Prof. Berwald so called it)(4) $dx$ which is associated to every point in the general manifold $X_n$.

And Mr. T. Hosokawa has studied on the various linear displace-ments in the Finsler's manifold.(5)

   " (I): Proceedings of the Imperial Academy VII. 1931. p. 211.
   " : Die Differentialgeometrie in der verallgemeinerten Mannigfaltig-
   keit, Rendiconti del Circolo Matematico di Palermo, Bd. 56.
   1932. p. 245.

(2) A. Kawaguchi: The foundation of the Theory of Displacements.

(3) P. Finsler : Ueber Kurven und Flächen in allgemeinen Räumen, Dissertation :
   Göttingen 1918. p. 121.

(4) L. Berwald: Untersuchung der Krümmung allgemeiner metrischer Räume auf
   Grund des in ihnen herrschenden Parallelismus, Mathematische
   Zeitschrift 25. 1926. p. 45.

(5) T. Hosokawa: On the Various Linear Displacements in the Berwald-Finsler's
   Manifold.
On the Linear Displacements in the Generalized Manifold.

On the other hand, E. Cartan\(^{(6)}\) has introduced a plane-element\(^{(7)}\) which is associated to every point in the metric manifold.

Then we shall see, in this paper, how the theory of linear displacements may be modified in the manifold in which every point is associated with plane-elements.

We shall denote this plane-element by \(u_\lambda\).

1. The manifold \(K_n\).

As in the theory of \(X_n\), we shall consider the point will be defined as any ordered set of \(n\) independent real variables \((x_1, x_2, \ldots, x^n)\).

To each point in the manifold \(X_n\), we shall associate a plane-element \(u_\lambda\) which is transformed as a covariant vector in \(X_n\), and then we shall call this associated point set "\(K_n\)." The points in \(X_n\) are called the fundamental points in \(K_n\).

2. Quantities in \(K_n\).

Let the transformation of the fundamental points be

\[
\begin{align*}
\rho^x' &= \rho^x(x'^1, x'^2, \ldots, x'^n) \\
\frac{\partial \rho^x'}{\partial x'^\mu} &= 0
\end{align*}
\]

and the transformation of the plane-elements in the manifold \(K_n\) be

\[
\begin{align*}
(a) & \quad u'_\nu = \frac{\partial \rho^\mu}{\partial x'^\nu} u_\mu \\
(b) & \quad u_\lambda \frac{\partial \rho^\lambda}{\partial x'^\mu} = u_\mu.
\end{align*}
\]

Now we can define contravariant and covariant vectors and scalars.

Any set of \(n\) functions \(\psi^\lambda(x, u) (\lambda = a_1, a_2, \ldots, a_n)\) which is transformed by the transformations (2.1) and (2.2) as follows:

\[
\psi^\lambda(x', u') = \frac{\partial \rho^\lambda}{\partial x'^\mu} \psi^\mu(x, u),
\]

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\(\text{The Science Reports of the Tôhoku Imperial University, series 1, vol. XIX. 1930. p. 37.}\)

\(\text{Connections in the Manifold Admitting Generalized Transformations.}\)

\(\text{Proceedings of the Imperial Academy VIII. 1932. p. 348.}\)

\(\text{Ueber nicht-holonomre Uebertragung in allgemeine Mannigfaltigkeit \(T_n\).}\)

\(\text{Journal of the Faculty of Science Hokkaidô Imperial University. series I. Mathematics vol. II. Nos. 1-2 (1934).}\)

\(\text{E. Cartan: Les Espaces Métriques Fondues Sur La Nation D'Aire, Paris 1933.}\)

\(\text{Cartan called it "l'élément d'appui," loc. cit. p. 8.}\)
where \( u_\lambda \) is the plane-element at the point \( x' \), and \( 'u_\lambda \) is the plane-element at the point \( 'x' \), is called a contravariant vector in \( K_n \).

And any set of \( n \) quantities \( w_k(x, u) \) which is transformed by the transformations (2.1) and (2.2) as follows:

\[
(2.4) \quad w'_k(x', u) = \frac{\partial x'^\mu}{\partial x^\lambda} w_\mu(x, u)
\]

is called a covariant vector in \( K_n \).

When a quantity is invariant by the transformations (2.1) and (2.2), the quantity is called a scalar in \( K_n \).

Similarly the transformations of contravariant, covariant and mixed affinos are defined as follows:

\[
\begin{align*}
\rho'_{\lambda \mu \kappa \lambda}(x', u) &= \frac{\partial x'^{\lambda}}{\partial x^\mu} \frac{\partial x'^{\mu}}{\partial x^\kappa} \frac{\partial x'^{\kappa}}{\partial x^\lambda} w(x, u) \\
\rho'_{\lambda \mu \kappa}(x', u) &= \frac{\partial x'^{\lambda}}{\partial x^\mu} \frac{\partial x'^{\mu}}{\partial x^\kappa} \frac{\partial x'^{\kappa}}{\partial x^\lambda} w(x, u) \\
\rho'_{\lambda \mu \kappa}(x', u) &= \frac{\partial x'^{\lambda}}{\partial x^\mu} \frac{\partial x'^{\mu}}{\partial x^\kappa} \frac{\partial x'^{\kappa}}{\partial x^\lambda} w(x, u).
\end{align*}
\]

When we introduce the fundamental tensor \( g_{\lambda \mu}(x, u) \) as in Riemannian geometry, we can get similar definitions about the length of a vector and the angle between two vectors, only different as to the point that all of these quantities are functions of \( x^\lambda \) and \( u_\mu \), say, as follows:

- the length of the contravariant vector \( v^\lambda \)
  \[ v^2 = g_{\lambda \mu} v^\lambda v^\mu \]
- the angle between two contravariant vectors \( v^\lambda \) and \( w^\mu \)
  \[ \cos \theta = \frac{g_{\lambda \mu} v^\lambda w^\mu}{v \cdot w} \]

In this case, as the length and the angle are functions of \( x^\mu \) and \( u_\lambda \), it is noticed that when the plane-element changes the length and the angle are changed.

Let us denote by \( |g|(\pm 0) \) the determinant of \( g_{\lambda \mu}(x, u) \)'s and by \( g^{\lambda \mu}(x, u) \) the cofactor of \( g_{\lambda \mu}(x, u) \) divided by \( |g| \), then we have

\[ g^{\lambda \mu} g_{\mu \nu} = A^\lambda_{\nu} \quad (A^\lambda_{\nu} = \delta^\lambda_{\nu}) \]

and we can get \( w^\mu \) corresponding to \( w_\lambda \) as follows:

\[ g^{\lambda \mu} w_\mu = w^\lambda, \quad g_{\lambda \mu} w^\mu = w_\lambda. \]

3. Tensor calculus in \( K_n \).
To consider the linear displacement in our manifold $K_n$, we shall follow Prof. J. A. Schouten(1) and Mr. T. Hosokawa(2) as follows.

We shall denote two contravariant, covariant, or mixed affinos of arbitrary order by $\mathcal{T}$ and $\mathcal{Q}$, and then,

I. A quantity and its differential are of same kind.

II. The differential is a linear function of linear elements

$$\delta \mathcal{Q} = dx^\mu \nabla_\mu \mathcal{Q}$$

where $\nabla_\mu \mathcal{Q}$ is the covariant derivative with respect to $x^\mu$.

III. The differential of the sum of any two quantities of same kind is the sum of the differentials of the quantities

(A) \[ \delta (\mathcal{Q} + \mathcal{T}) = \delta \mathcal{Q} + \delta \mathcal{T} \]

(B) \[ \nabla_\mu (\mathcal{Q} + \mathcal{T}) = \nabla_\mu \mathcal{Q} + \nabla_\mu \mathcal{T} \]

IV. For the differential of any product of two quantities

(A) \[ \delta (\mathcal{Q} \cdot \mathcal{T}) = (\delta \mathcal{Q}) \cdot \mathcal{T} + \mathcal{Q} \cdot (\delta \mathcal{T}) \]

(B) \[ \nabla_\mu (\mathcal{Q} \cdot \mathcal{T}) = (\nabla_\mu \mathcal{Q}) \cdot \mathcal{T} + \mathcal{Q} \cdot (\nabla_\mu \mathcal{T}) \]

V. The differential of any scalar is equal to the ordinary differential

(V. A) \[ \delta p = dp \]

where $p$ is a scalar.

VI. As to $du_\nu$, we shall define as follows:

$$du_\nu = u_\lambda A_\mu^\lambda dx^\mu$$

then it follows from (V)

(V. B) \[ \nabla_\mu p = \frac{\partial p}{\partial x^\mu} + u_\lambda A_\mu^\lambda \frac{\partial p}{\partial u_\nu} \]

When we differentiate partially components of a contravariant vector $\psi^\lambda(x,u)$ with respect to $u_\mu$, we obtain a tensor of second order as follows.

As $\psi^\lambda(x,u)$ is a vector, it is transformed to $'\psi^\lambda(x',u')$ by the following equations

$$'\psi^\lambda(x',u') = \frac{\partial x'^\lambda}{\partial x^\mu} \psi^\mu(x,u)$$

so, differentiating these equations with respect to $u_\mu$, we obtain

(1) J. A. Schouten, Der Ricci-Kalkül. pp. 63-64.

but, from \(2.2\) (a)

\[ \frac{\partial v^\lambda}{\partial u_\mu} = \frac{\partial x^\mu}{\partial u_\lambda} \]

From these equations we can see \(\partial v^\lambda/\partial u_\mu\) is a tensor of second order, so we shall denote it by \(v^{\lambda\beta}\) or \(\Delta^\beta v^\lambda\) and call it a contravariant derivative of the vector \(v^\lambda\) with respect to \(u_\mu\).

In general, \(\Phi\) being a afinor of arbitrary order \(\partial \Phi/\partial u_\mu\) is a afinor which has one more contravariant index, so we shall call it a contravariant derivative of the afinor \(\Phi\) with respect to \(u_\mu\) and denote it by \(\Phi^{\mu\nu}\) or \(\Delta^\nu \Phi\).

Next we shall define the covariant derivatives of the contravariant and covariant vectors, say,

\[
\begin{align*}
\delta v^\gamma &= d v^\gamma + \Gamma^\gamma_{\alpha\beta} v^\beta d x^\alpha \\
\delta w_\alpha &= d w_\alpha - \Gamma^\alpha_{\beta\gamma} w_\beta d x^\gamma
\end{align*}
\]

where \(\Gamma^\alpha\) and \(\Gamma^{\alpha\beta}\) are the parameters of the connection. In considering of (V. B) we have

\[
\begin{align*}
\nabla_\mu v^\gamma &= \frac{\partial v^\gamma}{\partial x^\mu} + \Gamma^\gamma_{\alpha\beta} v^\beta + u_\alpha A^\gamma_{\beta\mu} v^\beta \\
\nabla_\mu w_\alpha &= \frac{\partial w_\alpha}{\partial x^\mu} - \Gamma^{\alpha\beta}_{\gamma\mu} w_\beta + u_\alpha A^\beta_{\mu\alpha} w_\beta
\end{align*}
\]

As to the derivatives of afinors of higher order, we obtain, for example, the following equations:

\[
\begin{align*}
\delta v_{\alpha\beta}^{\gamma\delta} &= dv_{\alpha\beta}^{\gamma\delta} + \Gamma^\gamma_{\lambda\mu} v_{\alpha\beta}^{\lambda\delta} d x^\mu - \Gamma^\delta_{\alpha\beta} v_{\lambda\gamma}^{\lambda\delta} d x^\mu - \Gamma^\gamma_{\lambda\mu} v_{\delta\beta}^{\lambda\delta} d x^\mu \\
\nabla_\mu v_{\alpha\beta}^{\gamma\delta} &= \frac{\partial v_{\alpha\beta}^{\gamma\delta}}{\partial x^\mu} + u_\delta A^\gamma_{\beta\mu} v_{\alpha\beta}^{\lambda\delta} + \Gamma^\gamma_{\lambda\mu} v_{\alpha\beta}^{\lambda\delta} - \Gamma^\delta_{\alpha\beta} v_{\lambda\gamma}^{\lambda\delta} - \Gamma^\gamma_{\lambda\mu} v_{\delta\beta}^{\lambda\delta}.
\end{align*}
\]

Noticing to the axioms, \(A^{\alpha\beta}_{\lambda\mu}\), \(\Gamma^\gamma_{\lambda\mu}\), and \(\Gamma^{\alpha\beta}_{\lambda\mu}\) in these above equations must be transformed into \(\overline{A}_{\lambda\mu}\), \(\overline{\Gamma}_{\lambda\mu}\), and \(\overline{\Gamma}^{\alpha\beta}_{\lambda\mu}\) respectively by the transformation of fundamental points as follows:

\[
(3.1) \quad \overline{A}_{\lambda\mu} = P^\rho_{\alpha\lambda} Q^\beta_{\mu\beta} A^\gamma_{\alpha\beta}
\]
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\[ (3.2) \quad \Gamma_{\lambda \nu}^\sigma = P_{\lambda}^\sigma \Omega_{\nu}^\mu_0 \Gamma_{\mu \eta}^\nu - \Omega_{\nu}^\mu_0 \Omega_{\mu \eta}^\nu \frac{\partial P_{\lambda}^\sigma}{\partial x^\eta} \]

\[ (3.3) \quad \Gamma_{\lambda \nu}^\sigma = P_{\lambda}^\sigma \Omega_{\nu}^\mu_0 \Gamma_{\mu \eta}^\nu - \Omega_{\nu}^\mu_0 \Omega_{\mu \eta}^\nu \frac{\partial P_{\lambda}^\sigma}{\partial x^\eta} \]

where \( P_{\mu}^\lambda = \frac{\partial x^\lambda}{\partial x^\mu} \) and \( Q_{\mu}^\lambda = \frac{\partial x^\lambda}{\partial x^\mu} \).

Now, as we can see in (3.2) and (3.3)

\[ \nabla_{\mu} A_{\lambda}^\sigma = \Gamma_{\lambda \mu}^\sigma - \Gamma_{\mu \lambda}^\sigma = C_{\mu \lambda}^\sigma (x, u) \]

is an affinor and the case in which

\[ C_{\mu \lambda}^\sigma (x, u) = C_{\mu} (x, u) \]

is called "incident invariance," and \( C_{\mu \lambda}^\sigma (x, u) = 0 \) identically "contractional invariance." Also

\[ \frac{1}{2} (\Gamma_{\lambda \mu}^\sigma - \Gamma_{\mu \lambda}^\sigma ) = S_{\lambda \mu}^\sigma (x, u) \]

is an affinor and the case in which

\[ S_{\lambda \mu}^\sigma (x, u) = S_{\mu \lambda}^\sigma (x, u) \]

is called "covariant semi-symmetricity," and \( S_{\lambda \mu}^\sigma (x, u) = 0 \) identically "covariant symmetricity." Moreover, when

\[ \nabla_{\mu} g_{\lambda \sigma} (x, u) = Q_{\mu \lambda}^\sigma (x, u) = Q_{\mu} (x, u) g_{\lambda \sigma} (x, u) \]

this case is called "conformal," and when \( \nabla_{\mu} g_{\lambda \sigma} (x, u) = 0 \) identically "metric." All these names are due to Prof. J. A. Schouten.\(^{(1)}\)

Now we will consider the covariant derivatives of \( g^{\alpha \beta} \)

\[ \nabla_{\mu} g^{\alpha \beta} = Q_{\mu}^{\alpha \beta} + \frac{\partial g^{\alpha \beta}}{\partial x^\mu} + u \xi^{\mu \rho} g^{\alpha \beta} \]

contracting \( g_{\alpha \beta} g_{\lambda \sigma} \)

\[ g_{\alpha \beta} \Gamma_{\lambda \mu}^\sigma + g_{\beta \alpha} \Gamma_{\mu \lambda}^\sigma = -g_{\alpha \beta} g_{\lambda \sigma} \frac{\partial g^{\alpha \beta}}{\partial x^\mu} + g_{\alpha \beta} g_{\beta \lambda} Q_{\mu}^{\alpha \beta} - u \xi^{\mu \rho} g^{\alpha \beta} \]

On the other hand, differentiating \( g^{\alpha \beta} g_{\beta \alpha} = \delta^\alpha_\lambda \) with respect to \( x^\mu \)

\[ g_{\beta \alpha} \frac{\partial g^{\alpha \beta}}{\partial x^\mu} + g^{\alpha \beta} \frac{\partial g_{\beta \alpha}}{\partial x^\mu} = 0 \]

contracting \( g_{\alpha \beta} \)

\[ -g_{\alpha \beta} g_{\beta \alpha} \frac{\partial g^{\alpha \beta}}{\partial x^\mu} = \frac{\partial g_{\alpha \beta}}{\partial x^\mu} \]

\(^{(1)}\) J. A. Schouten, Der Ricci-Kalkül, p. 75
Similarly differentiating with respect to \( u_\eta \)

\[ g_{\beta \eta} g^\alpha \eta + g_{\alpha \eta} g^\beta \eta = 0 \]

contracting \( g_{\alpha \nu} \)

\[-g_{\alpha \nu} g_{\beta \mu} g_{\eta \eta} = 0\]

therefore

\[ g_{\alpha \nu} \Gamma^\eta_{\lambda \mu} + g_{\alpha \lambda} \Gamma^\eta_{\nu \mu} = \frac{\partial g_{\lambda \nu}}{\partial x^\mu} + g_{\alpha \lambda} g_{\beta \mu} Q^\eta_{\beta \mu} + \eta \eta \mu \lambda \nu \eta \eta \lambda \nu \eta \]

say,

\[ g_{\alpha \nu} \Gamma^\eta_{\lambda \mu} + g_{\alpha \lambda} \Gamma^\eta_{\nu \mu} = \frac{\partial g_{\lambda \nu}}{\partial x^\mu} + g_{\alpha \lambda} g_{\beta \mu} Q^\eta_{\beta \mu} + g_{\lambda \eta} g_{\beta \mu} Q^\eta_{\beta \mu} \]

similarly

\[ g_{\alpha \nu} \Gamma^\eta_{\lambda \mu} + g_{\alpha \lambda} \Gamma^\eta_{\nu \mu} = \frac{\partial g_{\lambda \nu}}{\partial x^\mu} + g_{\alpha \lambda} g_{\beta \mu} Q^\eta_{\beta \mu} + g_{\lambda \eta} g_{\beta \mu} Q^\eta_{\beta \mu} \]

\[-g_{\alpha \nu} \Gamma^\eta_{\lambda \mu} - g_{\alpha \lambda} \Gamma^\eta_{\nu \mu} = \frac{\partial g_{\lambda \nu}}{\partial x^\mu} + g_{\lambda \eta} g_{\beta \mu} Q^\eta_{\beta \mu} - g_{\lambda \eta} g_{\beta \mu} Q^\eta_{\beta \mu} \]

Summing up these equations and putting

\[ \frac{1}{2} (\Gamma^\eta_{\lambda \mu} - \Gamma^\eta_{\nu \lambda}) = S^\eta_{\lambda \mu} \]

we obtain

\[ 2g_{\alpha \nu} \Gamma^\eta_{\lambda \mu} = \left( \frac{\partial g_{\lambda \nu}}{\partial x^\mu} + \frac{\partial g_{\nu \lambda}}{\partial x^\mu} - \frac{\partial g_{\lambda \nu}}{\partial x^\mu} \right) \]

\[ + \left[ g_{\alpha \lambda} g_{\beta \mu} Q^\eta_{\beta \mu} + g_{\alpha \mu} g_{\beta \nu} Q^\eta_{\beta \nu} - g_{\alpha \eta} g_{\mu \eta} Q^\eta_{\mu \eta} \right] \]

\[ + 2g_{\alpha \lambda} S_{\lambda \mu} + 2g_{\alpha \mu} S_{\mu \nu} + 2g_{\alpha \eta} S_{\eta \nu} + u_{\gamma} (A^\alpha_{\beta \mu} g_{\gamma \nu} + A^\beta_{\gamma \nu} g_{\alpha \lambda} - A_{\lambda \eta} g_{\nu \mu} \eta) \]

hence,

\[ \Gamma^\eta_{\lambda \mu} = \{ \lambda \mu \} + T_{\lambda \mu}^\eta (\lambda \mu) \]

where

\[ \{ \lambda \mu \} = \frac{1}{2} g_{\lambda \mu} \left( \frac{\partial g_{\mu \lambda}}{\partial x^\nu} + \frac{\partial g_{\lambda \mu}}{\partial x^\nu} - \frac{\partial g_{\mu \lambda}}{\partial x^\nu} \right) \]

\[ T_{\lambda \mu}^\eta = \frac{1}{2} (g_{\lambda \mu} Q^\eta_{\nu} + g_{\mu \nu} Q^\eta_{\lambda} - g_{\gamma} g_{\lambda \mu} g_{\nu \eta} Q^\eta_{\gamma}) \]

\[ + S_{\lambda \mu}^\eta + g_{\alpha \mu} g_{\beta \nu} S_{\alpha \beta} + g_{\alpha \eta} g_{\beta \nu} S_{\alpha \beta} \]

\[ (\lambda \mu) = \frac{1}{2} \eta (A_{\lambda \mu} g_{\gamma \nu} + A_{\mu \nu} g_{\gamma \lambda} - A_{\gamma \nu} g_{\lambda \mu} \eta) \]

as

\[ \Gamma^\eta_{\lambda \mu} = C_{\lambda \mu}^\eta \]
similarly
\[ \Gamma^\nu_{\lambda\mu} = \{ \nu_{\lambda} \} + T^\nu_{\lambda\mu} + (s^\nu_{\lambda}) \]
where
\[ T^\nu_{\lambda\mu} = T^\nu_{\lambda\mu} - C^\nu_{\lambda\mu}^{(3)}. \]


To obtain the curvature tensors we shall calculate \( 2\nabla_{\mu} \nabla_{a} v^\lambda \). After some calculation, we obtain

\[ 2\nabla_{\mu} \nabla_{a} v^\lambda = -\left( \frac{\partial \Gamma^\lambda_{\alpha\mu}}{\partial x^a} \frac{\partial \Gamma^\lambda_{\alpha\mu}}{\partial x^a} + \Gamma^\lambda_{\alpha\mu} \Gamma^\alpha_{\mu\nu} \right) v^\nu \]
\[ -\Gamma^\lambda_{\alpha\mu} \Gamma^\alpha_{\mu\nu} + u_{\alpha\mu} A^\alpha_{\mu\nu} \frac{\partial \Gamma^\lambda_{\alpha\mu}}{\partial v^\nu} - u_{\alpha\mu} A^\alpha_{\mu\nu} \frac{\partial \Gamma^\lambda_{\alpha\mu}}{\partial v^\nu} \right) v^\nu \]
\[ -\left( \frac{\partial A^\beta_{\nu\mu}}{\partial x^a} + \frac{\partial A^\beta_{\nu\mu}}{\partial x^a} + \frac{\partial A^\beta_{\nu\mu}}{\partial x^a} \right) v^\nu \]
\[ + 2S^{\nu} \nu_{\mu\nu} \nabla_{a} v^\lambda. \]

Therefore if we put,

\[ R^{\alpha\beta\gamma}_{\mu\nu\lambda} = \frac{\partial \Gamma^\lambda_{\alpha\mu}}{\partial x^\nu} - \frac{\partial \Gamma^\lambda_{\alpha\nu}}{\partial x^\mu} + \Gamma^\lambda_{\alpha\mu} \Gamma^\alpha_{\nu\gamma} + u_{\mu\nu} \frac{\partial \Gamma^\lambda_{\alpha\mu}}{\partial v^\gamma} - u_{\nu\gamma} \frac{\partial \Gamma^\lambda_{\alpha\nu}}{\partial v^\mu} \]
\[ K^{\alpha\beta\gamma}_{\mu\nu\lambda} = \frac{\partial A^\alpha_{\mu\nu}}{\partial x^\lambda} + \frac{\partial A^\alpha_{\mu\nu}}{\partial x^\lambda} + \frac{\partial A^\alpha_{\mu\nu}}{\partial x^\lambda} \]

we obtain

(4.1) \[ 2\nabla_{\mu} \nabla_{a} v^\lambda = -R^{\alpha\beta\gamma}_{\mu\nu\lambda} v^\alpha \]

Similarly as to a covariant vector \( w_{\lambda} \) we can get

(4.2) \[ 2\nabla_{\mu} \nabla_{a} w_{\lambda} = R^{\alpha\beta\gamma}_{\mu\nu\lambda} w^\alpha \]

where

\[ R^{\alpha\beta\gamma}_{\mu\nu\lambda} = \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\lambda} - \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\gamma} + \Gamma^\alpha_{\mu\nu} \Gamma^\nu_{\gamma\lambda} - \Gamma^\nu_{\lambda\gamma} \Gamma^\alpha_{\mu\nu} + u_{\nu\gamma} \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial v^\lambda} - u_{\lambda\gamma} \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial v^\lambda} \]

As to a scalar we can get the following formulae,

(4.3) \[ 2\nabla_{\mu} \nabla_{a} p = -K^{\mu\nu\lambda\beta} w_{\alpha} p^\alpha + 2S^{\nu} \nu_{\mu\nu} \nabla_{a} v^\lambda \]

where \( p \) is an arbitrary scalar.

We will call above three tensors \( R^{\alpha\beta\gamma}_{\mu\nu\lambda}, R^{\alpha\beta\gamma}_{\mu\nu\lambda}, K^{\alpha\beta\gamma}_{\mu\nu\lambda} \) the curvature-tensors in our manifold \( K_{\alpha} \) which are, of course, the functions of \( z \) and \( u \).

By the covariant differentiation of (4.3) we obtain

(1) See T. Hosokawa, loc. cit. 1930.
and putting $\nabla p$ instead of $w_\mu$ in the equations which are reduced from (4.2)

$$2\nabla (\nabla p)\nabla p = R^{\mu\nu}_{\lambda\mu\nu}p + K_{\lambda\nu\gamma}u_\gamma p + 2S^\gamma_{\lambda\nu\gamma}n_\nu p,$$

we obtain

$$2\nabla (\nabla p)\nabla p = R^{\mu\nu}_{\lambda\mu\nu}p + K_{\lambda\nu\gamma}u_\gamma p + 2S^\gamma_{\lambda\nu\gamma}n_\nu p.$$ 

Summing up (4.4) and (4.5)

$$\nabla \nabla p - \nabla \nabla p = \nabla p - K_{\lambda\nu\gamma}u_\gamma p + 2S^\gamma_{\lambda\nu\gamma}n_\nu p + R^{\mu\nu}_{\lambda\mu\nu}p - K_{\lambda\nu\gamma}u_\gamma p + 2S^\gamma_{\lambda\nu\gamma}n_\nu p$$

similarly

$$\nabla \nabla p - \nabla \nabla p = \nabla p - K_{\lambda\nu\gamma}u_\gamma p + 2S^\gamma_{\lambda\nu\gamma}n_\nu p + R^{\mu\nu}_{\lambda\mu\nu}p - K_{\lambda\nu\gamma}u_\gamma p + 2S^\gamma_{\lambda\nu\gamma}n_\nu p.$$ 

Summing up these equations, we obtain

$$0 = \nabla p - K_{\lambda\nu\gamma}u_\gamma p + 2S^\gamma_{\lambda\nu\gamma}n_\nu p + R^{\mu\nu}_{\lambda\mu\nu}p - K_{\lambda\nu\gamma}u_\gamma p + 2S^\gamma_{\lambda\nu\gamma}n_\nu p$$

This is what corresponds to the identity which was obtained by T. Hosokawa (loc. cit. 1930) that is a generalization of the identity of Berwald.

For a special case in which

$$S_{\lambda\mu\nu} = 0$$

we have

$$\nabla (K_{\lambda\nu\gamma}u_\gamma p) + \nabla (K_{\lambda\nu\gamma}u_\gamma p) + \nabla (K_{\lambda\nu\gamma}u_\gamma p) + K_{\lambda\nu\gamma}u_\gamma p - K_{\lambda\nu\gamma}u_\gamma p + 2S^\gamma_{\lambda\nu\gamma}n_\nu p + R^{\mu\nu}_{\lambda\mu\nu}p - K_{\lambda\nu\gamma}u_\gamma p + 2S^\gamma_{\lambda\nu\gamma}n_\nu p = 0$$

as we can easily see from definition.

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