A Theorem Concerning the Derivatives of Meromorphic Functions.

By Kosaku Yosida.

(Read April 3, 1935.)

Given a meromorphic function \( y = f(x) \) we let \( x = g(y) \) be its inverse function defined on the Riemann surface \( R \).

We assume that the branches \( x = g_i(y) \), \( (i=1, 2, \ldots) \) of the function \( x = g(y) \) are expandable in regular power series \( P_i(t) \), \( (i=1, 2, \ldots) \):

\[
(1) \quad x = g_i(y) = P_i(t), \quad t = y^\frac{1}{\tau_i} \quad \text{in} \quad |t| < \delta_i^\frac{1}{\tau_i},
\]

where \( \tau_i \) denote integers \( \geq 1 \) and \( \delta \) a fixed positive constant not larger than 1.\(^a\)

For any positive constant \( \delta < \delta \) the circles \( |t| < \delta_i^\frac{1}{\tau_i} \) are thus mapped upon the schlicht finite domains \( B_i \) of the \( x \)-plane by \( x = P_i(t) \).

We denote by \( B_i \) the mapped images of the circles \( |t| < \delta_i^\frac{1}{\tau_i} \) by \( x = P_i(t) \).

If the origin \( x = 0 \) belongs to any one of the domains \( B_i \) or on its boundary, we choose it as \( B_i \). Then we have

Theorem. If we choose arbitrarily one point \( x \) within \( B_i \) \( (i=2, 3, \ldots) \) then

\[
(2) \quad \sum_{i=1}^{\infty} \frac{|f(x)|^\frac{1}{\tau_i} - 1}{|x|^{\frac{1}{\tau_i}} (1 + (\log 2\pi)^2 + \epsilon^2) |f(x)|^\frac{1}{\tau_i} < \frac{12\delta x}{(6-\delta)^2}}
\]

Proof. The functions \( x = P_i(t) \) are regular and schlicht in the circles \( |t| < \delta_i^\frac{1}{\tau_i} \), \( \delta_i^\frac{1}{\tau_i} < \delta \), where \( x = P_i(t) \). By "Verzerrungsmatr" the mapped images of the circles \( \delta_i^\frac{1}{\tau_i} < \delta_i^\frac{1}{\tau_i} \) by \( x = P_i(t) \) contain entirely the circles \( C_i \):

\[
(3) \quad |x - x_i| < \frac{1}{\tau_i} (\delta_i^\frac{1}{\tau_i} - \delta_i^\frac{1}{\tau_i}) P_i'(t_i).
\]

By the assumption the circles \( C_i \) do not contain the origin \( x = 0 \) in their interior, if \( i = 2, 3, \ldots \). Hence we have, by (3),

(1) These functions \( x = g(y) \) need not exhaust all the branches of \( x = g(y) \) at the point \( y = 0 \).

(2) The assumption \( \delta \leq 1 \) does not diminish the generality of the problem.
1935] Theorem Concerning the Derivatives of Meromorphic Functions. 171

and therefore

\[ |x| = \frac{1}{2\varepsilon_1} (\phi^i - \phi^1) |P'_i(t_i)|, \quad i \equiv 2, \]

(4) \[ |x| < |x_1| + \frac{1}{2} (\phi^i - \phi^1) |P'_i(t_i)| < 2|x| \] in \( C_i \), \( i = 2, 3, \ldots. \)

By the transformation \( z = \log x \) the \( z \)-plane cut along the positive real axis is monovaluedly (schlicht) mapped upon the strip \(-\pi < \Re(z) \leq \pi\) of the \( z \)-plane. The mapped image \( D_i \) of the circle \( C_i \) by this transformation, is not necessarily a connected domain. But these \( D_i \) are all contained in the strip and do not overlap each other.

Now consider a circular cylinder \( C \) of diameter 1, which touches the \( z \)-plane along the imaginary axis; and denote by \( G \) the generating line of \( C \) whose distance from the \( z \)-plane is 1. If we project the strip \(-\pi < \Re(z) \leq \pi\) upon \( C \) by the straight lines orthogonal to the line \( G \), the mapped image will have the area \( = 2\pi^2 \):

\[ \int \int \frac{d\Omega}{1 + |z|^2} = 2\pi^2, \quad d\Omega = \text{arclength on the } z\text{-plane}. \]

Hence we have

\[ \sum_{i=1} \int \int \frac{d\Omega}{1 + |z|^2} < 2\pi^2, \]

So that

(5) \[ \begin{cases} \sum_{i=1} \int \int \frac{d\omega}{1 + (\log |r_i|)^2 + \pi^2} < 2\pi^2 \\ d\omega = \text{arclength on the } x\text{-plane} \end{cases} \]

From (4) and (5), we obtain

\[ \sum_{i=1} \int \int \frac{d\omega}{4|x|^2 (1 + (\log |2x|)^2 + \pi^2)} < 2\pi^2. \]

Therefore by (3),

\[ \sum_{i=1} \frac{\pi \left( \frac{1}{4} (\phi^i - \phi^1) |P'_i(t_i)| \right)^2}{4|x|^2 [1 + (\log |2x|)^2 + \pi^2]} < 2\pi^2. \]

This proves the theorem, since we have

\[ (\phi - \phi_1) < \tau (\phi^i - \phi^1) < \log \frac{\delta_i}{\delta}; \quad (0 < \delta_i < \delta \leq 1, \tau_i \equiv 1), \]

\[ \phi_i \equiv \frac{1}{2} (\phi^i - \phi^1) |P'_i(t_i)|, \quad i \equiv 2, \]

\[ \phi \equiv \frac{1}{2} (\phi^i - \phi^1) |P'_i(t_i)|. \]
\[ P'(t_i) = \frac{|d y_i(y)|}{d y} \left. \frac{dy}{dt} \right|_{t_i} = \frac{\tau f'(x_i)^{r-1}}{f'(x_i)} \]

\section*{§ 2. Applications}

1. Ullrich's theorem. Let \( \tau_1 = 1 \) and \( |f'(x)| > \alpha \) positive constant \( \alpha \), for \( i = 2, 3, \ldots \), the above theorem shows that

\[ \sum_{i=2}^{\infty} \frac{1}{|x_i|^s} \text{ convergent}, s > 0. \]

This is the theorem 3 of E. Ullrich's paper\(^1\). The theorem 4\(^2\) of his paper is also contained in our theorem.

2. Selberg's theorem. Let the regular power series \( x = P_i(t), i = 1, 2, \ldots \), given in (1), represent all the branches \( x = g_i(z), i = 1, 2, \ldots \) of the inverse function \( x = g(y) \) which correspond to the point \( y = 0 \) on \( R \). We denote by \( B'_i \) the mapped images of the circles \( |t'_i| < \delta_i \) by \( z = P_i(t) \).

Let \( \tau(x) \) be the maximum of the integers \( \tau_i \) which correspond to the domains \( B'_i \) cut or touched by the circle \( |t'| = r \). Then we have, since \( \delta_i \leq 1 \), by (2),

\[ \frac{1}{f'(x)} \leq \tau(x) \frac{1}{r} \left( \frac{y_1^2 + (\log \frac{2}{\delta_i})^2}{\frac{r}{\tau r_i}} \right), r = |x|, \]

at the point \( x \) where \( f(x) \leq \delta_i \), \( \tau(x) \) denoting a positive function (\( \leq 1 \)) which tends to zero as \( x \) tends to \( \infty \).

This result is essentially the same as that obtained by H. Selberg\(^4\), by using an asymptotic estimation concerning modular functions.

3. A Generalisation of Collingwood-Cartan theorem\(^5\). The hypothesis is the same as above. We obtain

\[ m(r, \frac{1}{f}) \leq \tau(r) \left( m(r, \frac{f'}{f}) + O(\log r) \right), \]


by the inequality

$$m(r, \frac{1}{f}) \leq m(r, \frac{f'}{f}) + \frac{1}{2} \int_{\theta=0}^{\frac{2\pi}{r}} \log \frac{1}{|f'(r e^{i\theta})|} \, d\theta,$$

and the formula (7) as in Selberg's paper\(^{(8)}\).

The formula (8) shows that whenever \(v(r) = O(T(r, f)^{1-\eta})\), \(\eta > 0\), the defect \(\delta(0)\) of the function \(f(z)\) is \(0^{(7)}\). Mr. Shisuo Kakutani obtained elegantly a result more precise in the following paper.\(^{(8)}\)

Mathematical Institute,
Osaka Imperial University.

(Received 18. April, 1935.)

---

\(^{(6)}\) H. Selberg: loc. cit., p. 60.

\(^{(7)}\) Of course the function \(f(z)\) is supposed to be transcendental, not rational. Hence \(\lim_{r \to \infty} \frac{T(r, f)}{r} = 0\).