On the Exceptional Value of Meromorphic Functions.

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Let $y=f(x)$ be a meromorphic function, then $a$ is said to be exceptional (in R. Nevanlinna's sense) if the defect

$$
\delta(a) = \lim_{r \to \infty} \frac{m(r, a)}{T(r, f)}
$$

is positive.

R. Nevanlinna conjectured that an exceptional value in his sense will be at the same time an asymptotic value, i.e. a transcendental singular point of the inverse function $x=y(y)$ of $y=f(x)$.

In this paper we shall give some sufficient conditions in order that a given value $a$ may be normal (i.e. non-exceptional).

To make expressions simple, we shall state and prove our theorems in the case $a=0$.

Theorem I. If every branch $g_1(y)$ of $g(y)$ is regular for $|y|<k$, $k>0$, $k$ being fixed, then $y=0$ is not an exceptional value of $f(x)$.

Theorem II. If every branch $g_1(y)$ of $g(y)$ is of algebraic character for $|y|<k'$, and regular for $k' \leq |y|<k$, $0<k'<k$, $k$ and $k'$ being both fixed, and, furthermore, the number $\lambda$ of different branches of $g(y)$ attained by means of prolongation within $|y|<k$ from any one of them is collectively bounded ($\leq L$), then $y=0$ is not an exceptional value of $f(x)$.

H. Cartan proved the theorem I by showing

$$
\lim_{r \to \infty} \frac{m(r, 0)}{\log T(r, f)} \leq 1
$$

and H. Selberg gave the relation

$$
m(r, 0) = O(\log T(r, f)) + o(\log r)
$$

under the assumption that every $g_1(y)$ is regular for $0<|y|<k$ and of

References:

1. R. Nevanlinna: Le théorème de Picard-Borel et la théorie des fonctions méromorphes, p. 93, § 42.
algebraic character at \( y=0 \) and that its degree is collectively bounded.

Recently Mr. K. Yosida\(^{4}\) gave the good elementary proof of the same relation (without use of modular functions).

Our proof of theorem I is based on geometrical considerations and can be applied to theorem II so that we shall prove only theorem I.

**Proof of Theorem I.**

We consider as usual the Riemann sphere \( S \) of radius \( \frac{1}{2} \) touching \( y \)-plane at \( y=0 \).

Projecting \( y \)-plane stereographically on \( S \), let \( I_r \) be a curve on \( S \) corresponding to \( |x| = r \) by \( y = f(x) \).

The length \( L(r) \) of \( I_r \) is given by

\[
L(r) = \int_0^{2\pi} \frac{|f'(rc^{i\theta})|}{1 + |f'(rc^{i\theta})|^2} r d\theta.
\]

Then, by a geometrical consideration, we have

\[
(1) \quad u(r, k e^{i\varphi}) - u(r, 0) \leq \frac{L(r)}{2l} = \frac{1}{2l} \int_0^{2\pi} \frac{|f'(rc^{i\theta})|}{1 + |f'(rc^{i\theta})|^2} r d\theta
\]

for a fixed value \( k_0 \), \( 0 < k_0 < k \), and for any \( \varphi \), \( 0 \leq \varphi < 2\pi \), \( l \) being the distance (measured on \( S \)) between two circles on \( S \) corresponding to \( |y| = k_0 \) and \( |y| = k \) respectively.

Integrating the both sides of (1) with respect to \( \log r \), we have

\[
N(r, k e^{i\varphi}) - N(r, 0) \leq N(1, k e^{i\varphi}) - N(1, 0) + \frac{1}{2l} \int_1^r \int_0^{2\pi} \frac{|f'(rc^{i\theta})|}{1 + |f'(rc^{i\theta})|^2} r d\theta d\gamma,
\]

by Schwarz's inequality

\[
\leq N(1, k e^{i\varphi}) - N(1, 0) + \frac{1}{2l} \sqrt{\int_1^r \int_0^{2\pi} \frac{|f'(rc^{i\theta})|^2}{(1 + |f'(rc^{i\theta})|^2)^2} r d\theta d\gamma} \sqrt{\int_1^r \int_0^{2\pi} d\theta d\gamma}
\]

or

\[
(2) \quad N(r, k e^{i\varphi}) - N(r, 0) \leq N(1, k e^{i\varphi}) + \frac{1}{2l} \sqrt{\int_1^r \int_0^{2\pi} \frac{|f'(rc^{i\theta})|^2}{(1 + |f'(rc^{i\theta})|^2)^2} r d\theta d\gamma} \log r.
\]

Integrating once more the both sides of (2) with respect to \( \varphi \) from 0 to \( 2\pi \) and using II. Cartan's relation

\[
(4) \quad K. Yosida: A theorem concerning the derivatives of meromorphic functions, this Proceeding, June, p. 268 (1935).
\]

To prove Theorem II take \( k = k' \).

Then (2) becomes

\[
A(r, f) - A(r, 0) \leq \frac{L(r)}{2l}.
\]

(5) To prove Theorem II take \( k = k' \).

Theorem I follows from this at once if we use the inequality
\[ A(r) \leq (T(r, f))^{1+s}, \quad s > 0, \]
which is satisfied for all \( r \), except possibly in the intervals where the total variation of \( r \) is finite.

By the proof above we can obtain easily that, even if \( \lambda \) is not collectively bounded we have the same result if only the maximum \( \lambda(r) \) of \( \lambda \) such that \( \varphi(y) = r \) is satisfied for at least one element of its prolongation within \( |y| < k \) satisfies the inequality
\[ \int^r_0 \frac{\Lambda(t)^2}{t} \pi(t) = O((A(r))^{1-s}), \quad s > 0. \]

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Without loss of generality we can assume \( f(0) \neq \infty \).