On the Connections in $X_n$ Associated with the Points of $Y_m$

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(Read July 6, 1935.)

§1. Let us consider, in the first place, an $m$-dimensional manifold $Y_m$ which may be defined by a set of $m$ ordered real variables $y^a, y^b, \ldots, y^m$. With every point of the manifold $Y_m$, we associate an $n$-dimensional manifold $X_n$ which may be also defined by a set of $n$ ordered real variables $x^1, x^2, \ldots, x^n$. Then we shall call $Y_m$ "the fundamental manifold" and $X_n$ "the associated manifold," respectively.

With respect to the transformation of the coordinates in the fundamental manifold $Y_m$, we may choose the following:

$$y'^a = y''(y^a, y^b, \ldots, y^m)$$

Concerning the transformation of the coordinates in the associated manifold $X_n$, we can choose the transformation law of the form

$$x'^i = x''(x^1, x^2, \ldots, x^n; y^a, y^b, \ldots, y^m).$$

only when we consider the manifold $X_n$ associated with a fixed point of $Y_m$.

But with regard to the transformation of the coordinates in the manifold $X_n$ associated with a point $(y^a, y^b, \ldots, y^m)$ of the fundamental manifold $Y_m$, we must take the transformation law of the form

$$x'^i = x''(x^1, x^2, \ldots, x^n; y^a, y^b, \ldots, y^m).$$

Considering the manifolds $Y_m$ and $X_n$ simultaneously, we get an $(m+n)$-dimensional manifold $M_{m+n}$ so the theory of connections in the manifold associated with a fundamental manifold will be reduced to the one in an $(m+n)$-dimensional manifold, but the transformation law of the coordinates in the $(m+n)$-dimensional manifold is not the most general one, but the one of the form

$$y'^a = y''(y), x'^i = x''(x; y).$$

(1) Latin letters $i, j, k, \ldots$ take on the values $a_1, a_2, \ldots, a_m$.
(2) Greek letters $\lambda, \mu, \nu, \ldots$ take on the values $1, 2, \ldots, n$.
(3) The theory of linear displacements in the manifold admitting such a transformation was also studied by Mr. Shisanji, Hokari, see, "Ueber die Uebertragungen, die der erweiterten Transformationsgruppe angehören," Journal of the Faculty of Science, Hokkaido Imperial University, Series I. Mathematics Vol. III. No. 1, pp. 15-26 (1935).
§ 2. Next, we shall define the quantities in our fundamental manifold $Y_m$ by a quite analogous manner as in the ordinary tensor calculus. Of course, all the quantities in $Y_m$ are functions of $y$ only and are transformed under the transformation of the coordinates as follows:

- **Scalar**: $p, : p' = p$
- **Contravariant Vector**: $v^i, : v'^i = \frac{\partial p'^i}{\partial y^i} v^i$
- **Covariant Vector**: $w_a, : w'_a = \frac{\partial y^a}{\partial y'^i} w_a$
- **Affinor**: $v_{ij}, : v'_{ij} = \frac{\partial y^i}{\partial y'^j} \frac{\partial y^j}{\partial y'^i} v_{ij}$

Now we shall define affinors in our associated manifold $X_n$. By the consideration stated above, the quantities in $X_n$ must be not only functions of $x$, but also functions of $y$. We shall choose the following as the transformation law of the quantities in the manifold $X_n$.

- **Scalar**: $p, : p' = p$
- **Contravariant Vector**: $v^i, : v'^i = \frac{\partial x'^i}{\partial x^i} v^i$
- **Covariant Vector**: $w_a, : w'_a = \frac{\partial x}{\partial x'^a} w_a$
- **Affinor**: $v_{ij}, : v'_{ij} = \frac{\partial x'^i}{\partial x^j} \frac{\partial x'^j}{\partial x^i} v_{ij}$

Finally, we shall define the quantities in the manifold $M_{m+n}$. As a typical example, we may take a contravariant vector in $M_{m+n}$. Of course, a contravariant vector in $M_{m+n}$ must have $(m+n)$ components which are functions of $x'$'s and $y$'s and are transformed as differentials of coordinates. The set of differentials $(dx^a, dy')$ being transformed as follows:

$$dx'^a = \frac{\partial x'^a}{\partial x^a} dx^a + \frac{\partial x'^a}{\partial y'} dy',
$$
$$dy'^i = \frac{\partial y'^i}{\partial y'} dy',
$$

a contravariant vector in $M_{m+n}$ must have the components which are functions of $x'$'s and $y$'s.

1. The summation convention will be followed throughout the paper.
and these must be transformed under the transformation of the coordinates in \( M_{m+n} \) as follows:

\[
v' = \frac{\partial x'}{\partial x} v + \frac{\partial x'}{\partial y} v',
\]

\[
v'' = \frac{\partial y'}{\partial y} v'.
\]

When we introduce the following notation

\[
A'_{\mu} = \frac{\partial x'}{\partial x}, \quad A'_{\nu} = \frac{\partial x'}{\partial y}, \quad A''_{\mu} = \frac{\partial y'}{\partial x}, \quad A''_{\nu} = \frac{\partial y'}{\partial y},
\]

we can write the transformation law of a contravariant vector \((v', v')\) in \( M_{m+n} \) as follows,

\[
v' = A'_{\mu} v^\mu + A'_{\nu} v', \quad (2)
\]

Similarly, for a covariant vector \((w_\mu, w_\nu)\) in \( M_{m+n} \), we have

\[
w_\mu = A''_{\mu} w_\mu, \quad w_\nu = A''_{\nu} w_\nu. \quad (3)
\]

As it will be easily verified, the \( A'_{\mu} \)'s satisfy the following relations:

\[
A'_{\mu} = 0, \quad A'_{\nu} = 0,
\]

\[
A''_{\mu} A''_{\nu} = 8 \delta_{\mu}, \quad A''_{\mu} A''_{\nu} = 8 \delta_{\nu},
\]

\[
A''_{\mu} A'_{\nu} + A'_{\mu} A''_{\nu} = 0, \quad A''_{\mu} A''_{\nu} + A'_{\mu} A'_{\nu} = 0,
\]

\[
A''_{\mu} A''_{\nu} = 8 \delta_{\mu}, \quad A''_{\mu} A''_{\nu} = 8 \delta_{\nu}.
\]

Taking components of a covariant vector in the fundamental manifold \( Y_m \) and composing a set of \((m+n)\) functions \((0,0, \ldots 0; w_\mu, w_\mu, \ldots, w_\mu)\), we can consider that they are the components of a covariant vector in \( M_{m+n} \) by means of \((3)\).

Quite similarly composing a set of \((m+n)\) functions \((v', v', \ldots v'); 0, \ldots, 0\) where \( v' \) are components of a contravariant vector in \( X_n \), we can consider that they are the components of a contravariant vector in \( M_{m+n} \) by means of \((2)\).

As we can see easily from the transformation law \((3)\) of a covariant vector \((w_\mu, w_\nu)\) in \( M_{m+n} \), when \((w_\mu, w_\nu)\) are components of a covariant vector in \( M_{m+n} \), then we may consider that \( w_\mu \) are the components of
a covariant vector in \( X_n \).

Moreover, we can consider in \( M_{m+n} \) such an affinor which has the
tensor property not only in the fundamental manifold but also in the
associated manifold.

For example, we may consider a quantity \( K_\beta (x, y) \) whose trans-
formation law is as follows:

\[
K'_{\beta} = \frac{\partial \alpha_\beta}{\partial \alpha_\mu} \frac{\partial \alpha_\mu}{\partial \alpha_\gamma} K_\gamma,
\]

that is to say, when we transform only the coordinates in the funda-
mental manifold, \( K_\beta \) are transformed as the components of an affinor
of second degree with respect to the latin indices, \( i \) and \( j \), and under
the transformation of the coordinates in the associated manifold only, \( K_\beta \)
are transformed as the components of a contravariant vector with respect
to the greek index, \( \lambda \).

Remark. When \( a^\alpha \) and \( b^\lambda \)
are transformed as components of a contravariant vector in \( X_n \) for any
covariant vectors \( a^\alpha \) and \( b^\lambda \) in \( X_n \) and in \( Y_m \), respectively, then it
will be easily verified that the transformation law of \( a^\alpha \) and \( b^\lambda \) is the
followings,

\[
a'^\alpha = \frac{\partial a^\alpha}{\partial a^\gamma} a_\gamma,
\]
\[
b'^\lambda = \frac{\partial b^\lambda}{\partial b^\gamma} b_\gamma.
\]

This may be easily generalized.

§ 3. In the ordinary tensor calculus, a contravariant vector attached
to a fixed point in the manifold, determines a point in the tangential
manifold associated with the fundamental manifold, and a parallel
displacement of a vector may be considered as a certain correspon-
dence of these points in the tangential manifolds associated with the
fundamental manifold at the two near points.

So, also in our case, we shall define the relation between \( X_n \)
associated with the point \( y' \) of \( Y_m \) (we shall use the notation \( X_n(y') \))
and the one associated with the point \( y' + dy' \) of \( Y_m \), that is to say, the
manifold \( X_n(y + dy) \).

When a point \( x^\alpha \) in \( X_n(y) \) corresponds to a point \( x^\alpha + dx^\alpha \) in
\( X_n(y + dy) \), we shall define the relation between \( dx^\alpha \) and \( dy' \) by the
following equations:

\[
dx^\alpha = -\Gamma_\alpha^\gamma dy'
\]

where \( \Gamma_\alpha^\gamma \) are functions of \( x^\alpha \) and \( y' \).

Under the transformation of the coordinates in \( M_{m+n} \), these must
be transformed into

\[ d^x x' = -d^y y'. \]  (5)

But the relations between \( d^x x', d^y y' \) and \( d^x x'', d^y y'' \) are given respectively by the followings

\[ d^x x' = A^x \cdot d^x x'' + A^y \cdot d^y y', \]
\[ d^y y' = A^y \cdot d^y y''. \]

Substituting these equations into (5), we have

\[ A^x d^x x'' + A^y d^y y'' = -B^x A^y d^y y'. \]

Contracting \( A^y \),

\[ d^x x' = -B^y A^y d^y y'^1 - A^x d^x x'^1, \]
then, we have

\[ \Gamma^x y' = B^y A^y d^y y'^1, \]  (6)

similarly,

\[ \Gamma^x y' = \Gamma^x y A^y + A^x A^y. \]  (6')

When we carry the \( X_n \) at \( y \) along a closed curve which surrounds a surface element \( dS \) in \( Y_m \), we get a representation of \( X_n \) on itself.

Supposing that the point \( x \) corresponds to the point \( x' = x + d^x \) of \( X_n \), we have the following relations.

\[ \delta x^A = R_0 dS, \]

where

\[ R_0 = \frac{\partial \Gamma^A}{\partial y' y'} + \Gamma^A \frac{\partial y^A}{\partial x^A} - \frac{\partial y^A}{\partial x^A}, \]

and the transformation law of this quantity is

\[ R_0 = A^y A^x A^y A^x R_0. \]

This law may be verified directly, but we can also verify this law by using the following remark. Let us take an arbitrary afilinor \( \sigma^u \) in \( Y_m \), then

\[ R_0 \sigma^u dt = R_0^{\sigma u} \sigma^u dt, \]

and if we put

\[ dS^u = \sigma^u dt, \]

we have

\[ \delta x^A = R_0^{\sigma u} \sigma^u dt. \]

Then it follows that, for any afilinor \( \sigma^u \) in \( Y_m \), \( R_0^{\sigma u} \sigma^u \) is a contravariant vector in \( X_n \), so, by the remark, \( R_0 \) are transformed by the law mentioned above.

When \( R_0 = 0 \), the representation of the point \( x \) of \( X_n \) on itself is an identical one. So, in this case, the displacement of \( X_n(y_n) \) to \( X_n(y) \) does not depend upon the curve joining \( y_n \) to \( y \).

4. Let us consider an \( X_n(y) \), and, in this manifold \( X_n(y) \), consider two near points \( x \) and \( x + d^x x' \).
Now, when we transfer slightly the point \( y' \) to the point \( y' + dy' \), the point \( x^a \) comes to the point \( x^a + d^a x^a \), say, \( x^a = I_1 dy' \), and the point \( x^a + d^a x^a \) to the point
\[
\begin{align*}
x^a + d^a x^a &= I_1 \{ x^a + d^a x^a \}; y') dy', \quad \text{say,} \\
x^a + d^a x^a &= I_1 \{ x^a \}; y') dy' - \frac{\partial I_1}{\partial x^a} d^a x^a dy'.
\end{align*}
\]

Then we may consider that \( d^a x^a \) are transformed into \( d^a x^a - \frac{\partial I_1}{\partial x^a} d^a x^a dy' \) when the point \( y' \) comes to \( y' + dy' \), so we define that a contravariant vector \( v^a \) in \( X_a \) is transformed into \( v^a - \frac{\partial I_1}{\partial x^a} v^a dy' \) when the point \( y' \) comes to a near point \( y' + dy' \) of \( X_a \).

Now, we shall define the linear displacements of the vectors in the manifold \( X_a(y) \) as follows:

**Scalar** 
\[ \delta p = dp \]

**Contravariant Vector** 
\[ \delta v^a = \frac{\partial v^a}{\partial x^a} d^a x^a + \frac{\partial I_1}{\partial x^a} v^a dy' \]

**Covariant Vector** 
\[ \delta w_\mu (x) = \frac{\partial w_\mu}{\partial x^a} d^a x^a - \frac{\partial I_1}{\partial x^a} w_\mu dy' \]

where \( \delta v^a, \delta w_\mu \) should be, as in the ordinary tensor calculus, vectors, so the transformation law of \( I_1 \) and \( I_1 \) must be as follows:

\[
I_1^\nu = A^\nu_{\mu} A_{a, b} A_{c, d} I_1^\mu + \frac{\partial I_1}{\partial x^a} A_{a, b} A^\mu_{c, d},
\]

\[
T_1^\nu = A^\nu_{\mu} A_{a, b} A_{c, d} T_1^\mu + \frac{\partial T_1}{\partial x^a} A_{a, b} A^\mu_{c, d}.
\]

Then we can compare the vector \( v'(x) \) in \( X_a(y) \) with the vector \( v'(x + dx) \) in \( X_a(y + dy) \), \( x + dx \) being not necessarily the point corresponding to the point \( x' \).

First, we transfer the point \( x' \) to the point \( x' + d^a x' \), that is to say, \( x' - I_1' dy' \), then the vector \( v' \) becomes \( v' - \frac{\partial I_1'}{\partial x^a} v' dy' \). Next, in the manifold \( X_a(y + dy) \) associated with \( y + dy \), we transfer the vector \( v' - \frac{\partial I_1'}{\partial x^a} v' dy' \) at the point \( x' - I_1' dy' \) to the point \( x' + dx' \), by the parallel displacement defined above; then we have

\[
\left( v' - \frac{\partial I_1'}{\partial x^a} v' dy' \right) - I_1'(x' + d^a x', y' + dy') \left( v' - \frac{\partial I_1'}{\partial x^a} v' dy' \right) dx'
\]

where \( dx' = (x' + dx') - (x' - I_1' dy') \).
Calculating the above components of vector, we have
\[ v' = \frac{\partial f'}{\partial x} \delta x' + \Gamma \delta y'. \]

So the difference \( \delta v' \) between the above vector and the vector \( v'(x+dx) \) is as follows:
\[ \delta v' = dv' + \frac{\partial f'}{\partial x} \delta x' + \Gamma \delta y'. \quad (8) \]

From the transformation law of \( f' \), we have
\[ \delta x' = dx' + \Gamma \delta y'. \]

When we consider a vector field \( v'(x, y) \) then we get the covariant differentiation of this vector as follows:
\[ dv' = (\partial_{\mu}v')dx' + (\partial_{\nu}v')dy'. \]

Where
\[ \partial_{\mu}v' = \frac{\partial v'}{\partial x'}, \quad \partial_{\nu}v' = \frac{\partial v'}{\partial y'}. \]

We have
\[ \delta v' = \delta x' \Gamma_{\mu} + dy' \Gamma_{\nu}. \quad (10) \]

Where
\[ \Gamma_{\mu} = \partial_{\mu}v' + \Gamma_{\mu} \nu', \quad (10') \]
\[ \Gamma_{\nu} = \partial_{\nu}v' + \frac{\partial f'}{\partial x'} \nu' - (\partial_{\nu}v') \Gamma. \quad (10'') \]

From the transformation law of \( \Gamma_{\nu} \), we have
\[ \Gamma'_{\nu} = \delta_{\nu}v' + \frac{\partial f'}{\partial x'} v' - (\partial_{\nu}v') \Gamma. \quad (11) \]

\( \Gamma_{\mu}v' \) are evidently transformed as components of a mixed affinor of the second order, we have
\[ \Gamma_{\mu}v' = \Lambda_{\mu}^a \Lambda_{\nu}^b \Gamma_{mn}. \quad (12) \]

Taking
\[ \Gamma_{\mu}v' = \partial_{\mu}v' + (\partial_{\nu}f') \nu' - (\partial_{\mu}v') \Gamma. \]

And substituting the transformation law of \( \Gamma \) in the above equations, we have,
Then
\[ \delta t^\tau = \delta x^\tau \Gamma_{\tau\mu}^\nu v^\mu + \delta y^\gamma \Gamma_{\gamma\gamma}^\nu v^\nu. \]

Thus
\[ \delta t^\tau = A^\mu_{\nu} \delta x^\nu + \delta y^\gamma \Gamma_{\gamma\gamma}^\nu v^\nu. \]

\[ \delta t^\tau = A^\mu_{\nu} \delta x^\nu. \]

§ 5. In the first place, we take a curve \( y' = f'(t) \) in \( Y_m \), joining \( y'_0 \) to \( y'_0 + \Delta y' \), say,

\[ y'_0 = f'(t_0) \quad \text{to} \quad y'_0 + \Delta y'_0 = f'(t_0 + \Delta t), \]

and carry regularly a point \( x' \) of \( X_n(y'_0) \) and a vector \( v' \) at this point along the curve, then we have \( x'(t) \) and \( v'(t) \) in \( X_n(y(t)) \), where \( x'(t) \) and \( v'(t) \) satisfy the following relations,

\[ \frac{dx'^\mu}{dt} = - \Gamma^{\mu}_{\gamma\nu} x'(t), \quad \frac{d\Gamma^\mu_{\gamma\nu}}{dt} = \frac{\partial}{\partial x^\rho} \left[ \Gamma^{\mu}_{\gamma\nu} \right] \frac{df^\rho}{dt} \]

\[ x'(t_0) = x'_0, \quad v'(t_0) = v'_0. \]

Then the following relations are obtained neglecting the terms of higher order with respect to \( dt \),

\[ x'_0 + \Delta x' = x'_0 - \Gamma^\mu_{\gamma\nu} x'_0 \frac{df^\nu}{dt} \]

\[ + \frac{1}{2} \left[ - \Gamma^\mu_{\nu\gamma} \frac{df^\mu}{dt} + \left( \frac{\partial}{\partial x^\rho} \Gamma^\mu_{\gamma\nu} \frac{df^\rho}{dt} \right) \frac{df^\nu}{dt} \right] \Delta t^\nu. \]

\[ v'_0 + \Delta v' = v'_0 - \frac{\partial}{\partial x^\rho} \frac{df^\rho}{dt} \Delta t^\nu. \]

\[ + \frac{1}{2} \left[ - \frac{\partial}{\partial x^\rho} \frac{df^\rho}{dt} \frac{df^\mu}{dt} + \left( \frac{\partial}{\partial x^\rho} \frac{df^\rho}{dt} \right) \frac{df^\mu}{dt} \frac{df^\nu}{dt} \right] \Delta t^\nu \frac{df^\nu}{dt}. \]

Taking another curve \( y' = \phi'(t) \) in \( Y_m \) joining \( y'_0 \) to \( y'_0 + \Delta y' \), say,

\[ y'_0 = \phi'(t_0) \quad \text{to} \quad y'_0 + \Delta y' = \phi'(t_0 + \Delta t) \]

we obtain similarly,

\[ x'_0 + \Delta x' = x'_0 - \Gamma^\mu_{\gamma\nu} x'_0 \frac{df^\nu}{dt} \]

\[ + \frac{1}{2} \left[ - \Gamma^\mu_{\nu\gamma} \frac{df^\mu}{dt} + \left( \frac{\partial}{\partial x^\rho} \Gamma^\mu_{\gamma\nu} \frac{df^\rho}{dt} \right) \frac{df^\nu}{dt} \right] \Delta t^\nu, \]

\[ v'_0 + \Delta v' = v'_0 - \frac{\partial}{\partial x^\rho} \frac{df^\rho}{dt} \Delta t^\nu. \]

\[ + \frac{1}{2} \left[ - \frac{\partial}{\partial x^\rho} \frac{df^\rho}{dt} \frac{df^\mu}{dt} + \left( \frac{\partial}{\partial x^\rho} \frac{df^\rho}{dt} \right) \frac{df^\mu}{dt} \frac{df^\nu}{dt} \right] \Delta t^\nu \frac{df^\nu}{dt}. \]
Again, we carry the point and the vector in $X_n(y_0 + d_1y)$ along the curve
\[ y' = f'(t) + d_1y' \]
from \( y_0 + d_1y' \) to \( y_0 + d_2y' + d_3y' \).

In the former case, \( x_0 + d_1x' \) and \( v_0 + d_1v' \) are carried to
\[ x_0 + d_1x' + d_2x' + d_3x' \text{ and } v_0 + d_1v' + d_2v' + d_3v' \]
respectively, and in the latter case, \( x_0 + d_1x' \) and \( v_0 + d_1v' \) to
\[ x_0 + d_1x' + d_2x' + d_3x' \text{ and } v_0 + d_1v' + d_2v' + d_3v' \]
respectively.

\( x_0 + d_1x' + d_2x' + d_3x' \) may be obtained by writing \( x_0 + d_1x', \ y_0 + d_1y' \)
and \( v_0 + d_1v' \) instead of \( x_0, y_0 \) and \( v_0 \) respectively in \( x_0 + d_1x' \).

Neglecting the terms of higher order with respect to \( dt \), we have
\[
\begin{align*}
&x_{\gamma} + d_{\gamma}x' + d_{\gamma}d_{\gamma}x' = x_{\gamma} + d_{\gamma}x' - I'(x_0 + d_1x, y_0 + d_1y) \frac{d\varphi'}{d\xi} \frac{d\varphi'}{dt}
+ \frac{1}{2} \left[ -I'(x_0 + d_1x, y_0 + d_1y) \frac{\partial I'}{\partial x} \right] d\varphi' \frac{d\varphi'}{dt} \bigg| ds^2
+ \left( \frac{\partial I'}{\partial x} \right) \frac{d\varphi'}{d\xi} \frac{d\varphi'}{dt} \bigg| ds^2

&= x_{\gamma} + d_{\gamma}x' - I'(x_0, y_0) \frac{d\varphi'}{d\xi} \frac{d\varphi'}{dt} + \left( \frac{\partial I'}{\partial x} \right) \frac{d\varphi'}{d\xi} \frac{d\varphi'}{dt} \bigg| ds^2

&= x_{\gamma} + d_{\gamma}x' + d_{\gamma}d_{\gamma}x' + \left( \frac{\partial I'}{\partial x} \right) d\varphi' \bigg| ds^2

&= x_{\gamma} + d_{\gamma}x' + d_{\gamma}d_{\gamma}x' - d_{\gamma}d_{\gamma}x'

&= d_{\gamma}d_{\gamma}x'
\end{align*}
\]
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In the similar manner, we have

\[ d_1\delta + d_2\delta = \left( \left\{ \frac{\partial \Gamma^i}{\partial x^\beta} \frac{\partial \Gamma_i^\beta}{\partial x^\alpha} \Gamma_j^\beta - \frac{\partial \Gamma_i^\alpha}{\partial x^\beta} \frac{\partial \Gamma_j^\beta}{\partial x^\alpha} \right\} \delta \delta \right) dx^\alpha dx^\beta dx^\gamma dx^\delta dx^\gamma dx^\delta \]

\[ = R_{\alpha\beta}^\gamma dx^\alpha dx^\beta dx^\gamma dx^\delta dx^\gamma dx^\delta \]

In order to compare the two vectors, we must carry the one to the point to which the other is attached, so the difference is

\[ \delta v^\mu = d_1\delta + d_2\delta + d_3\delta + d_4\delta + d_5\delta - (\Gamma_{\mu\nu}^\rho dx^\alpha d_4 + d_5 d_2) \]

\[ = \left[ \left\{ \frac{\partial \Gamma^i}{\partial x^\beta} \frac{\partial \Gamma_i^\beta}{\partial x^\alpha} \Gamma_j^\beta - \frac{\partial \Gamma_i^\alpha}{\partial x^\beta} \frac{\partial \Gamma_j^\beta}{\partial x^\alpha} \right\} \delta \delta \right] \delta \delta \]

If we put

\[ \delta v^\mu = R_{\alpha\beta}^\gamma dx^\alpha dx^\beta dx^\gamma dx^\delta dx^\gamma dx^\delta \]

then, we have the following formulae,

\[ S_{\mu\nu} = \frac{1}{2} (\Gamma_{\mu\nu}^\rho - \Gamma_{\nu\mu}^\rho) \]

When we define the covariant differentiation in \( Y^\mu \) by the following equations

\[ \Gamma_{\alpha\beta} = \partial v^\alpha + \Gamma_{\alpha\beta}^\gamma v^\gamma \]

\[ \Gamma_{\alpha\beta} = \partial v^\alpha - \Gamma_{\alpha\beta}^\gamma v^\gamma \]

then, we have the following formulae,

\[ 2\Gamma_{\alpha\beta}^\gamma v^\gamma = v^\gamma \left( \frac{\partial \Gamma_i^\gamma}{\partial x^\beta} \frac{\partial \Gamma_i^\beta}{\partial x^\alpha} + \frac{\partial \Gamma_i^\alpha}{\partial x^\beta} \frac{\partial \Gamma_i^\beta}{\partial x^\gamma} - \frac{\partial \Gamma_i^\beta}{\partial x^\alpha} \frac{\partial \Gamma_i^\gamma}{\partial x^\beta} + \frac{\partial \Gamma_i^\gamma}{\partial x^\alpha} \frac{\partial \Gamma_i^\alpha}{\partial x^\beta} \right) \]

\[ + \frac{\partial \Gamma_i^\gamma}{\partial y^\beta} \frac{\partial \Gamma_i^\beta}{\partial x^\alpha} + \frac{\partial \Gamma_i^\alpha}{\partial y^\beta} \frac{\partial \Gamma_i^\beta}{\partial x^\gamma} - \frac{\partial \Gamma_i^\beta}{\partial x^\alpha} \frac{\partial \Gamma_i^\gamma}{\partial y^\beta} \right) = 2\delta_{\mu\nu} \delta_{\alpha\beta} \]

\[ = v^\gamma \Gamma_{\mu\nu}^\gamma v^\gamma - \frac{\partial \Gamma_i^\gamma}{\partial x^\beta} \delta_{\mu\nu} - 2\delta_{\mu\nu} \delta_{\alpha\beta} \]

\[ = v^\gamma R_{\mu\nu}^\gamma dx^\alpha dx^\beta dx^\gamma dx^\delta dx^\gamma dx^\delta \]

where

\[ 2\delta_{\mu\nu} = \Gamma_{\mu\nu} - \Gamma_{\nu\mu} \]
§ 6. Next we shall consider a covariant vector and the linear displacement of this vector. In the manifold $X_n(y)$, we may consider a covariant vector $v_a$ and the differentials $d**x_a$, then

$$v_ad**x_a = dt$$

is a scalar. When we displace the point $y$ to the point $y + dy$, then

$$x^a \rightarrow x^a - \Gamma^a_i dy^i$$

$$d**x^a \rightarrow d**x^a - \frac{\partial \Gamma^a_i}{\partial x^j} d**x^j dy^i.$$

So, we must have the transformation of a covariant vector

$$v_a \rightarrow v_a + \frac{\partial \Gamma^a_i}{\partial x^j} v_j dy^i.$$

Then we get quite analogously as in the case of contravariant vector

$$\delta v_a = d v_a - \frac{\partial \Gamma^a_i}{\partial x^j} v_j dy^i - \Gamma_a^b v^b \Delta x^a,$$ \hspace{1cm} (17)

where

$$\Delta x = \Delta x^a + \Gamma^a_i dy^i,$$

$$\delta v_a = \Lambda^b_a \delta v_b.$$

When we consider a vector field $v_a(x, y)$, we have

$$\Gamma_a^b x^b = \partial v_a - \Gamma_a^b v^b,$$ \hspace{1cm} (18)

$$\Gamma_a^b x^b = \partial v_a - (\partial_a \Gamma^a_i) v^i - (\partial_a v^i) \Gamma^i.$$ \hspace{1cm} (19)

§ 7. Some special cases.

1° At first, we consider a general manifold $\Gamma_m$ which is represented by $(y^1, y^2, ..., y^m)$. At a point $y^i$ of $\Gamma_m$, we may consider the manifold $X_m$ touching to $\Gamma_m$. Then we choose the coordinates $p^i$ in $X_m$ such a way that these coordinates will determine a point in the tangential manifold by the equations

$$p^i = dy_i.$$

Then obviously the transformation of the coordinates in $\Gamma_m$ is

$$y'' = y'(y^i, y^i, ..., y^m)$$

and the transformation law of the coordinates in $X_m$ is

$$p'' = A^i_k p^i.$$

This transformation law is evidently a special case of our transformation law represented by (1).

When, in the fundamental manifold $\Gamma_m$, the point $y^i$ is transferred slightly, say, to $y^i + dy^i$, the point $p^i$ in the tangential manifold $X_m$ touching to $\Gamma_m$ at the point $y^i$ will be also transferred to $p'^i + dp^i$. 
As we have already stated, we shall assume the relation between $dy_i$ and $dp_i$ as follows:

$$dp_i' = -p_i'\phi_{ik}\,dy_k'.$$

Putting $p_i'\phi_{ik} = I_k'$ we have

$$dp_i' = -I_k'dy_k'.$$

These equations define a displacement of a point $p'$ in $X_m(y)$ to the point $p'+dp'$ in $X_m(y'+dy').$

This case was treated by Prof. T. Hosokawa.\(^1\)

As in the case 1°, we consider, at first, a general manifold $Y_m$, say, the fundamental manifold $Y_m$.

In this case, we choose the coordinates $u_i$ in $X_m$ which is a tangential manifold attached to $Y_m$, in such a way that these coordinates will define a plane in the tangential manifold by the following equation,

$$u_idy'_i = 0.$$

Then, when the coordinates in the fundamental manifold are transformed by $y'' = y'(y)$, $u_i$ are transformed by $u_i' = -\Pi_i u_i$.

This law is also a special case of our transformation.

When $y'$ is transformed to $y'+dy'$, the plane represented by the above equation will be also transferred to another plane whose coefficients are $u_i + du_i$.

We assume the relation between $du_i$ and $dy_k'$ as follows,

$$du_i = u_i\eta_{ik}dy_k'.$$

Then we can discuss the theory of linear displacements quite analogously.

This case was treated by one of the present authors.\(^2\)

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(Received July 10, 1935.)
