On a Simultaneous Expansion of Several Functions.

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1. In Annals of Math. Vol. 35 (1934), Mr. P. W. Ketchum has obtained a sequence of functions, by which any two given functions $f_1(z)$ and $f_2(z)$, regular in the vicinity of $z=0$ and of $z=\infty$ respectively, are expanded simultaneously in the respective domains. The main theorem is as follows.

Theorem. Let $f_1(z)$ be regular for $|z| < \rho (\leq 1)$ and $f_2(z)$ be regular for $|z| > \rho$. Then the two functions are simultaneously expanded into a series of the form

$$
\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n z^{n+1}
$$

which represents $f_1(z)$ for $|z| < \rho$ and $f_2(z)$ for $|z| > \rho$. The expansion is unique and the convergence is absolute for $|z| < \rho$ and $|z| > \rho$, and is uniform for $|z| \leq \rho' < \rho$ and $|z| \geq \rho' > \frac{1}{\rho}$.

The proof of the theorem given by Ketchum is based on the actual calculation of the series, and is somewhat complicated. It can be much simplified by using a general theorem of expansion due to Mercer, D. V. Widder, and S. Takahashi, which runs as follows.

Let $h_n(z)$ be regular for $|z| < R$ and $h_n(0) = 0$, and

$$
\lim_{n \to \infty} \max_{|z| < R} |h_n(z)| = 0
$$

for every $R'$ less than $R$, then any function $f(z)$, regular for $|z| < R$, can be expanded uniquely into a series of the form

$$
f(z) = \sum_{n=0}^{\infty} c_n z^n [1 + h_n(z)] \quad \text{for } |z| < R
$$

which converges absolutely for $|z| < R$ and uniformly for $|z| \leq R' < R$.

We are going to show, in the next lines, how to utilize this theorem to simplify the proof of Ketchum.

2. If we put
\[ F_n(z) = \frac{z^n}{1 + z^{2n+1}} = z^n [1 + h_n(z)] \] (4)
we get
\[ h_n(z) = \frac{z^{2n+1}}{1 + z^{2n+1}} \] (5)
and hence \( h_n(z) \) evidently satisfies the condition (2) for any value of \( R \) less than or equal to 1.

We now take a function \( P(x, y) \) of two variables \( x \) and \( y \) which is assumed to be regular for the domain
\[ |y| < \text{Min} \left( |x|, \frac{1}{|x|} \right) \] (6)
namely
\[ |y| < |x| < \frac{1}{|y|}. \] (7)
Considering \( P(x, y) \), for a moment, as a function of \( y \) alone, we can expand it, according to Widder, into a series of the form
\[ P(x, y) = \sum_{n=0}^{\infty} A_n(x) F_n(y) \] (8)
in the domain (6), since (6) is contained in the domain \(|y|<1\). The series converges absolutely for (6) and uniformly for any closed domain contained in (6). Every coefficient \( A_n(x) \) is regular for (7). Its determinantal form is given by Takahashi in his paper previously mentioned.

If we put \( \frac{1}{y} \) in place of \( y \) of (6) and (8), we get
\[ P \left( x, \frac{1}{y} \right) = \sum A_n(x) y F_n(y) \] (9)
for the domain
\[ |y| > \text{Max} \left( |x|, \frac{1}{|x|} \right), \] (10)
since
\[ F_n \left( \frac{1}{z} \right) = \frac{z^{-n}}{1 + z^{-2n+1}} = \frac{z^{2n+1}}{1 + z^{2n+1}} = z^n F_n(z) \] (11)
And hence for the same domain (10), we have
\[ \frac{1}{y} P \left( x, \frac{1}{y} \right) = \sum A_n(x) F_n(y) \] (12)
The two series (9) and (12) has the same form, but are considered in

(1) Since \( A_n(x) \) does not contain \( y \), it is regular always except when \( x=0 \) and \( \infty \).
the different domains (6) and (10) respectively.

If we take another function $Q(x, y)$ which is also regular for (6) and which satisfies

$$Q(x, 0) = 0$$

(13)
then the function $\frac{1}{y} Q(x, y)$ is regular for (6), so that we can expand it into a series of the form

$$\frac{1}{y} Q(x, y) = \sum_{n=0}^{\infty} \overline{B}_n(x) F_n(y)$$

(14)

Hence we have, by the similar calculation as before,

$$Q(x, y) = \sum_{n=0}^{\infty} B_n(x) F_n(y)$$

(15)

for the domain (6), and

$$yQ\left(x, \frac{1}{y}\right) = \sum_{n=0}^{\infty} B_n(x) y F_n(y)$$

(16)

for the domain (10).

Until now, $P(x, y)$ and $Q(x, y)$ have been quite arbitrary other than the regularity. We now require that they should have the natures that

for (6)

$$P(x, y) + Q(x, y) = \frac{1}{x-y}$$

(17)

for (10)

$$\frac{1}{y} P\left(x, \frac{1}{y}\right) + yQ\left(x, \frac{1}{y}\right) = \frac{1}{x-y}$$

(18)

Solving these equations we easily see

$$P(x, y) = \frac{(x-1)-y(y+1)}{(y+1)(xy-1)(x-y)}$$

(19)

$$Q(x, y) = \frac{y(x+1)}{(y+1)(xy-1)(x-y)}$$

(20)

and both of these two functions are regular for (6) and $Q(x, y)$ satisfies the condition (13).

If we expand $P(x, y)$ and $Q(x, y)$, thus obtained, as (8) and (15), we have

$$B_0(x) = 0$$

(21)

since $Q(x, y)$ becomes zero of the second order at $y=0$. And from (12), (16), (17) and (18), we see that the series

$$S(x, y) = \sum_{n=0}^{\infty} A_n(x) F_n(y) + \sum_{n=1}^{\infty} B_n(x) y F_n(y)$$

(22)

converges to $\frac{1}{x-y}$ for the domains (6) and (10).
Now take two functions $f_1(z)$ and $f_2(z)$, regular for $|z| < \rho (\equiv 1)$ and $|z| > \frac{1}{\rho}$ respectively, mentioned in the theorem of Ketchum. Then, by Cauchy's theorem, we can easily see that the expression

$$
\frac{1}{2\pi i} \int_{r_p} \left( f_1(z) - \beta \frac{z}{1+z} \right) \frac{1}{x-z} \, dx - \frac{1}{2\pi i} \int_{r_p} \left( f_2(z) - \beta \frac{z}{1+z} \right) \frac{1}{x-z} \, dx
$$

(23)

where

$$
\beta = f_1(\infty)
$$

(24)

represents $f_1(z) - \beta \frac{z}{1+z}$ for $|z| < \rho$ and represents $f_2(z) - \beta \frac{z}{1+z}$ for $|z| > \frac{1}{\rho}$, since $f_2(z) - \beta \frac{z}{1+z}$ becomes 0 at $z = \infty$. Here we have assumed that $f_1(z)$ and $f_2(z)$ are continuous for the closed domain $|z| \leq \rho$ and $|z| \geq \frac{1}{\rho}$ respectively. If this is not the case, we are only to take $\rho'(\rho)$ instead of $\rho$ and make it tend to $\rho$ as the limit.

Substituting the series $S(x, z)$ in place of $\frac{1}{x-z}$ of (23) and integrating term by term, we get the series

$$
T(z) = \sum_{n=0}^{\infty} \alpha_n F_n(z) + \sum_{n=1}^{\infty} \beta_n z F_n(z)
$$

(25)

where

$$
\alpha_n = \frac{1}{2\pi i} \int_{r_p} \left( f_1(x) - \beta \frac{x}{1+x} \right) A_n(x) \, dx - \frac{1}{2\pi i} \int_{r_p} \left( f_2(x) - \beta \frac{x}{1+x} \right) A_n(x) \, dx
$$

$$
\beta_n = \frac{1}{2\pi i} \int_{r_p} B_n(x) \, dx
$$

(26)

which represents $f_1(z) - \beta \frac{z}{1+z}$ for $|z| < \rho$ and $f_2(z) - \beta \frac{z}{1+z}$ for $|z| > \frac{1}{\rho}$.

Consequently, adding the term

$$
\beta \frac{z}{1+z}
$$

(27)

to the series (25), we get the simultaneous expansion of Ketchum.

Absolute and uniform convergence of (25) follows immediately from Wiener's theorem. Uniqueness of the expansion is also evident. For, from (25), we must have

$$
T\left(\frac{1}{z}\right) = \sum \alpha_n F_n(z) + \sum \beta_n z F_n(z)
$$

(28)

and hence
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\[ \left\{ T(z) - \frac{1}{z} \right\} T\left( \frac{1}{z} \right) = \sum_{n} b_{n} F_{n}(z) \]  

so that \( b_{n} \), so also \( a_{n} \), is uniquely determined by Widder's theorem.

3. Transforming the origin of expansion, we can get a simultaneous expansion for two functions \( f_{1}(z), f_{2}(z) \) which are regular for \( |z-a| < \rho \) and \( z-a > \frac{1}{\rho} \) respectively. In particular, putting \( f_{2}(z)=0 \), we get an expansion which represents \( f_{1}(z) \) for \( z-a < \rho \) and represents 0 for \( z-a > \frac{1}{\rho} \).

If there are \( \mu \) functions \( f_{1}(z), f_{2}(z), \ldots, f_{\mu}(z) \) which are regular respectively within the circles

\[ |z-a_{m}| < \rho_{m} \leq 1, \quad m=1, 2, \ldots, \mu \]  

and if these circles satisfy the conditions

\[ \frac{1}{\rho_{\mu}} < |a_{\mu} - a_{\mu'}| \quad \text{for} \quad \mu \neq \mu' \]  

then we get an expansion representing \( f_{m}(z) \) within the \( m \)th circle \( z-a_{m} < \rho_{m} \) and 0 within \( |z-a_{m}| > \frac{1}{\rho_{m}} \) namely within all other circles.

Adding these together, we obtain a simultaneous expansion of the \( \mu \) functions \( f_{1}(z), f_{2}(z), \ldots, f_{\mu}(z) \) within the respective circles (30).

4. The sequence of functions (4), by which our expansion is deduced, can be generalized to a sequence for a wider range of functions having the form

\[ F_{n}(z) = z^{n}\{1 + h_{n}(z)\} \]  

without any change in the reasoning. Only we need for \( F_{n}(z) \) to put the assumption that \( h_{n}(z) \) satisfies the conditions

(a) \( h_{n}(z) \) is regular for \( |z| < \rho \) \((\leq 1)\) and \( |z| > \frac{1}{\rho} \)

(b) \( h_{n}(0) = 0 \)

(c) \( \lim_{n \to \infty} \max_{|z| = \rho'} |h_{n}(z)| = 0 \), for every \( \rho' < \rho \),

and, in addition to it, the assumption that

(d) \( F_{n}\left( \frac{1}{z} \right) = z F_{n}(z) \)

By means of such sequence \( \{F_{n}(z)\} \) and of \( \{z F_{n}(z)\} \), we can expand \( f_{1}(z) \) and \( f_{2}(z) \) simultaneously within their domains of regularity \( |z| < \rho \) and \( |z| > \frac{1}{\rho} \).
Mr. Ketchum has also remarked about such an extension as above. But his condition, corresponding to our (c), is weaker than ours, and consequently the domain of expansibility does not cover wholly the domain of regularity.

We may take, instead of (d), some other functional equation, for example, the equation \( F_a \left( \frac{1}{z} \right) = -z F_a(z) \), by a slight modification of the reasonings.

Also we may take two sequences \([F_a(z)]\) and \([G_a(z)]\) instead of \([F_a(z)]\) and \([zF_a(z)]\) of the above expansion, which satisfy the equations

\[
F_a \left( \frac{1}{z} \right) = z^\alpha F_a(z), \quad \alpha \geq 0
\]

\[
G_a \left( \frac{1}{z} \right) = z^\beta G_a(z), \quad \beta \leq 0, \quad \beta < \alpha
\]

The precise discussion on this extension will be made in the next occasion.

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