Mean Modulus of Analytic Function.

By

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The following interesting theorem is due to Mr. F. Riesz (1):

Among all analytic functions $f(x)$ regular for the domain $|x| \leq 1$, for which the values of $f(0), f'(0), \ldots, f^{(n-1)}(0)$ are given, there is one and only one function $F(x)$ which makes the value of

$$I = \int_{K} |f(x)| \, |dx|$$

minimum, integration being performed along the unit circle $K$. This function $F(x)$ has the form

$$e \prod_{i=1}^{n-1} (x - a_i) \prod_{i=1}^{n'-n-1} (x - \bar{a}_i)(\bar{z}, \bar{x} - 1)$$

(2)

where $\bar{a}$ denotes the conjugate of $a$. In other words the zeros of the function $F(x)$, which are greater than or equal to 1, have even multiplicity, and all other zeros occur at pair, one being the image of the other with respect to the unit circle.

Let us call such function as (2) a P-function.

Having been unable to see his paper, I have discussed (2) independently the existence of such P-function as above, but in a more general form, as a fundamental lemma of my own aim. It runs as follows:

Among all analytic functions $f(x)$ regular for $|x| \leq 1$, for which the values of

\[ (1) \] Acla Mathematica, 42 (1919).

The paper was sent to the American Mathematical Society, in August 1919. I think it will shortly appear in Trans. of Amer. Math. Soc. It was also read in the October meeting of Math. Club of the University of Chicago, in 1919.
are given, there is one and only one $P$-function whose degree does not exceed $2(\Sigma_l + \kappa - 1)$.

If we apply this theorem, we can get very easily a result which is an extension of the theorem of Mr. Riesz, which runs as follows:

Among all analytic functions $f(x)$ regular for $|x| \leq 1$ and for which the values of (3) are given, there exists one and only one function $F(x)$ which makes the value of the integral

$$I = \int_{-1}^{1} |f(x)| \, dx$$

smallest. This function $F(x)$ has the form

$$F(x) = \frac{P(x)}{\prod_{i=1}^{k} (x-a_i)^{\lambda_i + 1}}$$

where $P(x)$ is a $P$-function whose degree does not exceed $2(\Sigma_l + k - 1)$.

To prove this, we must know the following lemma given (1) by Mr. Riesz:

In order that

$$I(\lambda) = \int_{0}^{2\pi} |g(t) + \lambda h(t)| \, dt \quad (2)$$

should be smallest when $\lambda = 0$, it is necessary and sufficient that

$$\int_{0}^{2\pi} \overline{S}_g(t) h(t) \, dt = 0,$$

where $\overline{S}_g(x)$ means the conjugate of $\varphi(x)$ divided by $|\varphi(x)|$.

Now, that the values (2) are given for the function $f(x)$ is equivalent to that the values of the corresponding derivatives at $a_i$ are given for the function

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(1) See p. 159 of the paper l. c.
(2) Integral should be taken in the sense of Lebesgue.
\[ G(x) = f(x) \prod_{i=1}^{k} (\bar{a}_i x - 1)^{l_i + 1}. \]

Let the corresponding P-function of the degree not exceeding \(2(\Sigma l_i + \kappa - 1)\), whose derivatives at \(a_i\) have the same values as those of \(G(x)\) be the function (2). Then we get a function \(F(x)\) whose derivatives at \(a_i\) have the given values and is regular for \(|x| \leq 1\), which has the form

\[ F'(x) \prod (\bar{a}_i x - 1)^{l_i + 1} = c \prod (x - a_i)^{n - m - 1} (x - \bar{a}_i)(\bar{a}_i x - 1) \quad (5) \]

or

\[ xF(x) = \frac{c}{\prod (\bar{a}_i x - 1)^{l_i + 1} (x - a_i)^{l_i + 1}} \prod \left( \frac{x - a_i}{\bar{a}_i x - 1} \right)^{l_i + 1} \frac{x - a_i}{\bar{a}_i x - 1} \]

where \(\varphi(x)\) has a constant amplitude along the unit circle and other factors have the constant magnitude 1 along the unit circle.

Therefore we have, for the point of the unit circle \(x = e^{i\theta}\),

\[ \Re \varphi(x) = \text{const.,} \]

\[ \Re \left\{ \prod \left( \frac{x - a_i}{\bar{a}_i x - 1} \right)^{l_i + 1} \prod \frac{x - a_i}{\bar{a}_i x - 1} \right\} = \prod \left( \frac{\bar{a}_i x - 1}{x - a_i} \right)^{l_i + 1} \prod \frac{\bar{a}_i x - 1}{x - a_i}, \]

and hence

\[ \Re xF(x) = \text{const.} \prod (\bar{a}_i x - 1)^{l_i + 1} \prod \left( \frac{\bar{a}_i x - 1}{x - a_i} \right) \varphi(x), \quad \text{for } x = e^{i\theta}, \]

where \(\psi(x)\) denotes any function regular for \(|x| \leq 1\).

The right hand member of the above expression is a value which is attained for \(|x| = 1\) by a function regular for \(|x| \leq 1\), multiplied by \(x\), since \(|a_i| > 1\). Therefore by the theorem of complex integration

\[ \int_{0}^{2\pi} \Re xF(x)e^{i\theta} \prod (x - a_i)^{l_i + 1} \psi(x) d\theta = 0. \]

And hence by the lemma before mentioned.
Here evidently

\[ F(x) + \prod_{a_i}^k (x-a_i)^{l_i+1} \psi(x) \]

is the general form of function regular for \( |x| \leq 1 \) whose derivatives at \( a_i \) attain the same given values as those of \( F(x) \).

Therefore \( F(x) \), which is known from (5) to have the form stated in the theorem, gives the smallest value of the integral.

If there is another minimizing function \( F_1(x) \), then the function \( \frac{1}{2} \{ F(x) + F_1(x) \} \) must be also a minimizing function. Hence

\[ I = \int \frac{1}{2} \left( |F(x)| + |F_1(x)| \right) \, dx = \int \frac{1}{2} |F(x)| + \frac{1}{2} |F_1(x)| \, dx. \]

So the functions \( F(x) \) and \( F_1(x) \) must have the same amplitude along the unit circle, so that \( \frac{F(x)}{F_1(x)} \) is real along the unit circle. Therefore \( \frac{F(x)}{F_1(x)} \) must have the form

\[ \prod_{i}^{n-\alpha-1} \frac{(x-\beta_i)(\beta_i+1)}{(x-\gamma_i)(\gamma_i+1)}. \]

Hence \( F_1(x) \) is also a P-function of the degree not exceeding \( 2(\Sigma+\kappa-1) \) whose derivatives at \( a_i \) have the same values as those of \( F(x) \). Thus we see \( F(x) \) and \( F_1(x) \) to be identical.

Our theorem is thus proved.

I have once treated the similar problem (1), considering the maximum modulus \( M \) of \( f(x) \) along the unit circle instead of the integral \( I \). And we get unique solution which is rational. Starting from that investigation I have discussed (2) some general theorem about

\[ (1) \text{ See the paper "On the maximum modulus of an analytic function", Science Reports of the Tôhoku Imp. Univ. 4 (1915), pp. 257-311.} \]

\[ (2) \text{ Science Reports of Tôhoku Imp. Univ. 6 (1917), pp. 153-168.} \]
the method of finding the "upper and lower limits of some quantities regarding analytic function," relating more or less to the maximum modulus along a circle. Now it is to be remarked that the similar discussion can be performed about the method of finding the upper or lower limits of some quantities relating to the mean modulus of the function along a circle.

On Zero Points of Function defined by some Definite Integral.

By

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Recently Mr. G. Pólya (1) has found several interesting facts relating to the domain of zeros of the functions

\[ U(z) = \int_0^1 f(t) \cos zd\xi, \]

\[ V(z) = \int_0^1 f(t) \sin zd\xi, \]

under some conditions of the function \( f(t) \). Function of this kind is frequently used in Mathematical Physics. He gave two examples

\[ \frac{2}{\pi} \int_0^1 \frac{\cos zd\xi}{\sqrt{1-t^2}} = F(z), \]

\[ \int_0^1 \xi \sin zd\xi = \frac{\cos z}{z^2} (lgz - z). \]

It seems to me that this problem has quite intimate relation to my investigation about a system of integral equations. We can easily obtain the complete condition that the zero points of