The Operators in the Finite Calculus.

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§ 1. Introduction.

The object of this paper is to supplement to my previous paper "Notes on the Finite Calculus" various considerations which the author then omitted.

In the first place, we obtain the summation of \( f(x) \) from 0 to any number of times and by the aid of this expression we reach the result which corresponds to the Borel's theorem in Heaviside's operator\(^{(2)}\). The above expression and result are considered with regard the Milne-Thomson's operators deduced from forward difference operator and backward difference operator. We also explain the Cesaro's sum using the expression of summation.

In the second place, we extend the range of previous considerations of the Milne-Thomson's operator by the aid of the conception infinitely large and infinitely small spaces\(^{(3)}\), discuss the actions of the operator \( P^{-1} \) and the operator \( P \) on various operands and consider the behavior of \( x \) in the space of Milne-Thomson's operator, the method of solving the sum-difference equation is also added at the end of the paper.


We have already denoted the summation\(^{(4)}\) of \( f(x) \) from \( a \) to \( b \) by

\[
S_a^b f(x) \Delta x
\]

and defined by the following formula\(^{(4)\,(5)}\)

\[
S_a^b f(x) \Delta x = \omega [f(a) + f(a + \omega) + \ldots + f(b - \omega)].
\]

We know the formula for summation by parts\(^{(5)}\).

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We define here the function \( H(x|\omega) \) by the relation
\[
H(x|\omega) = \Gamma(x + \omega|\omega),
\]
where \( \Gamma(x|\omega) \) is the generalized Gamma function having the following properties\(^6\)
\[
\Gamma(x + \omega|\omega) = x \Gamma(x|\omega)
\]
and
\[
\Gamma(n\omega + \omega|\omega) = \omega^n n!,
\]
if \( n \) be a positive integer.

Since
\[
\Delta \frac{H(x-c|\omega)}{H(x-h+1|\omega)} = \frac{H(h+1)}{H(h)} \frac{H(x-c|\omega)}{H(x-h\omega-c|\omega)},
\]
we get
\[
\sum_{\omega} \frac{H(x-c|\omega)}{H(x-h\omega-c|\omega)} \Delta x = -1 \left[ \frac{H(h-c|\omega)}{H(h-h+1|\omega)} - \frac{H(a-c|\omega)}{H(a-h+1|\omega)} \right].
\]
(4)

Let us now evaluate expression
\[
\sum_{\omega} H(x-\omega-x'|\omega) \Delta x = (-1)^n \sum_{\omega} H(x-\omega-x'|\omega) \Delta x.
\]
which is equal to
\[
\sum_{\omega} H(x-\omega-x'|\omega) \Delta x.
\]
Applying the result (4) to (5.1), we have
\[
\sum_{\omega} \frac{H(x+\omega-x'|\omega)}{H(x+1|\omega)} \Delta x = (-1)^n \sum_{\omega} \left[ \frac{H(x+\omega-x'|\omega)}{H(x+1|\omega)} \right] \Delta x.
\]
(5.1)

Applying the result (4) to (5.1), we have
\[
\sum_{\omega} \frac{H(x+\omega-x'|\omega)}{H(x+1|\omega)} \Delta x = (-1)^n \sum_{\omega} \left[ \frac{H(x+\omega-x'|\omega)}{H(x+1|\omega)} \right] \Delta x.
\]
Hence (5) becomes
\[
\sum_{\omega} \frac{H(x-\omega-x'|\omega)}{H(x+1|\omega)} \Delta x = (-1)^n \sum_{\omega} \left[ \frac{H(x-\omega-x'|\omega)}{H(x+1|\omega)} \right] \Delta x.
\]
(6)

If we make use of the formula (2) and the result (6), we find
\[
\sum_{\omega} \frac{H(x-\omega-x'|\omega)}{H(x+1|\omega)} \Delta x = \frac{1}{H(x+1|\omega)} \int_{x'}^x f(x) dx.
\]
(6)

\(^6\) See Milne-Thomson, The Calculus of Finite Differences, p. 255

Since the last term in the right-hand side of (7) is obtained from the left-hand expression of (7) by replacing \( m \) by \( m + 1 \), applying again (2) and (6) to the last term of (7), we get

\[
\begin{align*}
& S \left\{ \frac{1}{H(m+1)} \frac{H(x|\omega)}{H(x-m+1\omega|\omega)} - \frac{1}{H(m+1)} \frac{H(x-\omega-\xi|\omega)}{H(x-m+2\omega-\xi|\omega)} \right\} \Delta f(\xi) \Delta \xi \\
& = \frac{1}{H(m+1)} \frac{H(x|\omega)}{H(x-m+1\omega|\omega)} \Delta \xi \left\{ \frac{f(0)}{H(m)} + S \frac{H(x-\omega-\xi|\omega)}{H(x-m+2\omega-\xi|\omega)} \Delta f(\xi) \right\} \\
& = \frac{\Delta f(0)}{H(m+2)} \frac{H(x|\omega)}{H(x-m+2\omega|\omega)} + S \frac{H(x-\omega-\xi|\omega)}{H(x-m+3\omega-\xi|\omega)} \Delta \xi.
\end{align*}
\]

(7)

After the repeated applications of (2) and (6) to the last term in the above result

\[
\begin{align*}
& S \frac{H(x-\omega-\xi|\omega)}{H(x-m+1\omega-\xi|\omega)} \frac{f(\xi)}{H(m)} \Delta \xi = S \sum_{r=0}^{\infty} \frac{\Delta f(0)}{H(m+r+1)} \frac{H(x|\omega)}{H(x-m+r+1\omega|\omega)} \\
& + S \frac{H(x-\omega-\xi|\omega)}{H(x-m+k+2\omega|\omega)} \Delta f(\xi) \frac{\Delta \xi}{H(m+k+1)}.
\end{align*}
\]

(8)

Hence if

\[
\lim_{k \to \infty} S \frac{H(x-\omega-\xi|\omega)}{H(x-m+k+2\omega|\omega)} \Delta f(\xi) \Delta \xi = 0,
\]

we have

\[
\begin{align*}
& S \frac{H(x-\omega-\xi|\omega)}{H(x-m+1\omega-\xi|\omega)} \frac{f(\xi)}{H(m)} \Delta \xi \\
& = S \sum_{r=0}^{\infty} \frac{\Delta f(0)}{H(m+r+1)} \frac{H(x|\omega)}{H(x-m+r+1\omega|\omega)};
\end{align*}
\]

(9)

or if we reflect that (5.1) is equal to (5), we have

\[
(-1)^m S \frac{H(x+\omega-x|\omega)}{H(x-\omega|\omega)} \frac{f(\xi)}{H(m)} \Delta \xi \\
= \sum_{r=0}^{\infty} \frac{\Delta f(0)}{H(m+r+1)} \frac{H(x|\omega)}{H(x-m+r+1\omega|\omega)}.
\]

(9.1)

We now suppose that $f(x)$ is expressible in the Newton's interpolation series of the following form

$$f(x) = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!} \frac{H(x, \omega)}{H(x-r\omega, \omega)}. \quad (10)$$

Since (9), we have by (10)

$$\frac{1}{P^{m+1}} \frac{1}{H(r)} \left( \frac{H(x, \omega)}{H(x-r\omega, \omega)} \right) = \frac{1}{H(m+r+1)} \frac{H(x, \omega)}{H(x-r\omega, \omega)}.$$

we have by (10)

$$\frac{1}{P^{m+1}} f(x) = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!} \frac{H(x, \omega)}{H(x-r\omega, \omega)}. \quad (11)$$

Since the right-hand series of (11) is equal to the right-hand series of (9), we have

$$\frac{1}{P^{m+1}} f(x) = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!} \frac{H(x, \omega)}{H(x-r\omega, \omega)} \frac{H(x, \omega)}{H(x-r\omega, \omega)} \Delta \xi. \quad (12)$$

But since we know (9) that $1/P^{m+1} f(x)$ is the summation of $f(x)$ from zero to $x-m+1$ times, we can write (12) in the form

$$\sum_{r=0}^{m+1} \frac{f^{(r)}(0)}{r!} \frac{H(x, \omega)}{H(x-r\omega, \omega)} \frac{H(x, \omega)}{H(x-r\omega, \omega)} \Delta \xi = \sum_{r=0}^{m+1} \frac{f^{(r)}(0)}{r!} \frac{H(x, \omega)}{H(x-r\omega, \omega)} \frac{H(x, \omega)}{H(x-r\omega, \omega)} \Delta \xi. \quad (13)$$

Let us now take the equation

$$\frac{1}{K(p)} = \cdots 0 - k(x)$$

and expand $1/K(p)$ in the following Laurent series

$$\sum_{r} \frac{r}{p^{r+1}} r - 0. \quad (14)$$

If we compare the relation (13) with the equation (12), we shall find the rule that the right-hand side

of (12) contains the factor which can be obtained by replacing \( x \) by \( x-\omega - \xi \) in the right-hand expression of (15). But since the operator in the left-hand side of (15) can stand for any term in the series (14), the above rule must hold for \( 1/K(p) \). Hence we have

\[
\frac{1}{K(p)} f(x) = \sum_{\xi} k(x-\omega - \xi) f(\xi) \Delta \xi.
\]

(16)

Suppose that the functions \( f(x) \), \( f_1(x) \) and \( f_2(x) \) are defined by the equations

\[
g(\mathbf{p}) \cdot 0 = f(x) \tag{17.1}
\]

\[
g(\mathbf{p}) \cdot 0 = f_1(x) \tag{17.2}
\]

\[
g(\mathbf{p}) \cdot 0 = f_2(x) \tag{17.3}
\]

and the functions \( g(\mathbf{p}) \), \( g_1(\mathbf{p}) \) and \( g_2(\mathbf{p}) \) satisfy the relation

\[
g(\mathbf{p}) = g_1(\mathbf{p}) \cdot g_2(\mathbf{p}),
\]

(18)

then we have by (18) and (17.1)

\[
g_1(\mathbf{p}) \cdot f_1(x) = g_1(\mathbf{p}) g_2(\mathbf{p}) \cdot 0 = g(\mathbf{p}) \cdot 0 = f(x) \tag{19.1}
\]

\[
g_2(\mathbf{p}) \cdot f_1(x) = g_2(\mathbf{p}) g_1(\mathbf{p}) \cdot 0 = f(x). \tag{19.2}
\]

On the other hand we have by (16)

\[
g_1(\mathbf{p}) \cdot f_2(x) = \sum_{\xi} f_1(x-\omega - \xi) f_2(\xi) \Delta \xi
\]

(20.1)

and

\[
g_2(\mathbf{p}) \cdot f_2(x) = \sum_{\xi} f_2(x-\omega - \xi) f_1(\xi) \Delta \xi.
\]

(20.2)

Hence combining (20.1) and (20.2) with (19.1) and (19.2), we finally obtain

\[
f(x) = \sum_{\xi} f_1(x-\omega - \xi) f_2(\xi) \Delta \xi = \sum_{\xi} f_1(x-\omega - \xi) f_1(\xi) \Delta \xi,
\]

(21)

which is the relation corresponding to Borel's theorem in Heaviside's operational methods.(13)

If we put \(^{14} \)

\[
\frac{1}{K(\mathbf{p})} \left[ \frac{1}{x} \right] = k^\#(x),
\]

where \( 1/K(\mathbf{p}) \) has the Laurent expansion (14). If we compare the relation

\[
\frac{1}{\mathbf{p}^{\omega+1}} \left[ \frac{1}{x} \right] = \frac{1}{\mathbf{H}(\mathbf{m})} \frac{\mathbf{H}(x-\omega, \omega)}{\mathbf{H}(x-m+1, \omega)}
\]

(15.1)

(12) \( g(\mathbf{p}) \cdot f_1(x) = g_1(\mathbf{p}) \cdot f_2(\mathbf{p}) \cdot 0 \) is the definition of operating with \( g_1(\mathbf{p}) \) on \( f_2(x) \).
(14) See (131) and (14) in this paper.
with the equation (12), we shall find the rule that the right-hand side of (12) contains the factor which can be obtained by replacing $x$ by $x-\xi$ in the right-hand expression of (15.1). But since the operator in the left-hand side of (15.1) can stand for any term in the series (14), the above rule must hold for $1/K(p)$. Hence we have

$$\frac{1}{K(p)}f(x) = \sum_{n} k^*(x-\xi) f(\xi) \Delta \xi,$$

(16.1)

By the similar reasoning as above the result (21) can be written

$$f(x) - \sum_{n} f_n^*(x-\xi) f(\xi) \Delta \xi = \sum_{n} f_n^*(x-\xi) f(\xi) \Delta \xi,$$

(22)

where

$$g_n(p) \left| \frac{1}{x} f_n^*(x), \quad g_n(p) \left| \frac{1}{x} f_n^*(x).$$

Let us now take the expression

$$\sum_{n} \frac{f(\xi)}{H(x-\omega-\xi)} \Delta \xi$$

(23)

instead of the left-hand side of (7). Since

$$\sum_{n} \frac{f(\xi)}{H(x-\omega-\xi)} \Delta \xi = \sum_{n} \frac{H(x-\xi)}{H(x-\omega+1 \omega-\xi)} \Delta \xi$$

(15) Sec. (III) and (64) in this paper.

(23) becomes

$$\sum_{n} \frac{f(\xi)}{H(x-\omega-\xi)} \Delta \xi = \sum_{n} \frac{H(x-\xi)}{H(x-\omega+1 \omega-\xi)} \Delta \xi$$

$$\sum_{n} \frac{H(x-\xi)}{H(x-\omega+1 \omega-\xi)} \Delta \xi$$

Hence after the repeated partial summation of the last term of the above result we get

$$\sum_{n} \frac{f(\xi)}{H(x-\omega-\xi)} \Delta \xi = \sum_{n} \frac{\Delta f(c)}{H(x-\omega+1 \omega-\xi)} \Delta \xi$$

(24)
If we suppose that $f(x)$ is expressible in Newton's interpolation series of the form

$$f(x) = \sum_{c=0}^{\infty} \frac{\Delta^c f(c)}{\Pi(c)} \frac{\Pi(x-c|\omega)}{\Pi(x-r \omega-c|\omega)}$$  \hspace{1cm} (25)$$

and remember the relation \(^{*}\),

$$\frac{1}{P_{m+1}} \left\{ \frac{1}{\Pi(\tau)} \frac{\Pi(x-c|\omega)}{\Pi(x-r \omega-c|\omega)} \right\} = \frac{1}{\Pi(m+r+1)} \frac{\Pi(x-c|\omega)}{\Pi(x-r \omega-c|\omega)}$$

we get from (25) and (24) following result

$$\frac{1}{P_{m+1}} f(x) = \frac{\frac{\Pi(x-c|\omega)}{\Pi(x-m+1 \omega-c|\omega)}}{\Pi(m)} \Delta \xi,$$  \hspace{1cm} (26)\(^{**}\)

which can also be written

$$\begin{align*}
\sum_{c=0}^{\infty} \frac{\Pi(x-c|\omega)}{\Pi(x-m+1 \omega-c|\omega)} \Delta \xi = \frac{\Pi(x-c|\omega)}{\Pi(x-m+1 \omega-c|\omega)} \frac{f(X)}{\Pi(m)} \Delta \xi.
\end{align*}$$

§ 3. The Borel's Theorem of the Milne-Thomson's Operators deduced from Backward Difference Operators.

In the case of backward difference operator the summation of $f(x)$ from $a$ to $b$ is

$$\sum_{a}^{b} f(x) \Delta \xi = \omega \{ u(a+\omega) + u(a+2 \omega) + \ldots + u(b) \},$$  \hspace{1cm} (27)\(^{**}\)

and for summation by parts is

$$\sum_{a}^{b} u(x) \Delta v(x) \Delta \xi = [u(x)v(x)]_{x=a}^{x=b} - \sum_{a}^{b} u(x-\omega) \Delta v(x) \Delta \xi.$$  \hspace{1cm} (28)

Since

$$\Delta \xi = \frac{\Pi(x+h+1 \omega-c|\omega)}{\Pi(x-c|\omega)} - \frac{\Pi(x+h \omega-c|\omega)}{\Pi(x-c|\omega)},$$

we get

$$\begin{align*}
\sum_{a}^{b} \frac{\Pi(x+h \omega-c|\omega)}{\Pi(x-c|\omega)} \Delta \xi
\end{align*}$$

\(^{**}\) The Milne-Thomson's operator in this equation differs from the operator which we so far used. The operator in this equation is defined by (33) and (34) on p. 23 of Proc. Phys. Math. Soc. Jap. 19 (1937).

Let us now evaluate the expression
\[ \frac{\xi}{o} \frac{H(x + m \omega - \xi | \omega)}{H(x - \xi | \omega)} \Delta \xi, \]
which is equal to
\[ (-1)^n \frac{\xi}{o} \frac{H(x - \omega - x | \omega)}{H(x - \omega - \xi | \omega)} \Delta \xi. \]  \hspace{1cm} (30.1)

Applying the result (29) to (30.1), we have
\[ (-1)^n \frac{\xi}{o} \frac{H(x - \omega - x | \omega)}{H(x - \omega - \xi | \omega)} \Delta \xi = \left( -1 \right)^n \frac{H(x - \omega | \omega)}{H(x - \omega - \xi | \omega)} \frac{H(x - \omega - x | \omega)}{H(x - \omega - \xi | \omega)} \Delta \xi. \]

Hence (30) becomes
\[ \frac{\xi}{o} \frac{H(x + m \omega - \xi | \omega)}{H(x - \xi | \omega)} \Delta \xi = \frac{1}{m + 1} \left[ \frac{H(x + m \omega - \xi | \omega)}{H(x - \omega - \xi | \omega)} - \frac{H(x + m \omega - \xi | \omega)}{H(x - \omega - \xi | \omega)} \right] \Delta \xi. \]  \hspace{1cm} (31)

If we make use of the formula (28) and the result (31), we find
\[ \frac{\xi}{o} \frac{H(x + m \omega - \xi | \omega)}{H(x - \xi | \omega)} \frac{f(\xi)}{H(x - \xi | \omega)} \Delta \xi = \frac{1}{m + 1} \left[ \frac{H(x + m \omega - \xi | \omega)}{H(m + 1)} \frac{H(x - \omega - \xi | \omega)}{H(x - \omega - \xi | \omega)} \right] \Delta f(\xi) \Delta \xi. \]

Since the last term in the right-hand side of (32) is obtained from the left-hand expression of (32) by replacing \( m \) by \( m + 1 \), applying again (28) and (31) to the last term of (32), we get
\[ \frac{\xi}{o} \frac{H(x + m \omega - \xi | \omega)}{H(x - \xi | \omega)} \frac{f(\xi)}{H(x - \xi | \omega)} \Delta \xi = \frac{1}{m + 1} \left[ \frac{H(x + m \omega - \xi | \omega)}{H(m + 1)} \frac{H(x - \omega - \xi | \omega)}{H(x - \omega - \xi | \omega)} \right] \Delta f(\xi) \Delta \xi. \]  \hspace{1cm} (32)

After the repeated applications of (28) and (31) to the last term of (32), we get
Hence if we have
\[ \lim_{k \to \infty} \sum_{r=0}^{k} \frac{\Delta f(0)}{H(m + r + 1)} = \frac{\Delta f(0)}{H(x - \omega \omega)} \]
we have
\[ \sum_{r=0}^{k} \frac{\Delta f(0)}{H(m + r + 1)} = \frac{\Delta f(0)}{H(x - \omega \omega)} \]
or if we reflect that (30.1) is equal to (30), we have
\[ (-1)^m \sum_{r=0}^{m} \frac{\Delta f(0)}{H(m + r + 1)} = \frac{\Delta f(0)}{H(x - \omega \omega)} \]
We now suppose that \( f(x) \) is expressible in the Newton's interpolation series of the following form
\[ f(x) = \sum_{r=0}^{m} \frac{\Delta f(0)}{H(m + r + 1)} \frac{H(x + r + 1)}{H(x - \omega \omega)} \]
Since
\[ \frac{1}{P^{m+1}} \sum_{r=0}^{m} \Delta f(0) \frac{H(x + r + 1)}{H(x - \omega \omega)} = \frac{\Delta f(0)}{H(x - \omega \omega)} \]
we have by (35)
\[ \frac{1}{P^{m+1}} f(x) = \sum_{r=0}^{m} \Delta f(0) \frac{H(x + r + 1)}{H(x - \omega \omega)} \]
Since the right-hand series of (36) is equal to the right-hand series of (34), we have
\[ \frac{1}{P^{m+1}} f(x) = \sum_{r=0}^{m} \Delta f(0) \frac{H(x + m + r \omega \omega)}{H(x - \omega \omega)} \]
But since we know that \( 1/P^{m+1} f(x) \) is the summation of \( f(x) \) from zero to \( x + 1 \) times, we can write (37) in the form
\[ \sum_{r=0}^{m+1} \Delta f(0) \frac{H(x + m + r \omega \omega)}{H(x - \omega \omega)} = \sum_{r=0}^{m+1} \Delta f(0) \frac{H(x + m + r \omega \omega)}{H(m + r + 1)} \]
Let us now take the equation

\[
\frac{1}{K(P)} \Delta \xi = f(x)
\]

where \(1/K(P)\) is expressible in the following Laurent series

\[
\sum_{r \geq 0} \frac{c_r}{P^{r+1}}
\]

If we compare the relation

\[
\frac{0}{P^{m+1}} = \frac{1}{H(m)} \frac{H(x + m - 1 \omega \omega)}{H(x - \omega \omega)}
\]

with the equation (37), we shall find that the right-hand side of (37) contains the factor which can be obtained by replacing \(x\) by \(x + \omega - \xi\) in the right-hand expression of (40). But since the operator in the left-hand side of (40) can stand for any term in the series (39), the above rule must hold for \(1/K(P)\). Hence we have

\[
\frac{1}{K(P)} \Delta \xi = \sum_{r \geq 0} \frac{c_r}{P^{r+1}}
\]

If we suppose that the relations (17.1), (17.2), (17.3) and (18) exist for the Milne-Thomson's operator deduced from the backward difference operator, we have by (11)

\[
g_i(P) f_i(x) = \sum_{r \geq 0} \frac{c_r}{P^{r+1}} \left( x + \omega - \xi \right) f_i(x) \Delta \xi.
\]

If we put

\[
\frac{1}{K(P)} \left[ \frac{1}{x} \right]_\omega = k^*(x),
\]

where \(1/K(P)\) has the Laurent expansion (39). If we compare the relation

\[
\]
with the equation (37), we shall find the rule that the right-hand side of (37) contains the factor which can be obtained by replacing $x$ by $x-\xi$ in the right-hand expression of (40.1). But since the operator in the left-hand side of (40.1) can stand for any term in the series (39), the above rule must hold for $1/K(p)$. Hence we have

$$\frac{1}{K(p)} f(x) = \frac{\hat{S}}{\hat{p}} \frac{\hat{K}^*(x-\xi)}{\hat{p}} f(x) \Delta \xi. \quad (41.1)$$

By the similar reasoning as above the result (43) can be written

$$f(x) = \frac{\hat{S}}{\hat{p}} f^*(x-\xi) f(x) \Delta \xi = \frac{\hat{S}}{\hat{p}} f^*(x-\xi) f(x) \Delta \xi, \quad (44)$$

where

$$g_p(p) \left[ \frac{1}{x} \right]_{m} = f^*(x), \quad g_p(p) \left[ \frac{1}{x} \right]_{m} = f^*(x).$$

If we denote by $p$ the Milne-Thomson's operator which is deduced from the fundamental relation

$$\frac{0}{p-\omega} = u(x),$$

where $u(x)$ is the solution of the difference equation

$$(\Delta - \omega) u(x) = 0$$

with the condition

$$u(\omega) = 1,$$

we get

$$\frac{1}{p^{m+1}} f(x) = \frac{\hat{S}}{\hat{p}} \frac{H(x+m\omega-\xi)}{\hat{p}} \frac{f(x)}{H(\xi)} \Delta \xi. \quad (45)$$

by the similar argument as used to obtain (26) from (23). The result (45) can be also written in the form

$$\frac{\hat{S}}{\hat{p}} \ldots \frac{\hat{S}}{\hat{p}} f(x) = \frac{\hat{S}}{\hat{p}} \frac{H(x+m\omega-\xi)}{\hat{p}} \frac{f(x)}{H(\xi)} \Delta \xi. \quad (46)$$

If we put $\epsilon = -\omega$ and suppose that $x = n\omega$, (46) can be written in the form

$$\omega^{m+1} \sum_{r=0}^{n} \ldots \sum_{r'=0}^{n} f(r\omega) = \omega \sum_{r=0}^{n} \frac{H(n+m-r\omega)}{H(n-r\omega)} f(r\omega); \quad (47)$$

and in consequence the arithmetical mean of $m$th degree of the series

(19) See (III) and (64) in this paper.
\[
\omega \sum_{r=0}^{n} f(r\omega) \tag{48}
\]
can be representable in the form
\[
\frac{\omega^{m+1}}{(n+1)^{m}} \sum_{r=0}^{n} \sum_{r_0}^{n} f(r\omega) - \frac{\omega}{(n+1)^{m}} \sum_{r=0}^{n} \frac{H(n+m-r\omega|\omega)}{H(n+r\omega|\omega)} f(r\omega). \tag{49}
\]

We introduce here the relation (47) the definition of the Cesàro's sum of \(m\)th degree for the interval \(\omega\) as follows
\[
\sigma(C, m|\omega) = \lim_{n \to \infty} \frac{H(n\omega|\omega)}{H(n+m\omega|\omega)} \sum_{r=0}^{n} \frac{H(n+m-r\omega|\omega)}{H(n+r\omega|\omega)} f(r\omega).
\]
The coincidence of the value of the Cesàro's sum \(\sigma(C, m|\omega)\) with that of (49) when \(n \to \infty\) is easily seen, if we reflect on
\[
\frac{H(n+m\omega|\omega)}{H(n\omega|\omega)} = (\omega)^{m} \frac{(1+\frac{1}{m})}{(1+\frac{1}{n})}.
\]

§ 4. The Interpretation of the Milne-Thomson's Operator using the Conception of the Infinitely Small Space.

Let us now begin with the fundamental relation
\[
\int_{0}^{1} (\mu-a)^{m+1} = \frac{1}{2\pi i} \int_{C} (1+\mu\omega)^{m} \frac{d\mu}{(\mu-a)^{m+1}}. \tag{50}
\]
Putting \(\mu = -(m+1)\), we get from (50)
\[
(\mu-a)^{m} 0 - \frac{1}{2\pi i} \int_{V} (\mu-a)^{m} (1+\mu\omega)^{m} d\mu. \tag{51}
\]

The paths of integration of the integrals (50) and (51) start at \(-1/\omega\) and describe a loop contour as shown in Fig. 1 and terminate at the starting point. We suppose that AB, CD coincide with the segment of the real axis between \(-1/\omega\) and \(a\) and that the circular part tends to zero.

If we start with arg \((a-\mu)\to 0\) along AB, we shall have arg \((\mu-a)\) \(-\pi\) along AB and arg \((\mu-a)\) \(\pi\) along CD.

Thus on AB,
\[
(\mu-a) = (a-\mu)e^{-\pi\omega},
\]
while on CD,
\[
(\mu-a) = (a-\mu)e^{\pi\omega}.
\]


\(^{(21)}\) Since the right-hand integral of (52) is the integral for Beta function, the right-hand expression of (52) results from (52).
If \( m > -1 \) in (51), the integral round the circle tends to zero when the radius tends to zero, so that we have

\[
(p - a)^m \cdot 0 = -\frac{\sin m\pi}{\pi} \int_{-\delta}^{\delta} \frac{(a - p)^m (1 + p\omega)^{\nu/\omega}}{p} \, dp
\]

Thus we get (21)

\[
(p - a)^m \cdot 0 = -\frac{\sin m\pi}{\pi} \frac{\Pi(x\omega)}{\Pi(x + m + 1\omega/\omega)} (1 + a\omega)^{\nu/\omega + m + 1}. \tag{53}
\]

If we put \( m = n \) in (53) where \( n \) is a positive integer or zero, (53) reduces to

\[
(p - a)^n \cdot 0 = (-1)^n \frac{\sin \pi}{\pi} \frac{\Pi(x\omega)}{\Pi(x + n + 1\omega/\omega)} (1 + a\omega)^{\nu/\omega + n + 1}. \tag{54}
\]

Remembering that \( \sin \pi/\pi \) is the measure of the infinitely small space (22), we can write (54) in the form

\[
(p - a)^n \cdot 0 = \left[ (-1)^n \frac{n!}{x^{(n+1)\omega}} (1 + a\omega)^{\nu/\omega + n + 1} \right]_{\infty} \tag{54.1}
\]

where

\[
x^{(n+1)\omega} = \frac{1}{(x + \omega)(x + 2\omega) \ldots (x + n + 1\omega)}.
\]

Putting \( \alpha = 0 \) in (54.1), we get

\[
p^n \cdot 0 = \left[ (-1)^n \frac{n!}{x^{(n+1)\omega}} \right]_{\infty}. \tag{55}
\]

If \( n = 0 \), (55) becomes

\[
p^0 \cdot 0 = \left[ \frac{1}{x + \omega} \right]_{\infty}. \tag{55.1}
\]

In the case of Milne-Thomson's operator deduced from the backward difference operator we begin with the integral

\[
0 = \frac{1}{2\pi i} \int_{-\delta}^{\delta} \frac{(1 - p\omega)^{-\nu/\omega}}{(p - a)^{n+1}} \, dp. \tag{56}^{(20)}
\]

Proceeding in exactly the same manner as we got (52) from (50), we obtain from (56)

\[
(p - a)^n \cdot 0 = -\frac{\sin \pi}{\pi} \int_{-\delta}^{\delta} \frac{(a - p)^n (1 - p\omega)^{-\nu/\omega}}{p} \, dp
\]

\begin{footnotesize}
\end{footnotesize}
from which we get

\[(p - a)^n \cdot 0 = \left[ \frac{(-1)^n n! (1 - \alpha \omega)^{-n+1}}{(x-\omega)(x-2\omega) \ldots (x-n+1\omega)} \right] \Delta^n 0, \tag{58} \]

If \( a = 0 \), this becomes

\[p^n \cdot 0 = \left[ \frac{(-1)^n n!}{(x-\omega)(x-2\omega) \ldots (x-n+1\omega)} \right] \Delta^n 0, \tag{59}\]

which becomes when \( n = 0 \)

\[p^0 \cdot 0 = \left[ \frac{1}{x-\omega} \right]. \tag{59.1}\]

Since the result of operating with \( 1/F(p) \) on zero where \( p \) denotes the Milne-Thomson's operator deduced from forward difference operator is obtained by operating with \( e^{\omega p} \) on \( f^*(x) \) given by

\[f^*(x) = \frac{1}{F(\omega)} 0, \tag{60} \]

where \( \Delta \) is the Milne-Thomson's operator in the space of the Heaviside's operator, the corresponding equations of (54.1), (55) and (55.1) becomes

\[(\Delta - a)^n \cdot 0 = \left[ (-1)^n n! (x-\omega)^{-n+1}(1 + a\omega)^{-n+1} \right], \tag{61.1} \]

\[\Delta \cdot 0 = \left[ (-1)^n n! (x-\omega)^{-n+1} \right], \tag{61.2} \]

\[\Delta^0 \cdot 0 = \left[ \frac{1}{x} \right]. \tag{61.3}\]

Since the relation (55.1) holds, the equation

\[f(x) = \frac{1}{F(p)} 0 \tag{62.1}\]

can be also written in the form

\[\text{(24) We have obtained (17) from (9) by multiplying the integrand of (9) by } e^{\omega p} \text{ on Proc. Phys. Math. Soc. Jap. 19, pp. 115-16 (1937), which means that the relation in the space of Milne-Thomson's operator is obtained by operating } e^{\omega p} \text{ on the corresponding relation in the space of Heaviside's operator.}

\[\text{(25) (60) is defined by the integral (3) and interpreted by (1) on Proc. Phys. Math. Soc. Jap. 19, p. 14 (1937).}\]
Thus we can speak:

(I) The operation with the function of the Milne-Thomson's operator deduced from the forward difference operator on the operand zero can be also regarded as the operation with the same function of the Milne-Thomson's operator on the operand

$$\left[ \frac{1}{x + \omega} \right]_0$$

Since the relation (59.1) holds, the equation

$$f(x) = \frac{1}{F(p)} \cdot 0$$

can be also written in the form

$$f(x) = \frac{1}{F(p)} \left[ \frac{1}{x - \omega} \right]_0.$$ 

Thus we can speak:

(II) The operation with the function of the Milne-Thomson's operator deduced from backward difference operator on zero can be also regarded as the operation with the same function of the Milne-Thomson's operator on the operand

$$\left[ \frac{1}{x - \omega} \right]_0$$

Since

$$\left[ \frac{1}{x} \right]_0 e^{-\omega x} \left[ \frac{1}{x + \omega} \right]_0,$$

we have by (62.2)

$$\frac{1}{F(p)} \left[ \frac{1}{x + \omega} \right]_0 = e^{-\omega p} \frac{1}{F(p)} \left[ \frac{1}{x + \omega} \right]_0 = e^{-\omega p} f(x).$$

(26) If we make use of the relation (19.1), the right-hand side of (62.2) can be varied as follows:

$$\frac{1}{F(p)} \left[ \frac{1}{x + \omega} \right]_0 = \frac{1}{\Lambda(p)} \left[ \frac{1}{p^{\omega + \omega}} \right]_0 = \frac{1}{\Lambda(p)} \cdot 0,$$

which asserts the coincidence of (62.2) with (62.1).

(27) Since $f(x) = e^{-\omega p} f(x)$, it is evident that $f'(x) = f'(x - \omega)$.

(28) Since the operator $\Delta$ can be regarded as $p$ when it operates on other than zero,

$$\Delta: \frac{1}{p^{\omega + 1}} \cdot 0 = \frac{1}{p^{\omega + 1}} \cdot 0.$$

This result is used to get the second expression in (66).
But owing to the argument in the foot-note (24) \( e^{-\omega f(x)} \) is equal to \( f^*(x) \) defined by (60), therefore we get

\[
\frac{1}{F(\omega)} \left[ \frac{1}{x} \right]_x = f^*(x) = \frac{1}{F(\Delta)} \cdot 0.
\]

This fact can be spoken as follows:

(III) The operation with the function of the Milne-Thomson's operator deduced from the forward difference operator on zero in the space of Heaviside's operator can be regarded as the operation with the same function of the Milne-Thomson's operator on the operand

\[
\left[ \frac{1}{x} \right]_x
\]

in the space of Milne-Thomson's operator.

Let us now obtain the various interesting result by the aid of the relation

\[
e^{\rho x} = \frac{(1 + \alpha \Delta)^{\rho/\Delta} - 1}{\rho} = \sum_{i=0}^{\infty} \frac{\Delta^i}{i!} \frac{\mu(h, \omega)}{H(h - \omega \omega)} \Delta^i.
\]

If we operate with both sides of (65) on the identity

\[
\frac{H(x|\omega)}{H(x-m|\omega)} = \frac{H(m)}{H(x+\omega|\omega)},
\]

we get\(^{(28)}\)

\[
\frac{H(x+h\omega)}{H(x+h-m\omega)} = \sum_{i=0}^{\infty} \left[ \frac{H(m)}{H(s)} \frac{H(h\omega)}{H(h-s\omega \omega)} \frac{H(x|\omega)}{H(x-m|\omega)} \right]_x \Delta^i,
\]

which is the Vandermonde's theorem in factorials analogous to the Binomial Theorem\(^{(29)}\). The directness with which this result is obtained with the use of the operator is striking.

If (65) operates on zero, owing to (61.2) and the relation\(^{(30)}\)

\[
\left[ \frac{1}{x+h} \right]_x = \frac{\sum_{i=0}^{\infty} (-1)^i \frac{\mu(h, \omega)}{H(h-s\omega \omega)} \frac{1}{(x+s\omega)(x+\omega)(x+\omega \omega) \ldots (x+\omega \omega \omega)} }{1}
\]

it results that

\[
\left[ \frac{1}{x+h} \right]_x = \sum_{i=0}^{\infty} \left( \frac{(-1)^i \mu(h, \omega)}{H(h-s\omega \omega)} \frac{1}{(x+s\omega)(x+\omega)(x+\omega \omega) \ldots (x+\omega \omega \omega)} \right)_{1}.
\]

Dividing both sides of (67) by \( \sin \pi/\pi \) which is the common measure


of the functions in infinitely small space, we have

\[ \frac{1}{x+h} = \sum_{i=0}^{\infty} \frac{(\pi)^i}{i!} \frac{\Pi(h | \omega)}{\Pi(h - s \omega | \omega)} \frac{1}{(x+\omega)(x-s-1\omega) \ldots (x+\omega)x}. \]

If we operate with \( e^{\alpha p} \) on both sides of (67.1) and using the notation \( x^{(-n+1)\omega} \), we obtain

\[ \frac{1}{x+h+\omega} = \sum_{i=0}^{\infty} \frac{(\pi)^i}{i!} \frac{\Pi(h | \omega)}{\Pi(h - s \omega | \omega)} 2^{(-i+1)\omega}. \]

This is the development of \( 1/(x+h) \) in the series of inverse factorials.

Upon multiplication of both sides of (65) by \( \Delta_n^\omega \), (65) becomes

\[ e^{\alpha p} \Delta_n^\omega = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\Pi(h | \omega)}{\Pi(h - s \omega | \omega)} \frac{x^{(-i+1)\omega}}{\Delta_n^\omega}. \]

If this operates on zero, it results by (61.2) that

\[ \left[ \sum_{i=0}^{\infty} (-1)^n \frac{1}{n!} \frac{\Pi(n+s)\Pi(h | \omega)}{\Pi(s)\Pi(h - s \omega | \omega)} \frac{1}{(x+n+\omega)(x+n+s-1\omega) \ldots (x+\omega)x} \right]_n. \]

By the same procedure as that by which we get (68) from (67) we obtain from this

\[ (x+h)^{(-n+1)\omega} = \sum_{i=0}^{\infty} (-1)^i \frac{(n+s)!}{n! s!} \frac{\Pi(h | \omega)}{\Pi(h - s \omega | \omega)} (x)^{(-i+1)\omega}. \]

which is again the development of \( (x+h)^{(-n+1)\omega} \) in the series of inverse factorials, and the case with which this is derived with the aid of the operators is remarkable.

\[ \text{§ 5. On the Logarithmic Singularities.} \]

Let us first define the Psi function for the interval \( \omega \) by the relation

\[ \Psi(x|\omega) = \log \omega - \gamma - \sum_{i=1}^{\infty} \left( \frac{\omega}{x + \omega s} - \frac{1}{s} \right) \]

having the following properties

(31) This series is obtained by the usual method on p. 291 of Milne-Thomson's book.

(32) The definition (71) differs slightly from the definition of \( \psi(x|\omega) \) on p. 256 of Milne-Thomson's book. \( \psi(x-\omega|\omega) \) defined by (71) is equal to \( \psi(x|\omega) \) in Milne-Thomson's book.
where $H'(x|\omega)$ is the derived function of $H(x|\omega)$ with respect to $x$.

If we now differentiate both sides of the relation (15) with respect to $m$, we get

$$\frac{\omega H'(x|\omega)}{H(x|\omega)} = \psi(x|\omega),$$  \hspace{1cm} (72)\(^{33}\)  

If we differentiate both sides of the relation (15) with respect to $m$, we have

$$\frac{\log p - \log p}{m+1} = \frac{1}{H(m)} \frac{H(x|\omega)}{H(x + m + 1|\omega)} \{ 1 + \psi(x + m|\omega) - \psi(m) \},$$  \hspace{1cm} (73)  

If we differentiate the relation

$$p^m = \frac{\sin (m+1)\pi}{\pi} H(m) \frac{H(x|\omega)}{H(x + m + 1|\omega)}$$  \hspace{1cm} (53.1)\(^{34}\)  

with respect to $m$, we have

$$p^m \log p = \cos (m+1)\pi H(m) \frac{H(x|\omega)}{H(x + m + 1|\omega)} = \left( \frac{\sin (m+1)\pi}{\pi} H(m) \frac{H(x|\omega)}{H(x + m + 1|\omega)} \right) \psi(x + m|\omega) - \psi(m),$$  \hspace{1cm} (74)\(^{35}\)  

from which we get

$$p^m \log p = \left( (-1)^{m+1} H(m) \frac{H(x|\omega)}{H(x + m + 1|\omega)} + (-1)^{m+1} \frac{H(x|\omega)}{H(x + m + 1|\omega)} \psi(x + m|\omega) - \psi(m) \right),$$  \hspace{1cm} (74)\(^{35}\)  

by putting $m$ equal to the positive integer $n$.

If we differentiate

$$\frac{\pi}{\sin (m+1)\pi} p^m = H(m) \frac{H(x|\omega)}{H(x + m + 1|\omega)}$$  \hspace{1cm} (33)  

with respect to $m$, we have

$$\left\{ \frac{\pi^2 \cos (m+1)\pi}{\sin (m+1)\pi} \frac{\pi}{\sin (m+1)\pi} p^m + \frac{\pi^2 \pi}{\sin (m+1)\pi} \right\} p^m \log p = - \frac{H(m)}{H(x + m + 1|\omega)} \psi(x + m|\omega) - \psi(m),$$  \hspace{1cm} (33)  

which becomes

(33) This relation can easily get from the relation

$$\frac{\omega H'(x|\omega)}{H(x|\omega)} = \psi(x|\omega)$$  \hspace{1cm} (33)  


(34) This relation is obtained by putting $a=0$ in (53).

\[
(-1)^n [\mathbf{P}^n \cdot 0]_\omega + (-1)^n [\mathbf{P}^n \log \mathbf{P} \cdot 0]_\omega \\
= -\mathcal{H}(n) \frac{\mathcal{H}(x|\omega)}{\mathcal{H}(x+n+1|\omega)} \left\{ \frac{1}{\omega} \psi(x+n+1|\omega) - \psi(n) \right\}.
\] (75)

It is remarkable that the left-hand side of (75) contains the function of the infinitely large space of the order \(\infty^2\).

If we put \(m=0\) in (73), we get

\[
\log \mathbf{P} \cdot 0 = -\frac{1}{\omega} \psi(x|\omega) - \psi(0).
\]

Hence we can conclude owing to the definition in the foot-note (12)

\[
\log \mathbf{P} \cdot 0 = -\mathbf{P} \left\{ \frac{1}{\omega} \psi(x|\omega) - \psi(0) \right\}.
\] (76)

On the other hand if we put \(n=0\) in (74), we have

\[
\log \mathbf{P} \cdot 0 = -\frac{1}{x+\omega} - \frac{1}{x+n} \left[ \frac{1}{\omega} \psi(x+n|\omega) - \psi(0) \right]_\omega.
\] (77)

Since we have by (55.1)

\[
\mathbf{P} \psi(0) = \left[ \frac{\psi(0)}{x+\omega} \right]_\omega,
\]

comparing (76) with (77), we get

\[
\mathbf{P} \psi(x|\omega) = \frac{1}{x+\omega} + \frac{1}{x+n} \left[ \frac{1}{\omega} \psi(x+n|\omega) \right]_\omega.
\] (78)

We now differentiate both sides of (65) with respect to \(h\) and get

\[
\rho e^{\mathbf{P} r} = \sum_{x=0}^\infty \frac{1}{\mathcal{H}(h|\omega)} \mathcal{H}(h-s\omega|\omega) \left[ \frac{1}{\omega} \psi(h|\omega) - \psi(h-s\omega|\omega) \right]_\omega.
\] (79)

If we operator with (79) on zero, by the same reason as we got (67), from (66) we get

\[
\left[ \frac{1}{(x+h)^\omega} \right]_\omega = \left[ \sum_{x=0}^\infty (-1)^x \frac{1}{\mathcal{H}(h-s\omega|\omega)} \psi(h|\omega) - \psi(h-s\omega|\omega) \right]_\omega.
\] (80)


(57) This relation corresponds to

\[
\rho \log x = \frac{1}{x} + \left[ \frac{\log x}{x} \right]_\omega
\]


(58) Since

\[
\rho e^{\mathbf{P} r} = \rho \left[ \frac{1}{x+h} \right]_\omega = \frac{d}{dx} \left[ \frac{1}{x+h} \right]_\omega,
\]

the left-hand side of (80) results from the left-hand side of (79).
Proceeding exactly in the same manner as we got (68) from (69), we get from (80)

\[ \frac{1}{(x+h)^2} = \sum_{n=0}^{\infty} \left( -1 \right)^{n+1} \frac{H(h \omega)}{H(h - 2n \omega)} \frac{1}{h} \left[ \Phi(h \omega) - \Phi(h - n \omega) \right] \left( (x - \omega)^{n+1} \right) \; \text{; (81)} \]

or since by (71)

\[ \frac{1}{\omega} \left[ \Phi(h \omega) - \Phi(h - n \omega) \right] = \frac{1}{h - \omega} + \frac{1}{h - 2\omega} + \ldots + \frac{1}{h - n\omega} + \frac{1}{h} \]

we have

\[ \frac{1}{(x+h)^2} = \sum_{n=0}^{\infty} \left( -1 \right)^{n+1} \frac{H(h \omega)}{H(h - 2n \omega)} \left[ \sum_{n=0}^{\infty} \frac{1}{h - n\omega} \right] \left( (x - \omega)^{n+1} \right) \; \text{; (81.1)} \]

Thus we have obtained the inverse factorial series of \( 1/(x+h)^2 \) with the aid of operators.

\[ \text{(6). Miscellaneous Considerations.} \]

In my previous paper the author concluded(39) that the operator \( p^{-1} \) plays the part of the summation operator from zero to \( x \) namely,

\[ p^{-1} f(x) = \sum_{n=0}^{x} f(x) \Delta x \; \text{; (82)} \]

But if we go into this conclusion we shall see that the result (82) is obtained by assuming that the function \( f(x) \) is capable of expansion in a Newton's series.

Hence the question naturally arises that the operator \( p^{-1} \) can also play the part of summation operator when the operand is not expandable in a Newton's series. We shall answer to this question by dividing following three cases.

(a) When the operand is

\[ x^{(-n-1)} = \frac{1}{(x + \omega)(x + 2\omega) \ldots (x + n\omega)} \; \text{; (83)} \]

where \( n \) is a positive integer greater than unity.

If we make use of the relation(40)

\[ x^{(-n-1)} = \left[ (-1)^{n+1} \frac{p^{n-1}}{(n-1)!} \right] \; \text{, (83)} \]

we have by the definition in the foot-note (12)


(40) This relation is obtained by putting \( n = 0 \) in (53.1).

(41) This relation is obtained by multiplying the both sides of (55.1) by \( \pi/\sin \pi \).
\( p^{-1} x^{(-n)\omega} = \left[ (-1)^{n-1} \frac{p^{n-1}}{(n-1)!} \right] 0 = \frac{(-1)^{n-1}}{(n-1)!} [p^{n-2}]\omega = -\frac{x^{(-n-1)\omega}}{n-1}. \)

Since we know
\[
-\frac{x^{(-n-1)\omega}}{n-1} = \tilde{S} x^{(-n)\omega} \Delta z,
\]
we can conclude that the operator \( p^{-1} \) plays the part of the summation operator from \( \infty \) to \( x \) when the operand is given by (83).

(b) When the operand is
\[ x^{(-1)\omega} \left( \frac{1}{x + \omega} \right). \] (84)

If we make use of the relation
\[ x^{(-1)\omega} = [p^0 0]_\omega, \] (85)
we have by the definition in the foot-note (12)
\[
p^{-1} \frac{1}{x + \omega} = \left[ \begin{array}{c} p^0 0 \\ \omega \end{array} \right]_\omega = \left[ \begin{array}{c} 1 0 \\ \omega \end{array} \right]_\omega = [1]_\omega. \] (84.1)

Hence we can conclude that the operator \( p^{-1} \) plays no longer the part of summation operator when the operand is \( x^{(-1)\omega} \) and the function in the infinitely large space results if we operate with \( p^{-1} \) on \( x^{(-1)\omega} \).

(c) When the operand is
\[ x^{(-1)\omega} \frac{\psi(x + \omega|\omega)}{\omega}. \] (86)

If we put \( n = 0 \) in (75), we have
\[ x^{(-n)\omega} \left\{ \frac{1}{\omega} \psi(x + \omega|\omega) - \psi(0) \right\} = -[p^0 0]_\omega - [\log p^0 0]_\omega. \]
But since by (85)
\[ x^{(-n)\omega} \psi(0) = [\psi(0) p^0 0]_\omega, \]
(86) reduces to
\[
\frac{1}{\omega} x^{(-1)\omega} \frac{\psi(x + \omega|\omega)}{\omega} = -[p^0 0]_\omega - [\log p - \psi(0) p^0 0]_\omega.
\]

Therefore by the definition in the foot-note (12) we have
\[
p^{-1} \left\{ \frac{1}{\omega} x^{(-1)\omega} \frac{\psi(x + \omega|\omega)}{\omega} \right\} = -[p^{-1} 0]_\omega - \left[ \frac{1}{p} \{ \log p - \psi(0) p^0 0 \} \right]_\omega
\]
\[ = -[1]_\omega + \left[ \frac{1}{\omega} \psi(x|\omega) \right]_\omega. \] (86.1)

Hence we can conclude that \( p^{-1} \) plays no longer the part of summation operator when the operand is given by (86) and the function
in the infinitely large space of the order $\infty^3$ results if we operate with $p^{-1}$ on (86).

The operation with $p^{-1}$ on almost other operands\(^{(42)}\) reduces to the above-mentioned three cases.

Take, for example, the operand

$$\frac{1}{x+h}$$

(87)

Since this function is developable in the series of inverse factorials given by (68), the operation with $p^{-1}$ on (87) reduces to the case (a) and the case (b).

Take further the operand

$$(-1)^{n-1}(n-1)! \cdot x^{\gamma-n}\left(1 + a\omega\right)^{x/\omega}.$$  \hspace{1cm} (88)

Since

$$(-1)^{n-1}(n-1)! x^{\gamma-n}\left(1 + a\omega\right)^{x/\omega} = [\left(p - a\right)^{n-1} \cdot 0]_{x},$$

(88.1)\(^{(45)}\)

if we denote the series which is got by expanding $(p-a)^{n-1}$ by the binomial theorem by

$$\sum_{r=0}^{n-1} c_r p_r,$$

(88.1) becomes

$$(-1)^{n-1}(n-1)! x^{\gamma-n}\left(1 + a\omega\right)^{x/\omega} = \left[ \sum_{r=0}^{n-1} c_r p^r \cdot 0 \right]_{x} = \sum_{r=0}^{n-1} \left[ (-1)^{r} ! c_r x^{\gamma-r+1}\right]_{x}.$$  \hspace{1cm} (89)

Since the right-hand series of (89) is the sum of the terms of the types (83) and (84), the operation with $p^{-1}$ on (88) reduces again to the case (a) and the case (b).

Let us now investigate the properties of the operator $p$ when it operates on various operands.

Since the operator $p$ plays the converse part of the operator $p^{-1}$, $p$ is equal to the difference operator $\Delta$ when it operates on the operands which are obtained as the results of operating with $p^{-1}$ where $p^{-1}$ plays the part of the summation operator, for example,

$$\hat{S} \Delta x \text{ or } \hat{S} \Delta x.$$  \hspace{1cm} \(\text{Hence we have only to examine as cases when } p \text{ operates on the operands obtained as the results of operating with } p^{-1} \text{ where } p^{-1} \text{ does}\)

\(^{(42)}\) The case when the operational expression of the operand has algebraic branch point is excluded here.

\(^{(43)}\) This is obtained by putting $m = n-1$ in (53).
not play the part of the summation operator.

(d) \textit{Remembering the result (84.1), we take unity as operand.}

Since by (15) \[1 = \frac{0}{p}\]
we have with the aid of the definition in the foot-note (12)
\[p \cdot 1 = p \cdot \frac{0}{p} = p^0 = \left[ \frac{1}{x + \omega} \right]_\infty\]  \hspace{1cm} (90.1)

On the other hand we have
\[\Delta \frac{1}{\omega} = 0.\]  \hspace{1cm} (90.2)

Hence we can conclude that the result (90.1) which is obtained operating with \(p\) on unity differs from the result (90.2) which is obtained operating with \(\Delta\) on unity only by the quantity in the infinitely small space.

(e) \textit{Remembering the result (86.1), we take \(\psi(x|\omega)\) as operand.}

We have by (78)
\[p \cdot \psi(x|\omega) = \frac{1}{x + \omega} + \frac{1}{x + \omega} \left[ \frac{1}{\omega} \psi(x + \omega|\omega) \right]_\infty\]  \hspace{1cm} (91.1)

On the other hand we have by (71)
\[\Delta \frac{\psi(x|\omega)}{\omega} = \frac{1}{x + \omega}.\]  \hspace{1cm} (91.2)

Hence we can conclude that the result (91.1) which is obtained operating with \(p\) on \(\psi(x|\omega)\) differs from the result (91.2) which is obtained operating with \(\Delta\) on \(\psi(x|\omega)\) only by the quantity in the infinitely small space.

Let us now investigate what parts play \(x\) and \(x^{-1}\) in the space of the Milne-Thomson's operator.

Take
\[x \frac{H(x|\omega)}{H(x - m\omega|\omega)},\]  \hspace{1cm} (92)

which can be varied as follows
\[x \frac{H(x|\omega)}{H(x - m\omega|\omega)} = (x - m\omega) \frac{H(x|\omega)}{H(x - m\omega|\omega)} \frac{H(x|\omega)}{H(x - m\omega|\omega)} + m\omega \frac{H(x|\omega)}{H(x - m\omega|\omega)} \]  \hspace{1cm} (93)

Owing to the relation (15) we have
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\[ \frac{H(x|\omega)}{H(x-m+1|\omega)} = \frac{H(m)}{p^{n+1}}. \] (94)

Hence we get from (92)

\[ \frac{H(x|\omega)}{H(x-m|\omega)} = \frac{H(m)}{p^{n+1}}. \] (92.1)

On the other hand we get from (93)

\[ \frac{H(x|\omega)}{H(x-m+1|\omega)} + m\omega \frac{H(x|\omega)}{H(x-m|\omega)} = \frac{H(m+1)}{p^{n+1}} + m\omega \frac{H(m)}{p^{n+1}}, \] (93.1)

Consequently it results from (92.1) and (93.1) that

\[ \frac{x H(m)}{p^{n+1}} + \frac{H(m+1)}{p^{n+1}} = m\omega \frac{H(m)}{p^{n+1}}. \] (95)

In order that the above identity holds, \( x \) must be regarded as the operator \(-d/dp (1+\omega p)\).

Since any usual function \( g(p) \) in the space of the Milne-Thomson's operator can be representable in the form

\[ g(p) = \sum a_n \frac{H(n)}{p^{n+1}}. \] (96)

and to each term of the series (96) the above rule holds. This rule must hold for \( g(p) \). Hence we have obtained the following very important result.

(IV) \( x \) plays the part of the operator

\[ -\frac{d}{dp} (1+\omega p) \]

in the space of Milne-Thomson's operator.

We now take

\[ \frac{1}{x} \frac{H(x|\omega)}{H(x-m+1|\omega)} \] (97)

(44) The notation means that the operational representation of the left-hand member of (94) is equal to the right-hand member of (94), namely, if we operate with the right-hand member of (94) on zero we get the left-hand expression of (94).

(45) This relation also holds in the case even when the function in the infinitely small space makes its appearance. If \( m \) is equal to \(-n\) where \( n \) is a positive integer greater than unity, the relation (96) becomes

\[ \frac{1}{x} \frac{H(x|\omega)}{H(x-m+1|\omega)} \]

and to each term of the series (96) the above rule holds. This rule must hold for \( g(p) \). Hence we have obtained the following very important result.

(IV) \( x \) plays the part of the operator

\[ -\frac{d}{dp} (1+\omega p) \]

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The operational representation of which is

\[ \frac{1}{x} \frac{H(x|\omega)}{H(x-m+1|\omega)} \] (97)

the operational representation of which is

\[ \frac{1}{x} \frac{H(x|\omega)}{H(x-m+1|\omega)} \]

Hence we can regard also in this case \( x \) as the operator \(-d/dp (1+\omega p)\).
which is equal to
\[ \frac{I(x|\omega)}{I(x-m\omega|\omega)} = m_\omega \cdot \frac{I(x|\omega)}{x I(x-m\omega|\omega)}. \] (97.1)

The function
\[ \frac{1}{x I(x|\omega)} \]
which is contained in the last term of (97.1) is obtained from (97) by putting \( m \) in place of \( m+1 \) in (97). Hence proceeding exactly in the same manner as we have obtained (97.1) from (97), the last term becomes
\[ \frac{m_\omega}{x I(x-m\omega|\omega)} I(x|\omega) = m_\omega \frac{I(x|\omega)}{x I(x-m\omega|\omega)} \frac{I(x|\omega)}{I(x-m-1\omega|\omega)}. \]

Substituting this result in (97) we have
\[ \frac{1}{x I(x|\omega)} \frac{I(x|\omega)}{I(x-m+1\omega|\omega)} = \frac{I(x|\omega)}{I(x-m\omega|\omega)} \]
\[ -m_\omega \frac{I(x|\omega)}{I(x-m-1\omega|\omega)} \frac{m_\omega(m-1)}{x I(x-m-1\omega|\omega)}. \]

By the repetition of the above process we finally reach the result
\[ \frac{1}{x I(x|\omega)} \frac{I(x|\omega)}{I(x-m+1\omega|\omega)} = \frac{I(x|\omega)}{I(x-m\omega|\omega)} \]
\[ -m_\omega \frac{I(x|\omega)}{I(x-m-1\omega|\omega)} + m_\omega(m-1) \frac{I(x|\omega)}{I(x-m-2\omega|\omega)} \]
\[ -m_\omega(m-1) \omega(m-2) \frac{I(x|\omega)}{I(x-m-3\omega|\omega)} + \ldots. \] (98)

The operational representation of (98) is
\[ \frac{1}{x} \frac{I(m+1)}{p^{m+2}} = -\int_p \frac{I(m+1)}{p^{m+2}} dp + \omega p \int_p \frac{I(m+1)}{p^{m+3}} dp \]
\[ -\omega p^2 \int_p \frac{I(m+1)}{p^{m+2}} dp + \omega p^3 \int_p \frac{I(m+1)}{p^{m+3}} dp - \ldots. \] (99)

The right-hand series of (99) is summed in the form
\[ \frac{1}{1 + \omega p} \int_p \frac{I(m+1)}{p^{m+2}} dp \]
by the binomial theorem. Hence we get
\[ \frac{1}{x} \frac{I(m+1)}{p^{m+2}} = \frac{-1}{1 + \omega p} \int_p \frac{I(m+1)}{p^{m+2}} dp. \] (100)}

(64) More strictly speaking, the note 2 as I have added to the integral (135) on 1069, (1937) of this Proceedings must be added to this integral.
In order that the identity (100) holds \(1/x\) must be regarded as the operator 
\[-(1 + \omega p)^{-1}\int p \, dp.\] 
Hence by the argument in all respect like that used to get (IV) from (95) we have

(V) \(1/x\) plays the part of the operator \([-1/(1 + \omega p)]\) in the space of Milne-Thomson's operator.

If we denote the operational representation of \(f(x)\) by \(F(p)\), we can easily prove that

\[
\frac{\Delta f(x)}{\omega} \equiv p^n F(p) - P^{n-1} f(0) - p^{n-2} f^{(1)}(0) - \ldots - P f^{(n-2)}(0) - f^{(n-1)}(0),
\]

where \(f^{(k)}(0)\) is the result of putting \(x=0\) in the \(k\)th difference of \(f(x)\).

If we combine (IV) with (101), we obtain the following useful results

\[
\begin{align*}
\frac{d}{dp} F(p) & = -\omega p \frac{dF(p)}{dp} - \omega F(p), \\
x \delta f(x) & = - \frac{d}{dp} (1 + \omega p) \{ p F(p) - f(0) \} \\
- \omega p \frac{dF(p)}{dp} & - F(p) - 2 \omega p F(p) - \omega p^2 \frac{dF(p)}{dp} - \omega f(0),
\end{align*}
\]

If we have to solve the difference equation whose coefficients are simple polynomials in \(x\), it is sometimes convenient to take the operational representation of this equation with the aid of (101) and (102). The operational representation thus obtained is usually the differential equation in the space of Milne-Thomson's operator which can be solved by the methods of solving the differential equation. If we operate with
the solution of this differential equation on zero, we get the required solution of the difference equation.

The fact that the solution of the difference equation is obtained by solving the differential equation is striking.

Let us now consider the sum equation of the type

$$u(x) = v(x) - \lambda \int_0^x k(x-\xi)u(\xi) \, d\xi$$  \hspace{1cm} (103)

which is similar to Volterra's integral equation.

If we use the notation \( U(p) \), \( V(p) \) and \( K(p) \) which are given by

\[
\begin{align*}
U(p) \cdot 0 &= u(x), \\
V(p) \cdot 0 &= v(x), \\
K(p) \cdot 0 &= k(x+\omega),
\end{align*}
\]  \hspace{1cm} (104.1, 104.2, 104.3)

the operational representation of (103) is by (21)

$$U(p) = V(p) - \lambda K(p) U(p).$$

Hence

$$U(p) = \frac{V(p)}{1 + \lambda K(p)} = \left[ 1 - \frac{\lambda K(p)}{1 + \lambda K(p)} \right] \cdot V(p).$$

If this operates on zero we have by (104.1), (104.2) and (21)

$$u(x) = v(x) - \lambda \int_0^x k(x-\xi) v(\xi) \, d\xi,$$  \hspace{1cm} (105)

where

$$h(x+\omega) = \frac{K(p)}{1 + \lambda K(p)}.$$  \hspace{1cm} (106)

\( u(x) \) given by the right-hand expression of (105) is the solution of the sum equation (103), and \( h(x-\xi) \) given by (106) is the solvent kernel of the kernel \( k(x-\xi) \) given by (104.3).

If we consider the sum-difference equation of the form

$$a_n \Delta u + a_{n-1} \Delta u + \ldots + a_1 \Delta u + a_0 u = v(x) - \int_0^x k(x-\xi)u(\xi) \, d\xi$$  \hspace{1cm} (107)

with the initial condition

$$\Delta u_{\omega} = u_{\omega-1} - u_{\omega}, \ldots, \Delta u_{\omega} = u_1, u = u_0 \text{ when } x = 0,$$  \hspace{1cm} (108)

the operational representation of (107) is by (21) and (101)

$$J(p) = U(p) \cdot \left\{ \sum_{r=0}^{n-1} a_r p^r + K(p) \right\} - \sum_{r=0}^{n-1} \left( \sum_{k=0}^r a_r \sum_{k=0}^r p^{r-k} u_k \right).$$

Hence

$$U(p) = \frac{J(p) + \sum_{r=0}^{n-1} \left( a_r \sum_{k=0}^r p^{r-k} u_k \right)}{\left[ \sum_{r=0}^{n-1} a_r p^r + K(p) \right]}.$$

If this operates on zero, it results that
\[ u(x) = \tilde{S} [h(x - \xi - \omega) v(\xi) \Delta^{\xi} + \sum_{r=0}^{n+1} \left( \sum_{k=0}^{r} \Delta_{k} h(x) \right)], \quad (109) \]

where \[ h(x) = 0 \int \left[ \sum_{r=0}^{n} a_{r} p^{r} + K(p) \right]. \]

\( u(x) \) given by (109) is the solution of the equation (n107) and satisfies the condition (108).

\section*{7. The Expression for the Milne-Thomson's Operator corresponding to Carson's Integral for Heaviside's Operator.}

Let us now try to get the expression which corresponds to Carson's integral in the case of Heaviside's operator.

By the direct calculation we have

\[ \Delta \left\{ (1 + a\omega) \right\}^{\xi - \omega - 1} (1 + p\omega)^{-\xi - 1} - \frac{p - a}{1 + p\omega} (1 + a\omega)^{\xi - \omega} (1 + p\omega)^{-\xi - 1}. \]

Hence

\[ \tilde{S} (1 + 2\omega + (1 + p\omega)^{-\xi - 1}) \Delta \xi = \frac{1}{p - a} \Delta \{ (1 + a\omega)^{\xi - \omega} (1 + p\omega)^{-\xi - 1} \}. \]

We now evaluate

\[ \tilde{S} \left[ \frac{H(\xi)}{H(n)H(\xi - n\omega)} \right] \left\{ (1 + a\omega)^{\xi - \omega} (1 + p\omega)^{-\xi - 1} \right\} \Delta \xi \quad (111) \]

using the formula (2). If we regard \( \Delta \xi \) as \( \omega \) in (2) and regard

\[ \frac{H(x + \omega)}{H(n)H(x - n\omega)} \]

\( v(\xi) \) in (2), then (111) can be varied by (110) as follows:

\[ \tilde{S} \left[ \frac{H(\xi + \omega)}{H(n)H(\xi - n\omega)} \right] \left\{ (1 + a\omega)^{\xi - \omega} (1 + p\omega)^{-\xi - 1} \right\} \Delta \xi
\]

\[ = \left[ \frac{H(\xi + \omega)}{H(n)H(\xi - n\omega)} \right] \left\{ (1 + a\omega)^{\xi - \omega} (1 + p\omega)^{-\xi - 1} \right\} \Delta \xi
\]

The first member in the right-hand side of (112) vanishes on account of

\[ \frac{H(x + \omega)}{H(n)H(x - n\omega)} \quad \text{when} \quad x = 0 \]

and the fact that
tends exponentially to zero as \( \xi \to \infty \) if we assume that 
\[(1 + a\omega) < (1 + p\omega).\]

Hence (112) becomes

\[
\frac{1}{p - a} \cdot \left[ - \frac{H(\xi \omega)}{H(\xi - n\omega)\omega} \cdot \{(1 + a\omega)^{\nu/\omega - n}(1 + p\omega)^{-\nu/\omega - 1}\} \right] \Delta \xi
\]

Repeating the same processes by which we have got the right-hand expression of (113) from the left hand side of (113), we finally reach the result

\[
\frac{1}{p - a} \cdot \left[ - \frac{H(\xi \omega)}{H(\xi - n\omega)\omega} \cdot \{(1 + a\omega)^{\nu/\omega - n}(1 + p\omega)^{-\nu/\omega - 1}\} \right] \Delta \xi
\]

if \( n \) is a positive integer.

If we apply (110) to the right-hand member of (114), we have

\[
\left[ - \frac{1}{(p - a)^{n+1}} \cdot \{(1 + a\omega)^{\nu/\omega}(1 + p\omega)^{-\nu/\omega}\} \right]_{\xi = 0}^{\xi = \infty} = \frac{1}{(p - a)^{n+1}}.
\]

Substituting this result in (114) we finally reach the following very important result

\[
\frac{1}{(p - a)^{n+1}} \cdot \left[ - \frac{H(\xi \omega)}{H(\xi - n\omega)\omega} \cdot \{(1 + a\omega)^{\nu/\omega - n}(1 + p\omega)^{-\nu/\omega - 1}\} \right] \Delta \xi = \frac{1}{(p - a)^{n+1}}.
\]

We know that

\[
\frac{1}{(p - a)^{n+1}} \cdot \left[ - \frac{H(\xi \omega)}{H(\xi - n\omega)\omega} \cdot \{(1 + a\omega)^{\nu/\omega - n}(1 + p\omega)^{-\nu/\omega - 1}\} \right] \Delta \xi = \frac{1}{(p - a)^{n+1}}.
\]

Hence comparing the result (115) with (116) we can say that the operational representation of

\[
\frac{H(\xi \omega)}{H(\xi - n\omega)\omega} \cdot (1 + a\omega)^{\nu/\omega - n}
\]

can be obtained if we multiply (116.1) by \((1 + p\omega)^{-\nu/\omega - 1}\) and sum up from zero to \( \infty \).

---

Since any usual function $f(x)$ is representable in the series
\[
f(x) = \sum_{n} \frac{c_n I (x|\omega)}{n! I (n|\omega) (1 + a \omega)^{x/n}} (1 + a \omega)^{x/n - n}
\]  
(117)
and to each term of the series (117) the above mentioned rule is applicable, this rule must be also applicable to $f(x)$. Hence we establish

(VI) The operation representation of $f(x)$ which is representable in the series (117) can be obtained, if we multiply $f(x)$ by
\[
(1 + p \omega)^{-x/n - 1}
\]
and sum up from zero to $\infty$.

If we denote the operational representation of $f(x)$ by $g(p)$, the rule (VI) is the same thing as the following relation
\[
g(p) = \sum_{n} (1 + p \omega)^{-x/n - 1} f(\xi) \Delta \xi,
\]  
(118)
which corresponds to the Carson's integral\(^{50}\) of the Heaviside's operator.

The method of getting the relation corresponding to Carson's integral in the case of the Milne-Thomson's operator deduced from backward difference operator is exactly the same as in the case of the Milne-Thomson's operator deduced from forward difference operator. Therefore we write here only the essence of the method in the case of Milne-Thomson's operator deduced from backward difference operator.

The expression corresponding to (111) is
\[
\sum_{n} \left[ \frac{I (\xi + n - 1|\omega)}{I(n|\omega) (1 + \omega)^{x/n - n}} (1 - a \omega)^{-x/n - n} (1 - p \omega)^{x/n - 1} \right] \Delta \xi
\]  
(111')
which we evaluate using the formula (28) and
\[
\sum_{n} \left[ (1 - a \omega)^{-x/n - n} (1 - p \omega)^{x/n - 1} \right] \Delta \xi = -\frac{1}{p - a} \sum_{n} \left[ (1 - a \omega)^{-x/n - n} (1 - p \omega)^{x/n - 1} \right] \Delta \xi
\]  
(110')
and from which we get the relation
\[
\sum_{n} \left[ \frac{I (\xi + n - 1|\omega)}{I(n|\omega) (1 + \omega)^{x/n - n}} (1 - a \omega)^{-x/n - n} (1 - p \omega)^{x/n - 1} \right] \Delta \xi
\]  
= \frac{1}{p - a} \sum_{n} \left[ \frac{I (\xi + n - 2|\omega)}{I(n - 1|\omega) (1 + \omega)^{x/n - n}} (1 - a \omega)^{-x/n - n} (1 - p \omega)^{x/n - 1} \right] \Delta \xi
\]  
(113')
which corresponds to (118).

The relation corresponding to (114) is here


the right-hand member of which is easily found to be equal to 
1/(p-a)^{n+1}. Hence we have

\[ \mathcal{S} \left[ \frac{\Pi(\xi + n - 1 \omega \omega)}{\Pi(n)\Pi(\xi - \omega \omega)} \right] \Delta \xi = \frac{1}{(p-a)^{n+1}}. \tag{114'} \]

The relation corresponding to (116) is here

\[ \frac{1}{(p-a)^{n+1}} \Pi(x + n - 1 \omega \omega) (1 - a \omega)^{-\xi/a - n}. \tag{116'} \]

Hence by the argument in all respect like that by which we deduced
(VI) from (115) and (116) we obtain from (115') and (116')
(VII) The operational representation of f(x) can be obtained, if we
multiply f(x) by

\[ (1 - p \omega)^{\xi/a - 1} \]

and sum up from zero to \( \infty \).

If we denote the the operational representation of f(x) by \( g(p) \), the
rule (VII) is the same thing as the following relation

\[ g(p) = \mathcal{S} \left[ (1 - p \omega)^{\xi/a - 1} f(\xi) \right] \Delta \xi \tag{118'} \]

which is the Carson's integral in the case of the Milne-Thomson's
operator deduced from backward difference operator.

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