The Distribution of Laminar Skin Friction on a Sphere placed in a Uniform Stream.

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I. Introduction.

§ 1. In a previous paper(1), one of the present writers has made a preliminary study on the laminar boundary layer on the surface of a sphere which is placed in a uniform stream. By employing the momentum integral equation for the boundary layer on a body of revolution, which corresponds with the well-known Kármán's momentum integral equation for the two-dimensional case, and assuming a quartic form for the velocity profile in the boundary layer, as done by Pohlhausen in the case of a two-dimensional stream, the differential equation for determining the thickness of the boundary layer has been solved and thus various characteristic quantities for the boundary layer have been discussed. For the velocity distribution outside the boundary layer, use has been made of the well-known theoretical distribution as well as a distribution found experimentally by O. Flachsbart when the Reynolds number of the stream was below the critical Reynolds number of a sphere.

However, no further theoretical investigations on the subject have been tried in the previous paper, because detailed experimental evidence concerning the boundary layer of a sphere was not available at that time which could be compared with theoretical results.

Recently, A. Fage(2), of the National Physical Laboratory, has made some detailed experimental researches on the boundary layer of a sphere, at Reynolds numbers below, within, and above the critical range over which the drag coefficient of a sphere experiences a large fall. The principal object of his experiments was to determine the influence of Reynolds number and turbulence in the free stream on the transition from laminar to turbulent flow in the boundary layer of a sphere, for Reynolds numbers within the critical range. In order to obtain the information on the matter, Fage measured distributions

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of normal pressure and surface friction over the surface of a sphere.

In view of the importance of the problem, the present writers have recently performed various calculations similar to those in the previous paper for the boundary layer of a sphere, by employing the actual distributions of velocity over the surface of a sphere which have been obtained from the distributions of normal pressure determined experimentally by Fage, and the results of the calculations have been compared with Fage's observations.

The object of the present paper is to describe a part of the results of our investigations. The paper deals with a case in which the Reynolds number of the stream is just within the critical range so that, according to Fage's observations, the flow in the boundary layer is everywhere laminar up to the point of separation. A more detailed description of the results, together with those for other cases of larger Reynolds numbers, will be given shortly elsewhere.

II. The Momentum Integral Equation for the Boundary Layer on a Body of Revolution.

§ 2. We first consider the general momentum integral equation for the boundary layer on a body of revolution. We assume that a body of revolution is placed in a uniform stream such that its axis of revolution is parallel to the direction of the undisturbed stream.

Let $x$ be the length of the generator of the body of revolution measured from the forward stagnation point, which coincides, in the present case, with the point where the axis of revolution cuts the surface of the body, and let $y$ be the distance of a point in the layer from the surface of the body measured along the normal to the surface. We denote the radius of the transverse cross-section of the body of revolution by $r$, which is a known function of $x$.

Further, let $\delta$ be the thickness of the boundary layer, and $u$ be the velocity in the $x$-direction inside the layer; while the velocity and pressure just outside the boundary layer will be denoted by $U$ and $p$ respectively. Also, we denote by $\rho$ the density of the fluid, which is assumed to be incompressible, and by $\tau$, the intensity of skin friction on the surface of the body.

Then, if we assume that $y$ is very small in comparison with the longitudinal radius of curvature of the body and that the thickness of the boundary layer $\delta$ also is very small compared with $r$, we get, with the aid of the theorem of momentum, the momentum integral equation
for the boundary layer on the body of revolution in the form:

\[ \frac{1}{r} \frac{d}{dx} \left( \int_{\delta}^{\infty} u^2 dy \right) - \frac{U}{r} \frac{d}{dx} \left( \int_{\delta}^{\infty} u dy \right) = -\delta \frac{1}{\rho} \frac{d}{dx} \frac{dp}{dx} - \tau. \tag{1} \]

If we use the relation:

\[ \frac{1}{\rho} \frac{d}{dx} \frac{dp}{dx} = -U \frac{dU}{dx}, \]

which follows immediately from Bernoulli's theorem, this equation can be written as:

\[ \frac{1}{r} \frac{d}{dx} \left( \int_{\delta}^{\infty} u^2 dy \right) - \frac{U}{r} \frac{d}{dx} \left( \int_{\delta}^{\infty} u dy \right) = \delta U \frac{dU}{dx} - \tau. \tag{2} \]

This momentum integral equation is applicable to both the laminar and turbulent boundary layers on a body of revolution. In the case of turbulent boundary layer, however, we must take for \( u, U \) and \( \frac{dU}{dx} \) their respective mean values.

Further, it will be noticed that at least for a blunt-nosed body of revolution the above integral equation (2) may legitimately be used, as was proved so by C. B. Millikan, to describe the laminar or turbulent boundary layer, even in the neighborhood of the forward stagnation point at the nose, where \( r \to 0 \) and therefore the condition that \( \delta \) is very small compared with \( r \) is not necessarily satisfied.

§ 3. In the case of laminar boundary layer, \( \tau_0 \) is the intensity of laminar skin friction and is given by

\[ \tau_0 = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0}, \tag{3} \]

where \( \mu \) is the coefficient of viscosity of the fluid concerned.

Thus, the momentum integral equation for the laminar boundary layer on a body of revolution is

\[ \frac{1}{r} \frac{d}{dx} \left( \int_{\delta}^{\infty} u^2 dy \right) - \frac{U}{r} \frac{d}{dx} \left( \int_{\delta}^{\infty} u dy \right) = \delta U \frac{dU}{dx} - \mu \left( \frac{\partial u}{\partial y} \right)_{y=0}. \tag{4} \]

where \( \nu = \mu/\rho \) is the kinematic coefficient of viscosity of the fluid.

It may be remarked here that this momentum integral equation for the laminar boundary layer on a body of revolution can also be derived, as done by Millikan in his paper cited above, from the

III. The Differential Equation for Determining the Thickness of the Laminar Boundary Layer.

§ 4. In the present paper we shall confine ourselves to the laminar boundary layer only, and therefore, the integral equation of the form (4) only will be considered.

Now, equation (4) can be put in the form:

\[
\frac{d}{dx} \int_0^x u^2 dy - U \frac{d}{dx} \int_0^x \frac{\partial u}{\partial y} dy + \frac{1}{\nu} \frac{d}{dx} \left[ \int_0^x u^2 dy - U \int_0^y u \partial y \right] = \delta \cdot U \frac{dU}{dx} - U \left( \frac{\partial U}{\partial y} \right)_{y=0}. \tag{5}
\]

To solve this equation approximately we assume for \( u \) a quartic form in \( y \), as assumed by Pohlhausen in the case of two-dimensional laminar boundary layer, namely:

\[
u = a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4, \tag{6}
\]

where \( a_1, a_2, a_3, a_4 \) are functions of \( x \).

The boundary conditions are

\[
\begin{align*}
\left. \begin{array}{l}
u = 0, \quad \frac{\partial^2 \nu}{\partial y^2} = - \frac{1}{\nu} \frac{dU}{dx}, \quad \text{at} \quad y = 0; \\
u = U, \quad \frac{\partial \nu}{\partial y} = 0, \quad \frac{\partial^2 \nu}{\partial y^2} = 0, \quad \text{at} \quad y = \delta.
\end{array} \right\}
\tag{7}
\]

The first condition that \( \nu = 0 \) at \( y = 0 \) is satisfied by the assumed expression for \( \nu \) and the remaining four conditions determine \( a_1, a_2, a_3, a_4 \). We thus have

\[
\begin{align*}
\left. \begin{array}{l}
\frac{\partial^2 \nu}{\partial y^2} = - \frac{1}{\nu} \frac{dU}{dx}, \quad \text{at} \quad y = 0; \\
\frac{\partial \nu}{\partial y} = 0, \quad \frac{\partial^2 \nu}{\partial y^2} = 0, \quad \text{at} \quad y = \delta.
\end{array} \right\}
\tag{8}
\]

where \( \lambda \) stands for the non-dimensional quantity \( \frac{\delta^2}{\nu} \frac{dU}{dx} \), namely:

\[
\lambda = \frac{\delta^2 U'}{\nu}. \tag{9}
\]

In this as well as in subsequent equations, dashes denote differentiation.
with respect to $x$.

The separation of the boundary layer from the surface of the body occurs at a point where $\partial u/\partial y$ becomes zero. Therefore, it will be seen from (6) and (8) that the point of separation is determined by

$$\lambda = -\frac{12}{29},$$

as in the two-dimensional case.

Making use of (8) we easily find that

$$\int u^2 dy = \frac{U \delta}{120} (84 + \lambda),$$

$$\int u d y = \frac{U \delta}{120} \left( 734 + \frac{71}{6} \lambda + \frac{5}{36} \lambda^2 \right),$$

and the substitution of these values in equation (5) gives

$$\frac{dz}{dx} = \frac{1}{U f(\lambda)} - \left( \frac{1}{r} \frac{dr}{dx} U \right) \frac{1}{U f'(\lambda)} + z^2 U'' g(\lambda),$$

where

$$z^* = \frac{\delta^2}{\nu},$$

and

$$f(\lambda) = \frac{7257.6 - 1336.32\lambda + 37.92\lambda^2 + 0.8\lambda^3}{213.12 - 5.76\lambda - \lambda^2},$$

$$f'(\lambda) = \frac{426.24\lambda - 3.84\lambda^2 - 0.4\lambda^3}{213.12 - 5.76\lambda - \lambda^2},$$

$$g(\lambda) = \frac{3.84 + 0.8\lambda}{213.12 - 5.76\lambda - \lambda^2}.$$  \hspace{1cm} (13)

This is the differential equation for determining the thickness $\delta$ of the laminar boundary layer on a body of revolution.

§ 5. At the forward stagnation point, i.e., at the origin $x=0$, the velocity $U$ vanishes and therefore it will be seen from equation (11) that unless the following quantity:

$$f(\lambda) - \lim_{x \to 0} \left( \frac{1}{r} \frac{dr}{dx} U \right) f'(\lambda)$$

becomes also zero at that point, no integral of (11) exists having a finite value at the origin. Thus, when $x = 0$,

$$f(\lambda) - \lim_{x \to 0} \left( \frac{1}{r} \frac{dr}{dx} U \right) f'(\lambda_0) = 0,$$  \hspace{1cm} (14)
where $\lambda_0$ is the value of $\lambda$ at the origin and is determined by this equation.

Now, the practically important bodies of revolution such as spheres and airship-shaped bodies have a blunt nose and in those cases the neighbourhood of the forward stagnation point at the nose can be generally approximated by a portion of the surface of a sphere with the longitudinal radius of curvature at the nose ($R_0$, say) as its radius. Thus, in the vicinity of the forward stagnation point we may write approximately

$$ x = R_0 \varphi, \quad r = R_0 \varphi, \quad U = c \varphi, $$

c being a constant, so that

$$ \lim_{x \to 0} \left( \frac{1}{r} \frac{d}{dr} \frac{U}{U'} \right) = 1. $$

Thus, remembering that the fundamental momentum integral equation adopted in the present paper can legitimately be used, as mentioned already, to describe the boundary layer on a body of revolution even in the vicinity of the forward stagnation point at the nose of the body when it has a blunt nose, if we confine ourselves to such a practically important blunt-nosed body of revolution, the equation for determining $\lambda$, i.e., the value of $\lambda = U'z_*$ at the origin, becomes

$$ f(\lambda_0) - f^*(\lambda_0) = 0, $$

or

$$ 7257.6 - 1762.56 \lambda_0 + 41.76 \lambda_0^2 + 1.2 \lambda_0^3 = 0. \quad \text{(15)} $$

It has been proved in the previous paper \(^{(1)}\) that this cubic equation has one negative root and two positive roots, 4.71601 and 21.14. Of these three roots, the negative one must be rejected owing to the fact that at the forward stagnation point $U' > 0$ so that $\lambda_0$ must necessarily be positive, while the greater one of the positive roots, i.e., 21.14, also cannot be adopted. Hence, the appropriate root of (15) is $\lambda_0 = 4.71601$ and the required integral of the differential equation (11) is the one defined by

$$ \lambda = U'z_* = 4.71601, \quad \text{at } x = 0. \quad \text{(16)} $$

Some of the values of the functions $f(\lambda), f^*(\lambda)$ and $g(\lambda)$ for values of $\lambda$ ranging from 5 to $-12$ have been tabulated in Table I in the previous paper, and use has been made of them in the present paper in performing the graphical integration of the differential equation (11).

\(^{(1)}\) S. Tomotika, loc. cit.
Next, we determine the value of \( \frac{dz^*}{dx} \) at \( x=0 \), i.e., \( (dz^*/dx)_e \). From (11) we have
\[
\left( \frac{dz^*}{dx} \right)_e = \lim_{x \to 0} \left\{ f(\lambda) \left( \frac{1}{r} \frac{dr}{dx} \frac{U'}{U'} \right) f^*(\lambda) \right\} + \left( \frac{dz^*}{dx} \right)_0 g(\lambda).
\]
But, in the case of a blunt-nosed body of revolution,
\[
\left( \frac{dz^*}{dx} \right)_e = \lim_{x \to 0} \left\{ f(\lambda) \left( \frac{1}{r} \frac{dr}{dx} \frac{U'}{U'} \right) f^*(\lambda) \right\} \left[ \frac{d(f-f^*)}{d\lambda} \frac{\lambda' + \frac{1}{2} \frac{U''}{U'} f^*}{U'} \right]_{x=0}.
\]
Inserting this in (17) and taking (16) into account, the value of \( (dz^*/dx)_e \) can be obtained by simple algebraic calculations. We thus find
\[
\left( \frac{dz^*}{dx} \right)_e = -3.2 \left( \frac{U''}{U'^2} \right)_e.
\]

With (16) and (18) as the initial conditions, the graphical solution of (11) can be conveniently carried out by the method described in the previous paper.

§ 6. In the particular case of a sphere, with which we are specially concerned in the present paper, if we introduce the central angle \( \theta \), we have
\[
x=a\theta, \; \; r=a \sin \theta,
\]
a being the radius of the sphere.

Then, if we write
\[
z = \frac{U_0}{a} z^* = \frac{U_0}{a^2} z^*, \quad (19)
\]
where \( U_0 \) is the velocity of the undisturbed stream, \( z \) is the non-dimensional function of \( \theta \) only and we have, from (11), the differential equation for \( z \) in the form:
\[
\frac{dz}{d\theta} = \frac{U_0}{U} f(\lambda) - \cos \theta \frac{U_0}{\sin \theta} \frac{f^2(\lambda)}{dU} + \frac{1}{U_0} \frac{d^2U}{d\theta^2} g(\lambda).
\]
One of the initial conditions is, by (16) and (19),
\[
\lambda = \frac{1}{U_0} \frac{dU}{d\theta} = 4.71601, \; \; \text{at} \; \theta = 0.
\]

In the vicinity of the forward stagnation point of a sphere, the velo-
city distribution outside the boundary layer is represented, as proved experimentally(1), by the well-known theoretical formula \( U = \frac{3}{2} U_0 \sin \theta \), irrespective of the values of the REYNOLDS number of the stream. Thus, we can always put

\[
\frac{dU}{d\theta} \bigg|_{\theta = \alpha} = \frac{3}{2} U,
\]

and therefore the above condition can also be written in the form:

\[
z = \frac{2}{3} \times 4.71601 = 3.14401, \quad \text{at } \theta = 0. \tag{22}
\]

The other condition is

\[
\frac{dz}{d\theta} = 0, \quad \text{at } \theta = 0, \tag{23}
\]

which follows immediately from (18), since in the case of sphere \( U'' \) is equal to zero at the forward stagnation point.

Our next problem is therefore to integrate the above differential equation (20) subject to these two initial conditions (22) and (23), and for this purpose we have to know the expression for \( U \), namely the velocity distribution over the surface of a sphere.

In the previous paper, use has been made of the theoretical velocity distribution \( U = \frac{3}{2} U_0 \sin \theta \) as well as an experimentally determined distribution, and some characteristic quantities for the laminar boundary layer on a sphere have been calculated in both cases(2). However, no comparison of the calculated results with observations has been made there, because at that time detailed experimental evidence concerning the boundary layer of a sphere was not available.

Recently A. FAGE(3) has carried out detailed measurements on the distributions of normal pressure and skin friction over the surface of a sphere, for the wide range of REYNOLDS number which included the so-called critical range, and thus he has discussed the influence of REYNOLDS number and turbulence in the free stream on the transition from laminar to turbulent flow in the boundary layer of a sphere.

In view of the importance of the problem, therefore, similar calculations to those in the previous paper have been repeated, using this time an actual velocity distribution on a sphere which has been

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(1) See, e.g., FAGE’S paper, loc. cit.
(2) On re-examining the calculations of the previous paper, it has recently been found that in Fig. 3 there the curve of \( z = \frac{1}{\alpha} \delta / U_0 \) for the experimental case was drawn unfortunately somewhat too steep for the range \( \theta \geq 70^\circ \), and therefore the values of \( \delta \) for \( \theta \geq 70^\circ \) were calculated a little too large in that case.
(3) A. FAGE, loc. cit.
obtained from the distribution of normal pressure determined experimentally by Fage in a case when the Reynolds number of the stream was just within the critical range so that the flow in the boundary layer was everywhere laminar up to the point of separation. The results of the calculations will be described in the following lines.

IV. Calculations using Experimentally Determined Velocity Distribution.

§ 7. The diameter, $D$, of the sphere used in Fage's experiments was 6 inches, and the observations were made in three wind tunnels at several wind speeds, $U_0$, ranging from 35 ft. per sec. to 135 ft. per sec. so that the range of the Reynolds number $DU_0/\nu$ covered, 110,000 to 424,500, included the critical range, 140,000 to 330,000.

![Graph showing pressure difference and angle]

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Fig. 1.
In one case when $U_0 = 50$ ft. per sec. and therefore the value of $DU_0/\nu$, being 157,200, was just within the critical range, Fage observed that the flow in the boundary layer was everywhere laminar up to the point of separation, which was found to occur at $\theta = 83^\circ$. In what follows this case will be subjected to mathematical analysis.

The observed values of the pressure coefficient $(p - p_\infty)/\frac{1}{2} \rho U_0^2$, where $p$ is the pressure on the sphere and $p_\infty$ the pressure in the undisturbed free stream, are reproduced in Fig. 1 for the range $0^\circ \leq \theta \leq 85^\circ$. It will be seen that the minimum pressure occurs at $\theta = 74^\circ$ approximately.

Using these values, the corresponding actual velocity distribution on the sphere has been found, with the aid of Bernoulli’s theorem:

$$\left(\frac{U}{U_0}\right)^2 = 1 - \frac{p - p_\infty}{\frac{1}{2} \rho U_0^2}$$

In the range $0^\circ \leq \theta \leq 85^\circ$, this can be expressed approximately by the formula:

$$\frac{U}{U_0} = 1.5 \theta - 0.43707 \theta^3 + 0.148097 \theta^5 - 0.042329 \theta^7,$$  \hspace{1cm} (24)

the curve of which is shown in Fig. 2.
Also, the pressure distribution corresponding with this velocity distribution has been calculated and is shown in Fig. 1 by a full line. It will be seen that in the case before us the above formula (24) represents, with sufficient approximation, the actual velocity distribution on the sphere for the range $0^\circ \leq \theta \leq 85^\circ$.

![Graph showing the pressure distribution](image)

The values of the three quantities:

\[
\frac{U}{U_0}, \quad \frac{1}{U_0} \frac{dU}{d\theta}, \quad \frac{1}{U_0} \frac{d^2U}{d\theta^2},
\]

occurring in the differential equation (20), have been calculated with the aid of the formula (24), and making use of them as well as the values of the functions $f(\lambda)$, $f^*(\lambda)$ and $g(\lambda)$, the graphical integration of the differential equation (20) has been carried out by the method described in the previous paper, subject to the initial conditions (22).
and (23). The result is shown in Fig. 3. It will be seen that the point of separation is approximately at \( \theta = 84^\circ \). This value should be compared with the observed one, \( 83^\circ \), and we find that the agreement between the calculation and the observation is quite satisfactory.

Next, the Reynolds number of the boundary layer, defined by \( \delta U/v \), is expressed, in general, in the form:

\[
\frac{\delta U}{v} = \frac{U}{U_0} \sqrt{\frac{z}{2}} \sqrt{\frac{D U_0}{v}},
\]

so that

\[
\frac{\delta U}{v} \sqrt{\frac{D U_0}{v}} = \frac{U}{U_0} \sqrt{\frac{z}{2}}.
\]

The curve of \((\delta U/v)/\sqrt{DU_0/v}\) is shown in Fig. 4. With the aid of this curve we can calculate, if necessary, the values of \( \delta U/v \) for any value of \( \theta \), since in the present case the value of \( DU_0/v \) is known to be 157,200.

![Graph](image)

Fig. 4.

Further, in order to see the way in which the non-dimensional
quantity \( \lambda \), i.e.,

\[
\lambda = \frac{1}{U_0} \frac{dU}{d\theta} z,
\]

changes with \( \theta \), the curve of \( \lambda \) is constructed in Fig. 5.

\[\text{\includegraphics[width=\textwidth]{fig5.png}}\]

**V. Comparison of the Calculated Distribution of Laminar Skin Friction with Fage's Observations**

§ 8. We now proceed to the calculation of the distribution of the intensity of laminar skin friction \( \tau_0 \) over the surface of the sphere(1). We have

\[
\tau_0 = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0},
\]

and therefore, by (6) and (8),

\[
\tau_0 = \frac{\mu U}{6\alpha} (12 + \lambda),
\]

(1) In Fage's paper (loc. cit.), the intensity of skin friction in general has been denoted by \( f \).
from which it follows, with the aid of (19),
\[
\frac{\tau_0}{\frac{1}{2} \rho U_0^2} = \frac{1}{3} \frac{1}{\sqrt{U_0/\nu}} \frac{U}{U_0} \frac{1}{y} (12 + \lambda).
\]

Making use of the known values of $z$, $\lambda$, $U_0/U_0$ and $DU_0/\nu$, the values of $\frac{\tau_0}{\frac{1}{2} \rho U_0^2}$ have been calculated for various values of $\theta$ ranging from $0^\circ$ to $84^\circ$, the latter corresponding with the theoretical point of separation. The distribution of the intensity of laminar skin friction $\tau_0$ thus calculated is shown in Fig. 6 by a full line. In this figure, Fage's experimental values of $\frac{\tau_0}{\frac{1}{2} \rho U_0^2}$ are also shown by $\times$ for comparison, which have been determined using a Stanton surface tube.

These observed points have been adjusted so as to fit the calculated curve near the point $\theta = 60^\circ$, because the observed points are relatively dense in the neighbourhood of this point. In other words, we have used the re-estimated value of the distance from the surface, $\bar{y}$, to which the speed, $u$, deduced from the velocity head at the mouth of the surface tube, must be related to obtain the intensity of skin friction $\tau_0$ by the relation $\tau_0 = \mu(u/\bar{y})$.

It will be seen that the agreement between the calculated distribution of laminar skin friction and the observations is quite satisfactory. In the present case, the flow in the boundary layer is everywhere

(1) It may be remarked that even if the observed points were adjusted so as to fit the calculated curve at $\theta = 55^\circ$, where a single observation only has been taken, only slight unappreciable modifications would be necessary for Fig. 6.
laminar and the layer separates from the surface of the sphere before the transition from laminar to turbulent flow can occur.

The fact that the calculated results agree quite satisfactorily with the observations seems to indicate that the momentum integral equation (4) for the laminar boundary layer on a body of revolution, together with the assumed quartic form for the velocity distribution in the layer, can be used, with sufficient approximation, to describe the laminar boundary layer on the surface of a sphere.

§ 9. On observing Fig. 6, it will be found that there are small deviations\(^{(1)}\) of the observed points from the calculated curve in the range from \(\theta =65^\circ\) to \(75^\circ\), which includes the observed point of minimum pressure, \(\theta =74^\circ\), for the present case.

These deviations seem to indicate that some kinds of accidental disturbances had occurred near the point of minimum pressure and thus the flow in the boundary layer, becoming somewhat irregular, had departed from the purely laminar state. The fact that when \(\theta >75^\circ\) the experimental points fall again on the calculated curve shows however that such disturbances had been unable to grow up so as to make the flow turbulent, but, on the contrary, they had soon decayed down and the motion had become again purely laminar, since the Reynolds number was not too large in the case under discussion.

This reminds us of the well-known similar phenomenon in the case of flow through a pipe of circular section in the Reynolds experiments. It is well known that when the Reynolds number is below a certain critical value, initial disturbances, if any, do not grow up so as to make the flow turbulent, but, on the contrary, they are soon obliterated and the flow becomes purely laminar.

Further, it is well established in the case of the Reynolds experiments that if the Reynolds number is greater than the critical value, the flow becomes sensitive to initial or accidental small disturbances, and the motion becomes ultimately turbulent.

A similar phenomenon may perhaps be expected to occur in the case of the flow in the boundary layer of a sphere. Thus, it may be expected that when the Reynolds number of the stream takes greater

\(^{(1)}\) It should be noticed that these deviations cannot be considered to be due to the inaccuracy of the formula (24) for \(CU_0\), because it can represent, with sufficiently good approximation, the observed pressure distribution, especially for the range from \(\theta =0^\circ\) to \(75^\circ\), as shown in Fig. 2.
values than a certain critical value, accidental disturbances, which are likely to be originated near the point of minimum pressure, would grow up more and more, and the flow in the boundary layer would become turbulent.

The present writers have lately carried out various calculations similar to those in the present paper for two other cases of greater Reynolds numbers, using, as before, the actual pressure distributions measured by Fage, and it has been ascertained that the expectation above mentioned is not erroneous. Full description of the results of our investigations will be given shortly elsewhere.

VI. Summary.

§10. The momentum integral equation for the boundary layer on the surface of a body of revolution is applied to the case of the laminar boundary layer of a sphere placed in a uniform stream. The quartic form is assumed for the velocity distribution in the layer, and for the velocity distribution just outside the boundary layer, use is made of an actual distribution which has been obtained from the pressure distribution measured by Fage in a case when the Reynolds number of the stream was just within the critical range so that, according to Fage, the flow in the boundary layer was laminar up to the point of separation.

The differential equation for determining the thickness of the laminar boundary layer is solved by the method of graphical integration, and thus various characteristic quantities for the layer are calculated.

It is found that the calculated point of separation is in good accordance with Fage's observation, and that the agreement between the calculated distribution of laminar skin friction and the observation is also quite satisfactory.

Further, some discussions are made on small deviations which have been found near the point of minimum pressure between the calculated distribution of laminar skin friction and the observed points.

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