On linear Differential Equation which admits a linear Differential Transformation.

By Sōichi Kareya.

(Road April 2, 1938).

1. A linear homogeneous differential expression of the form

\[ p_0(x)y^{(m)}(x) + p_1(x)y^{(m-1)}(x) + \cdots + p_{m-1}(x)y'(x) + p_m(x)y(x) \quad (1) \]

is usually denoted by a symbol, for example, \( P(y) \) or \( Py \). It may be observed as a transformation or an operation applied to a function \( y(x) \). And this operation has linear character, namely

\[ P\{ay(x) + bz(x)\} = aP\{y(x)\} + bP\{z(x)\}. \quad (2) \]

Equating \( P(y) \) to 0, we now consider a differential equation

\[ P(y) = p_0y^{(m)} + p_1y^{(m-1)} + \cdots + p_my = 0. \quad (3) \]

If for any solution \( y \) of (3), the function

\[ Q'y = q_0y^{(m)} + q_1y^{(m-1)} + \cdots + q_my, \quad (4) \]

which is transformed from \( y \) by \( Q \), is also a solution of (3), then we shall say that the equation (3) admits the transformation (4).

We are going to show some analytical relations between these two expressions \( P(y) \) and \( Q(y) \). Especially it is proved that the equation (3) can be perfectly integrated by solving the equations of the form

\[ Q(y) - \lambda y = f(x). \quad (5) \]

As a preliminary, we shall show a property of linear operations, which is essentially equivalent to a well known theorem of Frobenius.

2. Consider a general linear operation \( L(y) \), whose field of variable functions being suitably limited according to the necessity. We apply the operation to each of \( m \) arbitrary functions

\[ y_1(x), \ y_2(x), \ \ldots, \ y_m(x) \quad (6) \]

and we put

\[ L(y_k) = a_{1k}y_1 + a_{2k}y_2 + \cdots + a_{mk}y_m - t_k, \quad k = 1, 2, \ldots, m \quad (7) \]

where \( a_{ki} \) are given constants.

Corresponding to a polynomial

\[ A(z) = a_0 + a_1z + a_2z^2 + \cdots \quad (8) \]

we are accustomed to consider a linear operation \( A(L)y \) which means

\[ A(L)y = a_0y + a_1L(y) + a_2LL(y) + \cdots \quad (9) \]

And it is well known that the fundamental calculations of polynomials
can be similarly carried to the symbol $A(L)$ under the proper interpretations.

According to this notations, we can write the relation (7) in the following form
\[ a_{1k}y_1 + a_{2k}y_2 + \ldots + (a_{ik} - L)y_k + \ldots + a_{mk}y_m = t_k, \quad k = 1, 2, \ldots, m. \] (10)

Consider the determinant
\[ \Delta(L) = |a_{it} - \delta_{it}L| \quad (\delta_{it} \text{ is Kronecker's symbol}) \] (11)
as a polynomial of $L$, and denote the algebraic minors of it by $\Delta_{it}(L)$.
If $D(L)$ is the greatest common divisor of the $m^s$ minors $\Delta_{it}(L)$, then $D(L)$ is also a factor of $\Delta(L)$. Hence we can put
\[ \frac{\Delta(L)}{D(L)} = \Delta_0(L), \quad \frac{\Delta_{it}(L)}{D(L)} = \Delta_{it}(L), \] (12)
where $m^s$ functions $\Delta_{it}(L)$ have no more common divisor other than a constant.

If we operate $\Delta_{it}(L)$ to the both sides of (10) and add them together with respect to $k$, we get
\[ \Delta(L)y_1 + \Delta_{it}(L)t_1 + \ldots + \Delta_{it}(L)t_m, \quad l = 1, 2, \ldots, m \] (13)
since the calculation is performed as if (10) were a system of ordinary equations. But, if we calculate the result of operating a general polynomial $M(L)$ to $y_1$, we shall get in general, by successive substitutions of (7), the following form
\[ M(L)y_1 = c_1y_1 + \ldots + c_my_m + N_0(L)t_1 + \ldots + F_m(L)t_m, \quad l = 1, 2, \ldots, m. \] (14)

The vanishing of the terms of $y_1, \ldots, y_m$ in (13) is a special nature for the polynomial $\Delta(L)$.

Let us now inquire about the polynomial $M(L)$ of the lowest degree for which the terms of $y_1, \ldots, y_m$ vanish, namely
\[ M(L)y_1 = N_0(L)t_1 + \ldots + N_m(L)t_m, \quad l = 1, 2, \ldots, m. \] (15)

Operating suitable polynomials of $L$ to (13) and (15) respectively and adding them, we see that the greatest common divisor $E(L)$ of $\Delta(L)$ and $M(L)$ must have the same nature. Hence $M(L)$ of the lowest degree must be equal to $E(L)$ except a constant factor. In other words, $\Delta(L)$ is divisible by $M(L)$ of the lowest degree, so that
\[ \Delta(L) = M(L)R(L). \] (16)
Thus we get from (13) and (15) that
\begin{equation}
0 = [\Delta_u(L) - N_u(L)R(L)]t_1 + \cdots + [\Delta_m(L) - N_m(L)R(L)]t_m,
\end{equation}
\begin{equation}
l = 1, 2, \ldots, m.
\end{equation}

If, for example, the coefficient of \( t_1 \) in (17) is not identically zero and is of the degree \( v \) not less than the others, then we must obtain the equation of the form
\begin{equation}
S(L)y_1 = 0
\end{equation}
putting \( y_2 = \cdots = y_m = 0 \) in (17). Since, in this case, \( t_2, \ldots, t_m \) are constants and \( t_1 \) is linear with respect to \( L \), the polynomial \( S(L) \) must be of the degree \( v + 1 \). This contradicts the fact that the equation (18), which is identical for \( y_1 \) and \( L \), implies \( S(L) = 0 \). Hence all the coefficients of (17) must be identically zero. And therefore \( R(L) \) must be a common factor of all \( \Delta_{kl}(L) \), so that it must be a constant. Hence the required polynomial of the lowest degree is equal to \( \Delta(L) \) except an arbitrary constant factor.

That a general polynomial of the same nature is a multiple of \( \Delta(L) \) can be proved by the same method as above. The uniqueness of the representation of (14) can be also proved.

If we take a particular \( L \) for which \( t_1 = \cdots = t_m = 0 \) and take \( \{y_l\} \) which are linearly independent, then \( M(L)y_l = 0 \) only when \( M \) satisfies (15) for general operation. So we get the following theorem, which is essentially identical with the theorem of Frobenius on the equation of matrix.

A linear operation \( L \) for which
\begin{equation}
L(y_l) = a_{k1}y_1 + \cdots + a_{km}y_m, \quad k = 1, 2, \ldots, m
\end{equation}
satisfies
\begin{equation}
\Delta(L)y_l = 0, \quad l = 1, 2, \ldots, m.
\end{equation}
Other polynomial of \( L \) having the same nature as \( \Delta(L) \) is a multiple of \( \Delta(L) \). Especially we must have
\begin{equation}
\Delta(L)y_l = 0, \quad l = 1, 2, \ldots, m.
\end{equation}

3. We can apply the above result to the main subject of the paper, taking the equation (3) which admits the transformation (4).

Designate a fundamental system of solutions of (3) by
\begin{equation}
y_1(x), \quad y_2(x), \quad \ldots, \quad y_m(x).
\end{equation}
Since, by assumption, \( Q(y_1) \) is also a solution of (3), we must have a relation of the form
\begin{equation}
Q(y_k) = a_{k1}y_1 + \cdots + a_{km}y_m, \quad k = 1, 2, \ldots, m.
\end{equation}
Hence, by the preceding theorem, we can take a polynomial

\[ M(z) = a_0z^r + a_1z^{r-1} + \cdots + a_{r-1}z + a_r \] (24)

for which

\[ M(Q)y_l = 0, \quad l = 1, 2, \ldots, m. \] (25)

The general form of \( M(z) \) is a multiple of \( \Delta(z) \) before mentioned.

Thus we see that every solution of (3) must also be a solution of a differential equation of the form

\[ M(Q)y = (a_0Q^r + a_1Q^{r-1} + \cdots + a_{r-1}Q + a_r)y = 0, \] (26)

where \( \alpha_0, \alpha_1, \ldots, \alpha_r \) are some constants and \( r \) may be any multiple of the degree of \( \Delta(z) \). We may take \( M(z) = \Delta(z) \) in the sequel. Then \( r \) is a factor of \( m \).

4. We are now to insert a few remarks most of which are well known. (Heffter, J. f. Math. 116, (1896)).

Let \( A(y) \) and \( B(y) \) are two linear differential expressions and let \( u_1(x), \ldots, u_p(x) \) and \( v_1(x), \ldots, v_q(x) \) (27)
be the fundamental solutions respectively of the differential equations

\[ A(y) = 0 \quad \text{and} \quad B(y) = 0. \] (28)

If the two equations (28) have some common solutions not identically zero, then the set of all those common solutions forms the set of all solutions of a linear differential equation

\[ C(y) = 0. \] (29)

The form of \( C(y) \) can be obtained from \( A(y) \) and \( B(y) \) by a differential algorism, well known in the theory of differential equations. \( A(y) \) and \( B(y) \) can be put into the form

\[ A(y) = A_1C(y), \quad B(y) = B_1C(y), \] (30)

where the two equations

\[ A_1(y) = 0 \quad \text{and} \quad B_1(y) = 0 \] (31)

have no more common solution other than \( y = 0 \). Especially if all the solutions of \( B(y) = 0 \) are also the solutions of \( A(y) = 0 \), then we can put

\[ A(y) = A_1B(y). \] (32)

And, as a fundamental system of solutions of \( A(y) = 0 \), we may take

\[ v_1(x), \ldots, v_q(x), w_{q+1}(x), \ldots, w_p(x) \] (33)

where

\[ B(w_{q+1}), \ldots, B(w_p) \] (34)

forms a fundamental system of solutions of \( A_1(y) = 0 \).

Since the converse of (32) is evidently held, we see that the necessary and sufficient condition that all the solutions of \( B(y) = 0 \) should
be contained among the solutions of \( A(y) = 0 \) is the existence of \( A_1(y) \) of (32), namely that \( B(y) \) should be a right-hand factor of \( A(y) \). The existence or non-existence of such \( A_1(y) \) is of course determined by the method of indeterminate coefficients.

If \( P(y) = 0 \) admits a transformation \( Q(y) \), then \( PQ(y) = 0 \) whenever \( P(y) = 0 \), and conversely. Hence the necessary and sufficient condition that \( P(y) = 0 \) should admit the transformation \( Q(y) \) is the existence of a differential expression \( R(y) \) for which

\[
PQ(y) = RP(y).
\]

(35)

So we can determine, by the method of indeterminate coefficients, whether or not the equation \( P(y) = 0 \) admits the transformation \( Q(y) \).

In the case of the preceding paragraph, all the solutions of (3) are contained among the solutions of (26), so that there must exist an expression \( N(y) \) for which

\[
M(y)Q(y) = NP(y).
\]

(36)

The coefficients of \( M(y) \) are known to be constants and the order \( r \) of \( M(y) \) is known to be a factor of \( m \). We can determine the forms of such \( M(y) \) and \( N(y) \), by the method of indeterminate coefficients.

Differentiating \( A(y) \) and \( B(y) \) respectively \( q \) and \( p \) times, and eliminating \( y, y', \ldots, y^{p+q} \), we get a relation of the form

\[
UA(y) - VB(y) = 0,
\]

(37)

where \( U(y) \) and \( V(y) \) are of the orders \( q \) and \( p \) respectively.

If we assume that the both equations (28) have no common solutions, then \( U(y) \) and \( V(y) \) are of possibly smallest order and all other expressions \( \bar{U}(y) \) and \( \bar{V}(y) \) (not identically zero) having the same nature

\[
U(y) - \bar{V}B(y) = 0
\]

(38)

are of orders not less than \( q \) and \( p \) respectively. For the differential equation

\[
UA(y) = \bar{V}B(y) = 0
\]

(39)

satisfies \( p+q \) functions (27) which are linearly independent by the assumption of no common solutions.

Under the same assumption, differentiating \( A(y) \) and \( B(y) \) respectively \( q-1 \) and \( p-1 \) times and solving with respect to \( y, y', \ldots, y^{p+q-2} \), we get a relation of the form

\[
SA(y) - TB(y) = y,
\]

(40)

where \( S(y) \) and \( T(y) \) are at most of the orders \( q-1 \) and \( p-1 \) respectively. For if the solution were impossible or indeterminate we should have
a relation of the form (38) in which the orders of \( U(y) \) and \( \bar{V}(y) \) are at most \( q-1 \) and \( p-1 \) respectively.

\( S(y) \) and \( T(y) \) of (40) are determinate. For if there were another pair of them, we should have the same contradictory result as above, by subtracting those two relations.

In the general case where (28) have some common solutions, we take \( A_1(y) \) and \( B_1(y) \) of (30) and the corresponding \( S_1(y) \) and \( T_1(y) \), then we get the relation

\[
S_1A(y) - T_1B(y) = C(y),
\]

where the orders of \( S_1(y) \) and \( T_1(y) \) are less than those of \( B_1(y) \) and \( A_1(y) \) respectively. Such \( S_1(y) \) and \( T_1(y) \) are determinate.

From this, we can conclude that \( U(y) \) and \( \bar{V}(y) \) of (38) can be put into the form

\[
U(y) = WU(y), \quad \bar{V}(y) = W\bar{V}(y).
\]

For if the highest right-hand common factor of \( U(y) \) and \( U(y) \) is \( C(y) \), we get a relation of the form

\[
S\bar{U}(y) - T\bar{U}(y) = C(y).
\]

And hence, operating \( S(y) \) and \( T(y) \) to (37) and (38) respectively and subtracting, we get a relation of the form

\[
C\bar{U}(y) - DB(y) = 0.
\]

Hence the order of \( C(y) \) can not be less than \( q \), so that \( C(y) \) may be at most different from \( U(y) \) only by a left-hand factor not containing \( y \). Thus \( U(y) \) and consequently \( \bar{V}(y) \) can be put into the form (42).

5. We now return to the equation (26). The form of this equation, even in its lowest order, can be algebraically found without knowing the actual relation (23), as it is remarked in the preceding paragraph.

Decomposing \( M(z) \) into its linear factors, the equation (26) can be put into the form

\[
II(Q - \lambda_\mu)^m y = 0.
\]

Any two of the equations

\[
(Q - \lambda_\mu)^m y = 0.
\]

corresponding to different \( \mu \)'s can have no common solutions other than \( y = 0 \). For, by virtue of the relative primeness of \( (Q - \lambda_\mu)^m \) and \( (Q - \lambda_\mu)^m \) we can find two polynomials \( S(Q) \) and \( T(Q) \) for which

\[
\lambda_\mu^m y - T(Q)(Q - \lambda_\mu)^m y = y.
\]

Thus the problem of solving (45) is reduced to that of solving the
equations (46) for all different $\mu$'s. The last problem is evidently reduced to the problem of solving successively the equations of the form

$$(Q - \lambda)y = f(x),$$

(48)

where $f(x)$ is a known function.

After knowing all the solutions of (45), we can pick up from them all the solutions of (3) by trials. So we get the following theorem.

If a linear homogeneous differential equation $P(y) = 0$ admits a linear homogeneous differential transformation $Q(y)$, then the equation $P(y) = 0$ can be perfectly integrated by solving the equations of the form

$$(Q - \lambda)y = f(x).$$

(49)

6. One of the interesting particular cases of the preceding discussion is that where the transformation $Q(y)$ is of the first order, namely

$$Q(y) = u(x)y' + v(x)y$$

(50)

In this case, the reduced equation (49) is solved by quadratures. Hence we see that if a linear homogeneous differential equation $P(y) = 0$ admits a linear homogeneous differential transformation $Q(y)$ of the first order, the equation can be solved by quadratures.

In this case, the equation (26) is of the order $r$. Since (26) is satisfied by every solution of (3) whose order $m$ is a multiple of $r$, we must have $m = r$, and the both equations (3) and (26) must be identical.

If in this case, we transform the variables $x$ and $y$ by the relations

$$dx = u(x)\,d\xi, \quad y = \eta \, Exp \int \frac{-v(x)}{u(x)} \, dx,$$

(51)

then we evidently get

$$Q^* (y) = \frac{d^n}{d\xi^n} \eta \, Exp \int \frac{-v(x)}{u(x)} \, dx,$$

(52)

and consequently the equation (26) becomes

$$\alpha_0 \frac{d^n}{d\xi^n} \eta + \alpha_1 \frac{d^{n-1}}{d\xi^{n-1}} \eta + \ldots + \alpha_m \eta = 0.$$ 

(53)

Conversely the equation (3), transformed from (53) by (51), evidently admits the transformation $Q(y)$. For the equation (53) satisfies

$$\frac{d\eta}{d\xi} = Q(y) \, Exp \int \frac{v(x)}{u(x)} \, dx,$$

(54)

whenever it satisfies $\eta = \eta \, Exp \int \frac{v(x)}{u(x)} \, dx$, so that the transformed equation (3) satisfies $Q(y)$ whenever it satisfies $y$. Thus we get the following theorem of Fayet (Comptes rendus, 1937).
The necessary and sufficient condition that a linear homogeneous differential equation \( P(y) = 0 \) could be transformed into a linear equation of constant coefficients by a relation of the form (51) is that the equation \( P(y) = 0 \) should admit a transformation of the form (50).

7. It seems to be interesting to observe the case where the two given expressions \( A(y) \) and \( B(y) \) are mutually admitted, namely the equation \( A(y) = 0 \) admits \( B(y) \) and the equation \( B(y) = 0 \) admits \( A(y) \).

In this case, there exist two expressions \( G(y) \) and \( H(y) \) such that

\[
AB(y) = GA(y), \quad RA(y) = HB(y). \tag{55}
\]

If \( C(y) \) is the highest right-hand common factor of \( A(y) \) and \( B(y) \), mentioned in the paragraph 4, we can put them into the form (30). So we have from (55)

\[
AB_1(y) = GA_1(y), \quad BA_1(y) = HB_1(y). \tag{56}
\]

On the other hand, we can get the relation of the form

\[
U_1A_1(y) - V_1B_1(y) = 0 \tag{57}
\]

corresponding to the equation (37). And then, by (56), \( U_1(y), V_1(y) \) must be right-hand factors of \( B, A \) respectively, as mentioned in the end of the paragraph 4, namely

\[
A(y) = A_2V_1(y), \quad B(y) = B_2U_1(y). \tag{58}
\]

Conversely if the relation (58) holds for some \( A_2(y) \) and \( B_2(y) \), the (55) will hold for

\[
G(y) = A_2U_1(y), \quad H(y) = B_2V_1(y). \tag{59}
\]

Thus we get the theorem

In order that \( A(y) = A_2C(y) \) and \( B(y) = B_2C(y) \) should be mutually admitted, it is necessary and sufficient that \( U_1(y) \) and \( V_1(y) \) which are determined from (57) except a factor independent of \( y \), should be the right-hand factors of \( B(y) \) and \( A(y) \) respectively.

In the case where \( A(y) = 0 \) and \( B(y) = 0 \) have no common solutions, namely \( C(y) \equiv y \), \( A_2(y) \) and \( B_2(y) \) of (58) must be of the order zero. And the required condition can be easily reduced to

\[
BA(y) - AB(y) = 0. \tag{60}
\]

Thus we see that if \( A(y) = 0 \) and \( B(y) = 0 \) have no common solutions, then the necessary and sufficient condition for the mutual admittance is that the operations \( A(y) \) and \( B(y) \) should be permutable.

This last theorem can also be proved simply and directly as follows.

If \( A(y) \) and \( B(y) \) are permutable, namely
then (55) holds good for $H=A$, $G=B$. So the mutual admittance takes place. Conversely if $A(y)=0$ admits $B(y)$, then the equation

$$AB(y)=0$$  \hspace{1cm} (62)

satisfies not only the solutions of $B(y)=0$ but the solutions of $A(y)=0$. Since those two systems of solutions have no common elements, the general solution of (62) consists of the sum of those two kinds of solutions. The same is true for

$$BA(y)=0$$  \hspace{1cm} (63)

if $B(y)=0$ admits $A(y)$. Hence in the case of mutual admittance, the two equations (62) and (63) have the same solutions. Since their leading coefficients are equal, we get the relation (61).

Mathematical Department, Faculty of Science,
Tokyo Imperial University.

(Received April 7, 1936).