Supplementary notes on the central limit theorem of statistical mechanics.

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1. In a recent note(*) I have given a generalization of Darwin-Fowler method applicable to any system of arbitrary energy values. One point remains, however, obscure, namely we have given no exact formulation of the physical idea which permits us to avoid the contributions from the ghost cols in our main formula. Such a formulation will now be communicated in the following lines.

We start with the following lemma: Let \( f(\alpha) \) be the partition function (p. f.) of a (monotone increasing) phase volume \( j(\alpha) \). Then the p. f. of \( \int_0^\infty j(\alpha) d\alpha \) exists and is equal to \( f(\alpha)/\alpha \). The proof is an immediate consequence of the application of the rule of partial integration. Cf. for the following Doetsch, Theorie und Anwendung der Laplace-Transformation.

Next we propose that the correct interpretation of the smoothing out process in the previous paper consists in this: One integrates the total phase volume, gets the asymptotic expression for it, and comes back to the original function by formally differentiating the value thus obtained as such. By this process one can say that the discontinuity of the function has been removed. If we require the further smoothing, that is the continuity of the first derivative of the function, we must integrate the function twice and differentiate the asymptotic expression for it twice. A smoothing out of sufficiently high degree can be obtained thus \( \nu \)-times integration and differentiation.

Combining the result of this consideration with the aforesaid lemma we are entitled to take the asymptotic expression of

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} g(\alpha)[f(\alpha)e^{\alpha}]^{\nu} \frac{d\alpha}{\alpha^{1+\nu}}
\]

for a sufficiently large \( \nu \), and differentiate the expression \( \nu \)-times. As one sees, the contribution to the integral from the parts of the integration contour outside of the unit circle will be as small as we wish by selecting \( \nu \) large enough. Hence it is legitimate to discard the contributions

(*) These proceedings, 19, 1105, (1937).
from ghost cols in order to get a smoothed out formula. The asymptotic expression which is now supplied only from the principal col is

\[ \frac{g(\beta)[f(\beta)e^{i\eta}]^n}{\sqrt{2\pi n\varphi''(\beta)/\varphi(\beta)}} \beta^{1+n} \]

and the extra factor \( \beta^n \) can be removed by differentiating \( n \)-times with respect to \( n \), retaining only the leading terms in \( n \).

We have thus obtained a justification for our conjecture made in the previous note. One notes that this method avoids all difficulties connected with the almost periodic character of the partition function along the vertical line of the complex plane, whose detailed analysis requires presumably almost the same amount of complexities as the direct analysis of approaching infinite incommensurables by rationals with a common denominator (Diophantine problem in a general sense).

That the behaviour of the partition function for large imaginary arguments \( \alpha \sim \beta \pm i\infty \) reflects directly the continuity-discontinuity character of the determining function (phase volume), may be clearly seen in the facts that (1), if the phase volume is classical, i.e. is absolutely continuous, so that \( f(\alpha) = \int_0^\infty f'(\alpha) d\alpha \), then

\[ f(\alpha) \to 0 \text{ as } \alpha \to \beta \pm i\infty \]

by the Riemann lemma; (2), if the phase volume is totally quantized, i.e. is a step function, then \( f(\alpha) \) is almost periodic along the vertical line, so that in particular

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\beta + iy) e^{\epsilon y} dy = \text{salts of } e^{-\beta \epsilon} \text{ at } \epsilon \]

\[ = \begin{cases} (e^{-\beta \epsilon}) \text{ times the weight } w_i \text{ for } \epsilon = \epsilon_i, & \text{as } y \to \infty, \\ 0 & \text{otherwise} \end{cases} \]

by the Hadamard mean value theorem. . . . The last relation may be used as an inversion formula instead of the Mellin-Burkill's, but it is awkward to be dealt with.

We repeat the assertion that the above method alone can give the correct interpretation of any macroscopic quantity to which we apply freely the usual operations of analysis as if they were continuous, or rather regular, in the mathematical sense.

2. Ad the theorem of composition.

The theory of the previous paper will be complete if we have established the theorem of composition in its widest scope. We shall attempt
here such an extension..

We note first that the ordinary definition of Stieltjes integral presupposes the continuity of the first integrand, so that the Faltung-integral itself has no immediate meaning. A meaning can be given if we adopt the notion of generalized Stieltjes integral as given by Hardy, Young and Pollard (Cf. for the following Hobson, Theory of functions of a real variable, vol. 1, p. 546 et seqq.). After this it is necessary that at the accidentally coinciding points of discontinuity the two functions are continuous on the opposite sides. This is assured in the case of Faltung of two phase volumes, if we once for all normalize the phase volumes such as to be continuous to the right (or to the left ad libitum)...

We want to reduce the theorem of composition in the form here adopted to the analogous theorem in the theory of Laplace transformation (Cf. Doetsch loc. cit.), with the aid of the lemma formulated in § 1. Let us introduce the unit function $e(\alpha) = -\alpha$ and denote the formal operation $\int_{-\infty}^{\infty} e^{-\alpha x} \, dx$ by $L$. Then the lemma states that

$$L(j_1*\alpha e) = L(j_1) / \alpha$$

The left hand member of this equation is an ordinary Laplace transformation, so that, admitting the Faltung theorem for this transformation, we have

$$\frac{L(j_1) \cdot L(j_2)}{\alpha^2} = L(j_1 * \alpha e) \cdot L(j_2 * \alpha e) = L[(j_1 * \alpha e) * (j_2 * \alpha e)]$$

$$= L[(j_1 * \alpha e * j_2)] * \alpha = \frac{1}{\alpha^2} L(j_1 * j_2)$$

In this reduction use was made of the associative law of composition.

The last mentioned law, which is quite independent of the Laplace transformation and is based on the validity of the two operations..., partial integration and change of order of integration..., is usually concluded from the corresponding law of the genuine product in the function space of generating functions. (Cf. Doetsch, loc. cit. p. 162)

If we adopt this method in relation with the Laplace transformation, the reasoning would be circulating. But, since in this law the integration interval is every time finite, we may apply instead the Fourier transformation by cutting off the functions for large arguments. This brings the phase volumes into the distribution functions, so that the theorem is reduced to that in the probability theory. (Cf. e.g. Cramér, Random variables and probability distribution, p. 35).
Of course the direct proof of the law may be possible in different
generality according as the different definition of the integral adopted.
To my regret I have become only recently aware of the statements of
the Faltung theorems scattered over diverse mathematical journals. I
mention only the following two: Widder, Trans. Amer. Math. Soc.,
31, 694 (1929) states the theorem without the final modification to the
form in which we have written it here. Hille and Tamarkin, Proc.
Nat. Acad. Sci. U.S.A. 20, 140 (1934) gives it in the desired form
with the detailed convergence criterion, but without proof.

3. Quantal statistics.

The foregoing considerations are based on the classical Boltzmann
statistics, or that of localized systems in the terminology of Fowler.
We shall outline the well known results in quantal statistics, firstly in
order to show how the present method can afford all required fairly
smoothly and secondly to provide a formula necessary for a more
specific problem we intend to treat in a later section.....

The original suggestion of Bose of the trickery change of the
combinatorial rules cannot be maintained nowadays. The rational way
of treating the system of a number of identical particles is that in
which the particle number is regarded as an observable (second quanti-
zation). In the absence of interaction between the particles or in the
case of validity of Hartree approximation we say about the number of
particles in a certain state, but not about the behaviour of the indi-
vidual particles.

We shall neglect the annihilation and creation of particles, so that
the total number \( n \) of particles is a constant of motion. A transitive
or ergodic ensemble is specified not only by its energy \( \varepsilon \) but also by
its particle number \( n \). We now introduce the generalized phase volume
\( f(\varepsilon, n) \) as the number of states of the system having the energy \( \leq \varepsilon \)
and the particle number \( \leq n \).

In the composition of two systems with phase volumes \( f_1 \) and \( f_2 \),
the particle number as well as the energy goes over into that of the
composed system additively:

\[
\begin{align*}
  n &= n_1 + n_2, \\
  \varepsilon &= \varepsilon_1 + \varepsilon_2
\end{align*}
\]

and the phase volume of the latter is given by the two-dimensional
Faltung
Here the integral is to be taken in the sense of Lebesgue-Stieltjes. For the physical purposes more general regions of integration other than fundamental rectangles are unnecessary. Although the integral with respect to the particle number always reduces to a series, we have preferred this rather sophisticated representation in order to be able to apply the formula without formal change to other additive constants of motion, such as momentum, angular momentum, etc.

The Faltung theorem can be extended without difficulty to the two or higher dimensional case, so that the partition function defined by the two-dimensional Laplace transform

\[ f(\alpha, \xi) = \int_0^\infty \int_0^\infty e^{-(\alpha x + \xi n)} \, dj(\varepsilon, n) \]

is multiplicative in the process of composition:

\[ f(\alpha, \xi) = f_1(\alpha, \xi) \cdot f_2(\alpha, \xi). \]

The central limit theorem can likewise be formulated and proved for the composition of a large number of similar systems (Cf. the analogous problem in the probability theory, see e.g. Cramér, loc. cit.) But the problem we have now in view is slightly different from that hitherto considered. We consider the infinity of quantum states for a single particle problem as independent oscillators, each having the energy \( \varepsilon_i \) and the weight 1. The partition function for each single oscillator is particularly simple:

\[ f_1(\alpha, \xi) = \sum_{n=0,1,2,3,\ldots} e^{-(\alpha \varepsilon_i + \xi n)} \]

\[ = 1 + e^{-(\alpha \varepsilon_i + \xi)} \quad \text{(Fermi, } -\infty < \xi < \infty) \]

\[ = (1 - e^{-(\alpha \varepsilon_i + \xi)})^{-1} \quad \text{(Bose, } 0 < \xi < \infty) \]

An extension of the Faltung theorem for infinite factors gives us for the partition function of the total system of the infinite oscillators the expression

\[ f(\alpha, \xi) = \Pi i f_1(\alpha, \xi) \]

The convergence of the infinite product is secured if the corresponding classical partition function exists. To include the case of a continuum of oscillators, we have to set

\[ \log f(\alpha, \xi) = \int_0^\infty \pm \log (1 \pm e^{-(\alpha x + \xi)}) \cdot dj_i(x) \]
where \( j \) is the phase volume of the single particle problem.

One sees in the above derivation the particular merit of Darwin-Fowler method as compared with the more usual Gibbsian method (vid. e.g. Sommerfeld, ZS. f. Phys., 47, 1, 1928). It exhibits clearly the nature of the product form for the partition function.

Since \( n \) takes only integer values, \( f(\alpha, \xi) \) is in reality a power series in \( e^{-\xi} \):

\[
f(\alpha, \xi) = \sum_{n=0}^{\infty} f^{(n)}(\alpha) e^{-n\xi}
\]

The coefficient \( f^{(n)}(\alpha) \) is the partition function for the system of exactly \( n \) particles.

A method of calculating the partition function \( f^{(n)} \) is this: Supposing \( \mathcal{R}(\xi) > 0 \), we can develop thus:

\[
\pm \log (1 \pm e^{-(n+\sigma)}\xi) = \sum_{n=1}^{\infty} (\mp)^{n-1} e^{-n(n+\sigma)\xi}/n \cdot \{\text{upper sign: Fermi} \}
\]

\[
\pm \log (1 \pm e^{-(n+\sigma)}\xi) = \sum_{n=1}^{\infty} (\mp)^{n-1} e^{-n(n+\sigma)\xi}/n \cdot \{\text{lower sign: Bose} \}
\]

and a term by term integration gives

\[
\log f(\alpha, \xi) = \sum_{n=1}^{\infty} (\mp)^{n+1} \frac{e^{-n\xi}}{n} \int_0^\infty e^{-n\alpha} d\xi_n(\xi)
\]

\[
= \sum_{n=1}^{\infty} (\mp)^{n+1} \frac{e^{-n\xi}}{n} f_n(n\alpha)
\]

Hence we have the identity in \( z = e^{-\xi} \),

\[
\sum_{n=0}^{\infty} f^{(n)}(z)x^n = \exp \left\{ \sum_{n=1}^{\infty} (\mp)^{n+1} \frac{z^n}{n} f_n(n\alpha) \right\}
\]

By means of the formula \( f^{(n)}(\alpha) \) can be determined in terms of \( f \). They are polynomials in \( \sigma = f(\rho\alpha) \), homogeneous in the sense that in every term \( \Pi \sigma \), the index sum \( \sum p \) is constant. They satisfy, as one easily sees, the recurrence formula

\[
\frac{\partial f^{(n)}}{\partial \sigma_p} = (\mp)^{n+1} f^{(n-p)}(\alpha)/p
\]

Rellying on this relation or otherwise we can determine \( f^{(n)} \) successively. Thus we get

\[
f^{(0)} = 1
f^{(1)} = \sigma_1 = f_2(\alpha)
\]

\[
f^{(2)} = \frac{1}{2} \sigma_1^2 = \frac{1}{2} \sigma_2
\]
1938] Supplementary note on the central limit theorem. 383

etc. In general

\[ f^{(n)} = \frac{1}{6} \sigma_1^1 + \frac{1}{2} \sigma_1 \sigma_2 + \frac{1}{3} \sigma_3, \]

\[ f^{(n)} = \frac{1}{24} \sigma_1^4 + \frac{1}{4} \sigma_1^2 \sigma_2 + \frac{1}{3} \sigma_1 \sigma_3 + \frac{1}{8} \sigma_2^2 + \frac{1}{4} \sigma_4, \]

etc. In general

\[ f^{(n)} = \sum_{(k)} \chi^{(k)(n)} \frac{\sigma_1^{k_1} \sigma_2^{k_2} \ldots}{k_1! \ldots k_2! 2^{k_2} \ldots} \]

summed over all the partitions \((k)\) of \(n:\)

\[ k_1 + 2k_2 + \ldots = n \]

\(\chi^{(k)}\) is the class character of the symmetric group corresponding to the Bose (=1) resp. Fermi (=±1) statistics... They can again be written in the more elegant determinantal form (Cf. for all these circumstances the systematic group-theoretical treatment of Kofink. Ann. d. Phys., (5), 28, 264, 1937). They are, however, expressions not quite convenient for a general discussion esp. for the computation for larger \(n\).

For these purposes there is another, for our main consideration more natural method of calculating \(f^{(n)}\), namely that one utilizing the Cauchy integral:

\[ f^{(n)}(\alpha) = \frac{1}{2\pi i} \oint f(\alpha, \xi) e^{\xi} d\xi \]

integrated around the origin of the \(e^{-\xi}\)-plane. The phase volume of the system of \(n\) particles is now given by

\[ f^{(n)}(\xi) = \frac{1}{2\pi i} \oint_{\xi=i} f^{(n)}(\alpha) e^{\xi} d\alpha / \alpha \]

Now a form of the central limit theorems consists in the asymptotic evaluation of this phase volume for large \(n\) and \(\xi\).

Let us try again the method of steepest descent for this evaluation in spite of the different form of the integrand as compared with our classical case. It is necessary to examine the general behaviour of \(f(\alpha, \xi)\) near the col of the integrand \(f \cdot e^{\xi} \cdot e^{\alpha}\), that the method may have any significance. The simultaneous equations which determine the position of the col

\[ \xi = -\frac{\partial \log f}{\partial \alpha}, \quad n = -\frac{\partial \log f}{\partial \xi} \]

define inversely \(\xi, n\) as functions of the two parameters \(\alpha, \xi\). From the monotony of the right hand expressions we see that \(\xi, n\) increases (in fact up to infinity) as, for example, the temperature parameter \(\alpha\) tends
Lemma of Abelian type: Supposing our phase volume \( j = j^{(1)} \) for the single particle problem to be of the asymptotic form \( A \exp(a \cdot t) \) (\( A, a, t \): positive constants) the integral expression \( \log f(\alpha, \xi) \) has the asymptotic form

\[
\log f(\alpha, \xi) = \int_0^\infty \pm \log (1 \pm e^{-a \cdot t}) \, d(A \exp(a \cdot t)) \quad \text{for} \quad \alpha \to 0;
\]

in short, we can replace in \( f \) the phase volume \( j \) by its asymptotic approximation.

Remark: The integral representing \( \log f(\alpha, \xi) \) can be considered as a variant of the Laplace integral (for \( \xi \to \infty \), it becomes, as known, just to the classical partition function of Laplace type), and the lemma is a modification of the corresponding Abelian theorem in the theory of Laplace transformation, Cf. Doetsch, loc. cit.).

Proof: Put \( j(\alpha) = A \exp(h(\alpha)) \) and choose a certain \( E \) such that \( h(\alpha) < H \) for \( \alpha \geq E \), \( H \) being an arbitrary positive number. Divide the region of integration into two parts \( (0, E), (E, \infty) \). In the former interval we get the estimation for the difference between the true and the assigned integrals

\[
\left| \int_0^E \right| < \pm \log (1 \pm e^{-a}) \int_0^E \left| d_j(\alpha) + d(A \exp(a \cdot t)) \right| < \text{const},
\]

and in the latter interval

\[
\left| \int_E^\infty \right| < H \cdot \log f(\alpha, \xi) = \frac{H \log f(1, \xi)}{\alpha}
\]

For sufficiently small \( \alpha \), const. \( \times a < H \log f(1, \xi) \), so that

\[
\left| \frac{\log f(\alpha, \xi)}{\log f(1, \xi)} - 1 \right| < 2H.
\]

From the arbitrariness of \( H \) follows the statement of the lemma.

The lemma can also be stated thus: Under the hypothesis mentioned it holds

\[
f(\alpha, \xi) = [f(\alpha, \xi)]^{\exp(a \cdot \alpha \cdot \xi)}
\]

with \( g \to 0 \), as \( \alpha \to 0 \).

Here we have modified the definition of \( f^* \) in order to show the explicit dependence on the volume constant \( A \) of the previous \( f^* \). By the way it may be remarked that the integral representing \( f^* \) is in a certain sense an extension of the Riemann zeta-function. In fact we have in the Bose case
The identity
\[ \log (1 + e^{-\varepsilon}) = -\log (1 - e^{-\varepsilon}) + \log (1 - e^{-2\varepsilon}) \]
permits the analogous statement in the Fermi case, too.

After this digression we come back to our original problem of rational formulation and proof of the central limit theorem in quantal statistics. It is important to realize that, in distinction to the classical case, the simple requirement \( \varepsilon, n \to \infty \) in the same order of magnitude does not lead to a definite consequence. This relates to the degeneration phenomenon of the quantal gases, and can be in the particular case of Fermi interpreted in the following manner: In the case of extreme degeneration the gas might not have but a few number of states available in spite of its numerous particles and its high energy content. For such a condensed system with few degrees of freedom we can not of course expect a thermodynamically normal behaviour. To attain the latter property we must have some sort of relaxation in order to supply more freedom. For this purpose one may let the energy content \( \varepsilon \) tend to \( \infty \) more rapidly than the particle number \( n \). More elegant is the following scheme, suggested by Fowler: We supply more energy levels for the single particle system (we consider a series of fictitious models with varying size). At the same time we choose \( \varepsilon \) and \( n \) large. To be precise we suppose a certain means of modifying the structure of the system, such that all the energy levels are lowered at the same rate. This corresponds to replacing the phase volume \( j_0(\varepsilon) \) by \( j_0(\lambda \varepsilon) \) and letting \( \lambda \) tend to infinity. This is again equivalent to lowering the unit of energy, so that the above lemma is applicable for the corresponding series of the partition functions \( f \). We have thus, since \( j_0(\lambda \varepsilon) \) has the asymptotic form \( A \lambda^\gamma \varepsilon^\gamma \),
\[ f(\alpha, \xi) = [f^*(\alpha, \xi)]^{g^\gamma + \nu} \text{ with } g \to 0 \text{ as } A \to \infty, \]
where we have let \( \lambda \) be absorbed in \( A \) and suppose now \( A \), the size, growing more and more.

Now, if we let all the capacity factors \( A, \varepsilon, n \) increase at the same rate, the position of the col, whose coordinates must be intensity factors, does not move appreciably, and in this final form the usual method of steepest descent is applicable directly. . . .

The final position of the col \((\beta, \eta)\) is given by
The asymptotic value of \( j^{(n)} \) is

\[
j^{(n)} \sim \int \phi^2(\epsilon, \beta, \eta) \frac{d\epsilon}{2\pi(AD)^{1/2}}.
\]

with \( \Delta \) the determinant of the second order derivatives of \( \log f^* \). If we want the level density instead of the phase volume, the factor \( 1/\beta \) must be omitted. The degree of approximation is much worse than that given by Fowler. This can of course be improved by laying more stringent conditions on the asymptotic behaviour of \( j^{(n)} \) itself. In the above form the correcting factor may be arbitrarily large. The logarithm only has the indicated asymptotic form.

In comparison with the classical case we examine the required growth of the energy in terms of the energy absolute unit. \( A=O(n) \) means the scale of energy diminishing at a rate \( O(n^{-\gamma}) \), so that the energy is really increasing as \( O(n)/O(n^{-\gamma})=O(n^{1+\gamma}) \). I have some only partially successful attempts to apply the method of col directly, that is, without diminishing the energy scale. This is possible for the Bose case with \( \gamma>1 \), where I have confirmed actually that \( \beta, n \to 0 \) at the same rate means \( \epsilon=O(\beta^{-1-\gamma}), n=O(\beta^{-\gamma}), \log f=O(\beta^{-\gamma}) \).


If the particle in question has internal degrees of freedom, weakly coupled with the external ones, the phase volume of the single particle problem has the structure \( j^* = i^*k^* \), and the corresponding classical partition functions are connected multiplicative: \( f^* = g \cdot h \). Now how do these properties reflect themselves on the quantal partition functions \( f(\alpha, \xi) \) etc.? Instead of dealing with \( f \) itself we shall prefer to consider the simpler integrals

\[
F_{\pm} = -\frac{d}{d\xi} \log f(\alpha, \xi) = \int_{e^{\pm\xi}} \frac{d\xi}{e^{\pm\xi + 1}} G_{\pm}, H_{\pm}
\]

They may be conceived as an extension of the Laplace integrals, and we want to call them the Fermi resp. Bose transformations of the corresponding phase volumes.

Developing \( H(\xi) > 0 \), as we have done previously, we have

\[
F_{\pm} = \sum_{n=1}^{\infty} (\mp)^{n+1} e^{-n\xi} f_{\pm}(n\alpha) = \sum_{n=1}^{\infty} (\mp) e^{-n\xi} g_{\pm}(n\alpha) h_{\pm}(n\alpha)
\]
and for the other two

\[ G_{\pm} = \sum_{n=1}^{\infty} (\mp)^{n+1} e^{-n^{2}g_{4}(n\alpha)} \]

\[ H_{-} = \sum_{n=1}^{\infty} e^{-n^{2}}g_{4}(n\alpha). \]

We shall only make use of the Bose transformation for the internal motion; owing to symmetry it can likewise be the external motion. As one sees, the three power series of \( e^{-x} \) are connected in such a way that the coefficients of the first series are the products of the corresponding coefficients of the other two series. Now the well known Parseval theorem gives a general relation between such a trio of power series, which we write here in the complex Faltung form:

\[ F_{\pm}(\alpha, \xi) = \frac{1}{2\pi i} \int H_{-}(\alpha, \xi') G_{\pm}(\alpha, \xi - \xi') d\xi' \]

integrated along a certain vertical line in the \( \xi \)-plane, which is the map of a circle around the origin in the \( e^{-x} = z \)-plane. The circle must be so chosen that \( |x|/\rho_{a} < |\xi'| < \rho_{a} \), where \( \rho^{\prime}s \) are the radii of convergence of \( G \) and \( H \) and are equal to \( e^{\pi\alpha_{\pm}e_{0}} \) with \( \alpha_{0} \) the lowest energy values. Thus we have established a Faltung theorem for the Fermi and Bose transformations. According to this, to a Faltung of two phase volumes corresponds again a Faltung of their transforms, but this time in the complex sense.

Referring back to the original functions \( \log f \), etc. we get

\[ \log f_{\pm}(\alpha, \xi) = -\frac{1}{2\pi i} \int \frac{\partial \log h_{-}(\alpha, \xi')}{\partial \xi'} \log g_{\pm}(\alpha, \xi - \xi') d\xi' \]

The expression appears more elegant if we again introduce the Stieltjes integrals formally.

\[ \log f_{\pm}(\alpha, \xi) = -\frac{1}{2\pi i} \int \log g_{\pm}(\alpha, \xi - \xi') d\xi' \log h_{-}(\alpha, \xi'). \]

As an illustration we consider the case of a paramagnetic electron gas. This rather trivial example elucidates some features of the theorem. The only two internal states are supposed to have the energies \( \pm \varepsilon_{0} \). Then we have

\[ -\frac{\partial \log h_{-}(\alpha, \xi)}{\partial \xi} = \sum_{\pm} \frac{1}{e^{x_{\xi} + i\xi} - 1} \rho_{a} = e^{-ae_{0}} \]

Let the Fermi transform for the external motion be of the asymptotic form
Allowing the validity of changing the order of integrals it remains to calculate

\[ \frac{1}{2\pi i} \oint \frac{d\chi'/\chi'}{(e^{\pm \alpha \xi}/\chi' - 1)(e^{\pm \alpha \xi}/\chi' + 1)} \]

integrated over the circle \(|\chi'| < |\chi'| < e^{-\alpha \xi}\). From the theorem of residue follows immediately the value

\[ (e^{\pm \alpha \xi + \alpha \xi + \xi} + 1)^{-1}. \]

Hence we get

\[ \log f_+ (\alpha, \xi) = \sum \int_0^\infty \frac{\alpha \varepsilon' d\varepsilon}{e^{\pm \alpha \varepsilon + \alpha \xi + \xi} + 1}. \]

This trivial result can be extended to the large negative values of \(\xi\).

Let us introduce the notation \(Q_{\pm}(\alpha, \xi) = Q_{\pm}(\alpha + \xi) = (e^{\pm \xi} + 1)^{-1}\). Then the above example shows the relation

\[ Q_{\pm}(\alpha + \beta, \xi) = \frac{1}{2\pi i} \int Q_-(\alpha, \xi') Q_\pm(\beta, \xi - \xi') d\xi' \]

with \(\Re(\xi + \beta) > \Re \xi' > -\Re \alpha\). This is a special case of our Faltung theorem and can serve to prove the latter reciprocally.

Many of the beautiful properties of the Laplace transformation are based on this additivity property and, once an analogous addition theorem being discovered, we can expect the other similar properties, too, holding in the theory of the Fermi-Bose transformations.

The inversion formula, which we shall in the following establish, comes as a special example under the reign of the above general point of view. The Mellin-Burkill inversion formula of the Laplace transformation depends essentially on the existence of the discontinuous integral

\[ \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} e^{-\alpha z} d\alpha = \begin{cases} \frac{1}{\alpha} & \text{if } \varepsilon < 0, \\ \frac{1}{2} & \text{if } \varepsilon = 0, \beta > 0, \\ 0 & \text{if } \varepsilon > 0 \end{cases} \]

apart from the above indicated addition theorem. We now try to discover a similar discontinuous integral itself in the theory of our Fermi-Bose transformations. Consider for this purposes the function

\[ Q_{\pm}(\alpha \xi, \xi)/\alpha \]

with a real parameter \(\varepsilon, \xi\) (\(\Re(\xi) > 0\)) complex fixed, \(\alpha\) complex variable.
Near the point at infinity $a \sim \infty$, the function does not much differ from $e^{-\frac{(a+\xi)^2}{\alpha}}$, and in particular its integrals over a left ($\epsilon < 0$) or right ($\epsilon > 0$) semi-circle of infinite radius vanishes. Apart from the simple pole at the origin it has an infinite array of simple poles $(a+\xi = 2n\pi i$ or $(2n+1)\pi i)$ along a vertical line with abscissa $-\frac{\mathcal{R}(\xi)}{\epsilon}$. Consider now the integral

$$\frac{1}{2\pi i} \int_{a-\infty}^{b+\infty} \frac{Q_{\pm}(a\epsilon + \xi)}{\alpha} \, d\alpha$$

with $\beta > 0$ fixed. If $\epsilon > 0$, we complete a closed path of integration by adding the right hand circle, and we get the value zero, since there is no singularity right to the straight path. If $\epsilon < 0$, the singularity line of $Q$ passes to the right half plane, but this time we take the left hand circle to complete the path. The residue comes only from the origin and is equal to the constant value $Q_{\pm}(\xi)$, so long as $-\frac{\mathcal{R}(\xi)}{\epsilon} > \beta$. We shall not use the integral in the exceptional case $-\frac{\mathcal{R}(\xi)}{\epsilon} < \beta$.

The discontinuous factor thus obtained turns out to be useful in the inversion problem in question. The Fermi-Bose transforms are of the form

$$F_{\pm}(\alpha, \xi) = \int_0^\infty Q_{\pm}(a\epsilon, \xi) \, dj_\epsilon(\epsilon').$$

The Faltung with $Q_{-}(-a\epsilon, \xi)$ gives

$$\frac{1}{2\pi i} \int_0^\infty Q_{-}(-a\epsilon, \xi') F_{\pm}(\alpha, \xi - \xi') \, d\xi = \int_0^\infty Q_{\pm}(a(\epsilon' - \epsilon), \xi) \, d\epsilon'$$

and the discontinuous integral gives next, at least formally,

$$\frac{1}{2\pi i} \int_{a-\infty}^{b+\infty} \frac{d\alpha}{\alpha} \cdot \frac{1}{2\pi i} \int_0^\infty Q_{-}(-a\epsilon, \xi') F_{\pm}(\alpha, \xi - \xi') \, d\xi'$$

$$= \frac{Q_{\pm}(\xi)}{2} \left[ j_\epsilon(\epsilon + 0) + j_\epsilon(\epsilon - 0) \right]$$

and completes the inversion process. We have supposed $0 < \beta \epsilon < \mathcal{R}(\xi') < \mathcal{R}(\xi)$.

The inversion of a special Bose transformation

$$\int_0^\infty Q_{-}(a\epsilon, 0) \varphi(\epsilon) \, d\epsilon$$

has been solved by applying the Parseval theorem in the theory of Fourier transformations by Wiener (Cf. Paley and Wiener, Fourier
transforms in the complex domain). The method, regarding the problem as an integral equation of Faltung type, involves integrations of Fourier type twice and a division by the zeta-function. The insertion of the degeneration parameter $\xi$ seems to round off all circumstances.

5. Summary.

In the sections 1 and 2 we have discussed some unsatisfactory points of our previous memoir, concerning the contributions from the ghost cols to our main integral and the theorem of composition in its most general form. The section 3 outlines the chief results of quantal statistics in our style and treats esp. the limit theorem somewhat more precisely than we find in the current literatures. In the last section 4 we set up two theorems concerning the Fermi and Bose transformations.

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