On some Integro-differential Equations obtained operationally.

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The integro-differential equations of the form

\[ u(x) = \int_0^\infty \left( \frac{d}{dt} \right) g(t) \, dt \]  

is easily deducible with the aid of Heaviside's operator.

In this paper I denote by \( \mu \) the Heaviside's operator with the operand zero

\[ g(p) \]  

and assume that \( g(p) \) is analytic in the neighbourhood of the infinity in the complex \( p \) plane.

If \( f(x) \) and \( g(p) \) are combined by

\[ f(x) = g(p) \cdot 0, \]

there exists the integral relation

\[ f(x) = \frac{1}{2\pi i} \int_0 \, e^{\lambda x} g(p) \, dp \]  

due to Bromwich, where the integration is positively around a large circle \( C \) with its center at the origin of the complex \( p \) plane and includes all the singularities of \( g(p) \) inside it.

If we make the change of the variable in the integral (3) by the relation

\[ p = \frac{-1}{s}, \]  

it becomes

\[ f(x) = \frac{1}{2\pi i} \int_\gamma \, e^{s x} g \left( \frac{-1}{s} \right) \frac{ds}{s^3} \]

\[ = \frac{1}{2\pi i} \int_\gamma \, e^{s x} \frac{1}{s^3} \left\{ -s^{m-1} g \left( \frac{-1}{s} \right) \right\} ds, \]

where we suppose for a while that \( m \) is an integer (or zero). Since the path \( \gamma \) is the representation of the circle \( C \) into the complex \( s \) plane by the transformation (3), the integration of (4) is positively around

(1) This Proceedings, 18, pp. 356-371 (1936).
(3) In this discussion we assume that

\[ \left\{ -s^{m-1} g \left( \frac{-1}{s} \right) \right\} \]

has no singularities at the origin of the complex \( s \) plane. Hence the value of \( m \) must be so chosen that the above assumption is satisfied.
a small circle \( \Gamma' \) which excludes all the singularities \( s^{m-1}g(-1/s) \).

We know the integral relation\(^{(4)}\)

\[
\int_0^\infty f(t)g(t)dt = \frac{1}{2\pi i} \int_{\Gamma_1} g(s) g(-1/s) ds, \tag{5}
\]

when

\[
f(t) = g(p) \cdot 0, \quad f(t) = g(p) \cdot 0. \tag{5'1}
\]

If we put

\[
g(p) = \frac{1}{p^{m+1}} e^{-\frac{s}{s}}, \tag{6'1}
\]

\[
g(p) = -(-p)^{m-l} g\left(\frac{1}{p}\right), \tag{6'2}
\]

\( f_1(x) \) and \( f_3(x) \) become

\[
f_1(t) = \frac{t^m}{x^n/n!} J_m(2\sqrt{xt}) \tag{7'1}
\]

\[
f_3(t) = \left[\left(-(-p)^{m-l} g\left(\frac{1}{p}\right)\right) \cdot 0. \tag{7'3}
\]

Hence if we replace \( p \) in (6'1) and (6'2) by \( s \) and substitute (6'1), (6'2), (7'1) and (7'2) in (5), (5) becomes

\[
\int_0^\infty \frac{t^m}{x^n/n!} J_m(2\sqrt{xt}) f(t) dt = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{1}{s^{m+1}} e^{-\frac{s}{s}} (-s^{m-1}) g\left(\frac{-1}{s}\right) ds. \tag{8}
\]

Since the path\(^{(4)}\) \( \Gamma_1 \) can be deformed into the path \( \Gamma \) in (4) continuously without passing over any singularities of the integrand in the right-hand integral of (8), (8) becomes

\[
\int_0^\infty \frac{t^m}{x^n/n!} J_m(2\sqrt{xt}) f(t) dt = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{1}{s^{m+1}} e^{-\frac{s}{s}} (-s^{m-1}) g\left(\frac{-1}{s}\right) ds. \tag{8}
\]

If we combine this with the equation (4), we obtain

\[
f(x) = \int_0^\infty \frac{t^m}{x^n/n!} J_m(2\sqrt{xt}) f(t) dt. \tag{9}
\]

If the function \( g(p) \) satisfies the relation

\[
g(p) = p^{-n} g\left(\frac{1}{p}\right), \tag{10}
\]

we have by (7'2)

\[
f_3(t) = (-1)^m \frac{d^{m+n-1}}{dt^{m+n-1}} \alpha(t). \tag{11}
\]

If we substitute (11) in (9), (9) becomes

\[
\int_0^\infty \frac{t^m}{x^n/n!} J_m(2\sqrt{xt}) \frac{d^{m+n-1}}{dt^{m+n-1}} \alpha(t) dt. \tag{12}
\]

This equation is an integro-differential equation of the type (1).

If we assume the relations similar to (10) and proceed exactly in the same way as we have obtained (12) from (9), we can easily deduce from (9) other integro-differential equations of the type (1).

Let us now consider the examples of the method which we so far discussed.

We first take

\[
g(p) = \frac{1}{(p+1)^{m+1}}, \tag{12}
\]

where \( n \) is a positive integer or zero. Since

\[
g\left(\frac{1}{p}\right) = \frac{p^{m+1}}{(1+p)^{m+1}},
\]

\( g(p) \) satisfies

\[
g^{(m+1)}(p) = g\left(\frac{1}{p}\right). \tag{13}
\]

(4) This Proceedings, 20, p. 168 (1938).

(5) This comes from the fact that both \( \Gamma_1 \) and \( \Gamma \) includes singularities of

\[
\frac{1}{s^{m+1}} e^{-\frac{s}{s}}
\]

and excludes all singularities of

\[
(-s^{m-1}) g\left(\frac{-1}{s}\right).
\]

(6) \( C \) may be any circle whose center is at the origin of \( p \) plane and whose radius is greater than unity. The integrations in (14), (17), (19), (21) and (23) are positively around the circle \( C \).
Since the relation (13) is of the same meaning as the relation (10), it might be excepted that the integro-differential equation of the type (1) can be deduced from (13).

If we make use of (12), the integral (2) becomes

\[ \frac{x^n e^{-x}}{n!} = \frac{1}{2\pi i} \int_C \frac{e^{ps}}{(p+1)^{n+1}} dp, \quad (14) \]

where the circle \( C \) having its center at the origin of \( p \) plane is any one whose radius is greater than unity.

The right-hand integral of (14) can be varied with the aid of (3) and (5) as follows:

\[ \frac{1}{2\pi i} \int_C \frac{1}{(p+1)^{n+1}} dp = \frac{1}{2\pi i} \int_I e^{-s} \frac{s^{n+1}}{1-s} ds = \frac{1}{2\pi i} \int_R \frac{e^{-sI}}{(1-s)^{n+1}} ds = \int_0^\infty J_n(2\sqrt{it}) \frac{dt}{dt} e^{-it} dt. \quad (15) \]

Hence combining this result with (14), we get the integro-differential equation

\[ x^n e^{-x} = \int_0^\infty J_n(2\sqrt{it}) \frac{dt}{dt} e^{-it} dt, \quad (16) \]

If \( \mathcal{G}(p) = \frac{1}{p^{n+1}} \), the equation (2) becomes

\[ \sin x = \frac{1}{2\pi i} \int_C \frac{e^{ps}}{p^{n+1}} dp, \quad (17) \]

which can be varied with the aid of (3) and (5) as follows:

\[ \frac{1}{2\pi i} \int_C \frac{e^{ps}}{p^{n+1}} dp = \frac{1}{2\pi i} \int_R \frac{s^{n+1}}{1+s} ds = \frac{1}{2\pi i} \int_R \frac{e^{-sI}}{1+s} ds = \int_0^\infty J_n(2\sqrt{it}) \frac{dt}{dt} e^{-it} dt. \quad (18) \]

Hence we get the integro-differential equation

\[ \sin x = \int_0^\infty J_n(2\sqrt{it}) \frac{dt}{dt} e^{-it} dt. \]

If we take

\[ \mathcal{G}(p) = \frac{p}{p^{n+1}} \]

the relation (2) becomes

\[ \cos x = -\frac{1}{2\pi i} \int_C \frac{e^{ps}}{p^{n+1}} dp = \frac{1}{2\pi i} \int_R \frac{e^{-sI}}{1+s} ds = \int_0^\infty \sqrt{-i} \sqrt{x} J_n(2\sqrt{it}) \cos dt. \quad (19) \]

Hence combining (19) and (20), we have

\[ \sqrt{-x} \cos x = \int_0^\infty J_n(\sqrt{2} |t|) \sqrt{-i} \cos dt, \]

which is also an example of the integro-differential equation (1).

If \( \mathcal{G}(p) = \frac{1}{\sqrt{p^{n+1}}} \), the equation (2) becomes

\[ J_n(x) = \frac{1}{2\pi i} \int_C \frac{e^{ps}}{\sqrt{p^{n+1}}} dp. \quad (21) \]

This can be varied as follows:

\[ \frac{1}{2\pi i} \int_C e^{ps} \frac{1}{\sqrt{p^{n+1}}} dp = \frac{1}{2\pi i} \int_R \frac{e^{-sI}}{\sqrt{1+s}} ds = \frac{1}{2\pi i} \int_R \frac{e^{-sI}}{\sqrt{1+s}} \frac{1}{\sqrt{1+s} ds = \int_0^\infty J_n(2\sqrt{it}) \frac{dt}{dt} e^{-it} dt. \quad (22) \]

Hence we get the integro-differential equation

\[ J_n(x) = \int_0^\infty J_n(2\sqrt{it}) J_n(t) dt. \]

If we put

\[ \Gamma \]

(7) Since \( \Gamma \) is the representation of \( C \) into the \( s \) plane by the transformation (3), \( \Gamma \) is a circle having its center at the origin of \( s \) plane and the radius of this circle is less than unity. The integrations with respect to \( s \) in (15), (18), (20), (22) and (23) are positively around the circle \( \Gamma \).
with the aid of (3) and (5) as follows:

\[ g(p) = \frac{(p-1)^n}{(p+1)^{n+1}} \]

the equation (2) becomes

\[ e^{-\frac{1}{2}L_n(2t)} = \frac{1}{2\pi i} \int_C e^{\frac{1}{2}x} \frac{(p-1)^n}{(p+1)^{n+1}} dp, \]

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where \( L_n(x) \) is Laguerre's polynomial defined by

\[ L_n(x) = \frac{e^x \frac{d^n}{dx^n} (e^{-x})}{n!}. \]

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The right-hand integral of (23) can be varied

\[ \frac{1}{2\pi i} \int_C e^{\frac{1}{2}x} \frac{(p-1)^n}{(p+1)^{n+1}} dp \]

\[ = \frac{1}{2\pi i} \int_C e^{-\frac{a}{2}} (1+s)^n ds \]

\[ = (-1)^n \int_0^\infty J_d(2\sqrt{xt}) e^{-\frac{a}{2}L_n(2t)} dt. \]

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Hence we get the integral equation

\[ e^{-\frac{1}{2}L_n(2t)} = (-1)^n \int_0^\infty J_d(2\sqrt{xt}) e^{-\frac{a}{2}L_n(2t)} dt. \]

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(8) G. Doetsch, Laplace-Transformation, p. 186.