Note on the Influence of Vortices upon the Resistance of a Circular Cylinder moving through a Fluid.

By Susumu Tomotika and Nobuo Sugawara.

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I. Introduction.

§ 1. In 1913, L. Föppl(1) discussed the equilibrium as well as the stability of a vortex pair behind an infinite circular cylinder moving, with a constant velocity \( U \), through an unlimited mass of an incompressible perfect fluid in a direction perpendicular to the axis of the cylinder. The two vortex filaments, \( A \) and \( B \), were assumed to be parallel to the axis of the cylinder and also assumed to be situated symmetrically with respect to a diametral line parallel to the undisturbed stream. The fluid motion discussed was therefore two-dimensional. Denoting the radius of the cylinder by \( a \), the distance of \( A \) and \( B \) from the axis, \( O \), of the cylinder by \( c \), and the angle \( AOB \) by \( 2\gamma \), it has been shown that the vortices are in relative equilibrium, provided that they lie on the curve whose equation is

\[
2c \sin \gamma = c - \frac{a^2}{c},
\]

with the strength \( \kappa \) given by

\[
\kappa = 2Uc \left( 1 - \frac{a^2}{c} \right) \sin \gamma.
\]

It has been shown, however, that the equilibrium of the vortex pair is stable for symmetric disturbances, but not for anti-symmetric disturbances.

By employing the theorem of momentum, Föppl also calculated the resistance, \( P_r \), experienced by the cylinder(2). For the case when the


\( (2) \) In the general case when the strength of vortices is variable, Föppl's result is

\[
P_r = 3\rho \frac{d}{dt} \left\{ \kappa \left( c - \frac{a^2}{c} \right) \sin \gamma \right\},
\]

and this result is also referred to in W. Müller's book, "Mathematische Strömungslehre, (1926), 110".

It may be remarked here that Föppl used \( C \) in place of \( \pi/2 \) in the present paper, so that his original expression for the resistance is of the form:

\[
P_r = 6\pi \rho \frac{d}{dt} \left\{ C \left( c - \frac{a^2}{c} \right) \sin \gamma \right\}.
\]
strength $\kappa$ of the vortices is invariable, in accordance with the well-known fundamental characteristics of the vortex motion in a perfect fluid, his result is, in the notation of the present paper,

$$P_a = 3\kappa p\frac{d}{dt}\left((c - \frac{a^2}{c})\sin\gamma\right),$$

where $\rho$ being the density of the fluid concerned.

§ 2. Fifteen years later, W. G. Bickley\(^{1}\) has also calculated the drag experienced by a circular cylinder in a uniform stream, assuming a vortex pair behind it as in Föppl's case. In calculating the resistance, Bickley employed however a quite different procedure, i.e., he used a direct method of summing up the fluid pressures acting on the cylinder. Assuming the strength $\kappa$ of the vortices to be constant, his expression for $P_a$ may be written in the form:

$$P_a = 2\kappa p\frac{d}{dt}\left((c - \frac{a^2}{c})\sin\gamma\right),$$

where $a$, $c$ and $\gamma$ have the same meaning as before.

It is seen that Bickley's expression (4) is not in accord with Föppl's (3). However, it should naturally be expected in the problem under discussion that we would arrive at one and the same expression for the resistance experienced by a circular cylinder, irrespective of the method of calculation.

Thus, it seems to be rather interesting to re-investigate the matter; and to the discussion of this problem the present short note is mainly directed.

II. Calculation of the Resistance by Direct Summation of Fluid Pressures acting on a Circular Cylinder.

§ 3. First, we have calculated the resistance experienced by an infinite circular cylinder moving with a constant velocity $U$ through a perfect fluid, with a vortex pair in its wake, by using a direct method which was adopted by Bickley\(^{2}\), i.e., by directly summing up the fluid pressures acting on the cylinder.

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(2) W. G. Bickley loc. cit.
Adopting the usual artifice we have considered a stationary cylinder in a uniform stream of velocity $U$ flowing in the positive direction of the $x$-axis, the origin of the coordinate-axes ($x, y$) being taken on the axis, $O$, of the cylinder. Then, assuming that the two vortices, $A$ and $B$, are situated symmetrically with respect to the $x$-axis, we have carefully repeated the analysis similar to that of Bickley.

Although the results obtained are, for the most part, the same as those of Bickley, except for some obvious misprints on his part, yet a brief summary of the principal results may be given for reference in the following lines.

Denoting the radius of the cylinder by $a$, the distance of $A$ and $B$ from the axis $O$ of the cylinder by $c$, the angle $AOB$ by $2\gamma$, and the strength of the vortices by $\kappa$, we have, for the velocity components $u_A, v_A$ of the vortex $A$,

$$u_A = U \left(1 - \frac{a^2 \cos 2\gamma}{c^2}\right) - \frac{\kappa}{2\pi} \left[\frac{c \sin \gamma}{c^2 - a^2} + \frac{1}{2c \sin \gamma}\right] - \frac{c(c^2 + a^2) \sin \gamma}{c^4 + a^4 - 2a^2 c^2 \cos 2\gamma},$$

$$v_A = -U \frac{a^2 \sin 2\gamma}{c^3} + \frac{\kappa}{2\pi} \left[\frac{c \cos \gamma}{c^2 - a^2} - \frac{c(c^2 - a^2) \cos \gamma}{c^4 + a^4 - 2a^2 c^2 \cos 2\gamma}\right].$$

On account of the symmetry, the velocity components $u_B, v_B$ of the vortex $B$ are given by

$$u_B = u_A, \quad v_B = -v_A. \quad (6)$$

Also, the velocity components of the image vortices, $C$ and $D$, at the inverse points (Fig. 1) are

$$u_C = -\frac{a^2}{c^3} (u_A \cos 2\gamma + v_A \sin 2\gamma),$$

$$v_C = \frac{a^2}{c^3} (v_A \cos 2\gamma - u_A \sin 2\gamma),$$

and

$$u_D = u_C, \quad v_D = -v_C. \quad (8)$$

From (5) it follows immediately that if the vortices are stationary, i.e., if $u_A = v_A = 0$, the vortices must lie on the curve whose equation is

$$2c \sin \gamma = c - \frac{a^2}{c}, \quad (9)$$

and that for any position of the vortex on this curve the value of $\kappa$
is given by

$$\kappa = 2UC\left(1 - \frac{\alpha^2}{c^2}\right)\sin \gamma.$$  \hspace{1cm} (10)

These results (9) and (10) are in accordance with Föppl’s.

Next, the resistance experienced by the cylinder has been calculated by directly summing up the fluid pressures acting on the cylinder. We get, with Bickley,

$$P_\alpha = 2\rho \psi \left\{ u_A\frac{a^2}{c}\sin 2\gamma + v_A\left(1 - \frac{a^2}{c}\cos 2\gamma\right)\right\},$$  \hspace{1cm} (11)

or

$$P_\alpha = 2\rho (u_A - v_c),$$  \hspace{1cm} (12)

where $\rho$ is the density of the fluid concerned.

Substituting the values of $u_A$ and $v_A$ given by (5) in (11), we have, after some reduction,

$$P_\alpha = \frac{\rho a^2 \psi^2 \cos \gamma 4a^4 \sin^2 \gamma - (c^2 - a^2)^2}{\pi c^2 (c^2 + a^4 - 2a^2 c^2 \cos 2\gamma)}.$$  \hspace{1cm} (13)

It will be seen therefore that the cylinder experiences no resistance when the vortices are in relative equilibrium on the curve (9).

Also, the expression for $P_\alpha$ can be put in another form, as follows. Since

$$v_A = \frac{d}{dt} \left(c \sin \gamma\right), \quad v_c = \frac{d}{dt} \left(\frac{a^2}{c} \sin \gamma\right),$$

we have

$$P_\alpha = 2\rho \psi \frac{d}{dt} \left(\frac{c - a^2}{c} \sin \gamma\right).$$  \hspace{1cm} (14)

This is what has been referred to as Bickley’s result in §2. However, the expression of this form is not given explicitly in his paper.

III. Application of the Theorem of Momentum to the Calculation of the Resistance.

§4. We next proceed to the calculation of the resistance, $P_\alpha$, of the circular cylinder under consideration, by applying this time the theorem of momentum, which was employed by Föppl in his paper(1).

We assume that the cylinder is moving with a constant velocity $U$ through a perfect fluid, with a vortex pair in its wake as before. We take the coordinate-axes $(x, y)$ as shown in Fig. 1.

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(1) L. Föppl, loc. cit.
Then, with the same notation as before, the complex velocity potential \( w \) for the flow outside the cylinder is

\[
 w = -U \frac{a^2}{z} + \frac{ik}{2\pi} \log \left( \frac{z-z_a}{z-z_b} \right),
\]

where \( z_a, z_b, z_c \) and \( z_d \) are the complex coordinates of the vortices \( A, B, C \) and \( D \) respectively.

Let \( P \) be any point in the fluid and let the distances \( AP, BP, CP \) and \( DP \) be denoted by \( r_1, r_2, r_3 \) and \( r_4 \) respectively. Also, we denote the angles \( \angle APB \) and \( \angle CPD \) by \( \alpha \) and \( \beta \) respectively. Then

\[
 \log \left( \frac{z-z_c}{z-z_d} \right) = \log \frac{r_3 r_4}{r_1 r_2} + i(\beta - \alpha).
\]

Thus, the velocity potential \( \phi \) is

\[
 \phi = -U a^2 \frac{x}{x^2+y^2} + \frac{k}{2\pi} (\alpha - \beta).
\]

For convenience we write

\[
 \phi = \phi_1 + \phi_2,
\]

where

\[
 \phi_1 = -U a^2 \frac{x}{x^2+y^2}, \quad \phi_2 = \frac{k}{2\pi} (\alpha - \beta).
\]

Now, if we denote by \( Q_x \) the \( x \)-component of the momentum of the fluid, we have

\[
 Q_x = \rho \int \int u dx \, dy,
\]

where the integral is taken over the whole fluid region outside the cylinder. However, since \( u = -\frac{\partial \phi}{\partial x} \),

\[
 Q_x = -\rho \int \int \frac{\partial \phi}{\partial x} \, dx \, dy.
\]

We denote, as before, by \( P_x \) the resistance experienced by the cylinder. Then, the fluid experiences the reaction \(-P_x\) from the body according to the law of action and reaction. Therefore, by the theorem of momentum, we have
\[ \frac{dQ_x}{dt} = -P_x. \]  

(21)

Combining this with (20), we get

\[ P_x = \rho \frac{d}{dt} \int \int \frac{\partial \phi}{\partial x} \; dx \; dy. \]  

(22)

Since, however, \( U \) is assumed to be constant in the present paper, \( \int (\partial \phi / \partial x) \; dx \; dy = \text{const.} \), and therefore

\[ \frac{d}{dt} \int \int \frac{\partial \phi_x}{\partial x} \; dx \; dy = 0. \]  

(23)

Thus,

\[ P_x = \rho \frac{d}{dt} \int \int \frac{\partial \phi_x}{\partial x} \; dx \; dy, \]  

(24)

\( \phi_x \) being given by (18).

§ 5. Our next problem is to evaluate the integral \( \int \int (\partial \phi_x / \partial x) \; dx \; dy \). By Green's theorem, this surface integral can be transformed into a line integral taken along an appropriate contour which surrounds the fluid region concerned. Taking a contour as shown in Fig. 2, we have

\[ \int \int \frac{\partial \phi_x}{\partial x} \; dx \; dy = \oint \phi_x \cos(n, x) \; ds, \]  

(25)

\( n \) denoting the outward normal at a point on the contour, and \( ds \) a line element of the contour reckoned positive in the counter-clockwise sense.

This contour consists of various parts, namely: the surface of the cylinder; two small circles, \( K_1 \) and \( K_2 \), of radius \( a \) with the points A and B as their centres respectively; a large circle \( \Sigma \) of radius \( R \) surrounding the cylinder and the vortices; and some straight paths connecting these circles. The straight paths ab, cd and ef connecting the two small circles \( K_1 \) and \( K_2 \) are taken to be perpendicular to the \( x \)-axis.

Then, remembering that \( \cos(n, x) = 0 \) along the straight paths parallel to the \( x \)-axis, we have\(^{(1)}\)

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\(^{(1)}\) In this equation, \( \phi_{ab} \) denotes the value of the function \( \phi_x \) at any point on the paths ab and ef, which are on the left-hand side of AB, while \( \phi_{cd} \) is the value of \( \phi_x \) at the corresponding point with the same ordinate on the path cd, which lies on the right-hand side of AB. Further, \( \theta' \) denotes the central angle of the circle \( K_1 \) and \( \theta'' \) that of the circle \( K_2 \).
But, the first integral \( \iiint \phi_2 \cos \theta \cdot R d\theta \) evidently vanishes when \( R \to \infty \), and also we have, when \( \varepsilon \to 0 \),

\[
\oint \phi_2 \cos (\pi + \theta') \cdot \varepsilon d\theta' + \oint \phi_2 \cos (\pi + \theta'') \cdot \varepsilon d\theta'' = 0.
\]

Therefore, arranging the terms, we get

\[
\oint \phi_2 \cos (n, x) ds = \oint_{a}^{b} \phi_{a} ds + \oint_{c}^{d} \phi_{c} ds - \oint_{c}^{d} \phi_{c} ds - a \int_{-\pi}^{\pi} \phi_2 \cos \theta d\theta.
\]
However, it is easily proved that $\phi \equiv \theta - \kappa$. Thus, remembering that $ds = dy$ on $ab$ and $ef$, and $ds = -dy$ on $cd$, and that the ordinates of the points $a$, $b$, $c$, $d$, $e$, and $f$ are $0$, $c \sin \gamma - \varepsilon$, $c \sin \psi - \varepsilon$, $-(c \sin \gamma - \varepsilon)$, $-(c \sin \psi - \varepsilon)$, and $0$ respectively, we have

$$\int_a^b \phi \, ds + \int_a^t \phi \, ds - \int_a^d \phi \, ds = \int_{-\varepsilon}^{\varepsilon} \phi \, dy - \int_{-\varepsilon}^{\varepsilon} \phi \, dy$$

and this value tends to $2 \kappa c \sin \gamma$ in the limit when $\varepsilon \to 0$.

Thus,

$$\int \phi z \cos (n, x) \, ds = 2 \kappa c \sin \gamma - \int_{\varepsilon}^{\gamma} \phi z \, d\theta. \quad (26)$$

or, since, by (18), $\phi = \frac{\kappa}{2\pi} (\alpha - \beta)$,

$$\int \phi z \cos (n, x) \, ds = 2 \kappa c \sin \gamma - \frac{\kappa a}{2\pi} \int_{-\varepsilon}^{\varepsilon} (\alpha - \beta) \cos \theta \, d\theta. \quad (27)$$

$\alpha$ and $\beta$ in this formula denote, of course, the values of the angles $APB$ and $CPD$ respectively when the point $P$ lies on the surface of the cylinder. Thus, we easily have

$$\alpha = 3 \log \left( \frac{ae^{iv} + ce^{iv}}{ae^{iv} - ce^{iv}} \right), \quad \beta = 3 \left[ 1 - \log \left( \frac{ae^{iv} - c}{ae^{iv} + c} e^{iv} \right) \right], \quad (28)$$

where $\Im (z)$ means the imaginary part of $z$.

§ 6. Next, we shall evaluate the following two integrals:

$$I_1 = \int_{-\varepsilon}^{\varepsilon} \alpha \cos \theta \, d\theta, \quad I_2 = \int_{-\varepsilon}^{\varepsilon} \beta \cos \theta \, d\theta. \quad (29)$$

We first consider the integral $I_1$. If we put

$$I_1^{(1)} = \int_{-\varepsilon}^{\varepsilon} \log \left( \frac{ae^{iv} - ce^{iv}}{ae^{iv} - ce^{iv}} \right) e^{iv} \, d\theta,$$

$$I_1^{(2)} = \int_{-\varepsilon}^{\varepsilon} \log \left( \frac{ae^{iv} - ce^{iv}}{ae^{iv} - ce^{iv}} \right) e^{-iv} \, d\theta,$$

then

$$I_1 = I_1^{(1)} + I_1^{(2)}. \quad (30)$$
we have, by taking (28) into account,

\[ I_i = \frac{1}{2} \Im \left[ I_i^{(1)} + I_i^{(2)} \right]. \]  

(31)

The evaluation of the integrals \( I_i^{(1)} \) and \( I_i^{(2)} \) can be carried out immediately. Putting \( \zeta = ae^{i\theta} \), we get

\[ I_i^{(1)} = \frac{1}{a} \left\{ \oint \log(\zeta - ce^{i\gamma}) d\zeta - \oint \log(\zeta - ce^{-i\gamma}) d\zeta \right\} = 0, \]  

(32)

since \( |\zeta| = a < c \). In this as well as in subsequent equations, the integrals are all taken along the boundary contour of the cylinder in the sense indicated. Also, with \( \zeta = ae^{i\theta} \), we have, since \( |\zeta| = a < c \),

\[ I_i^{(2)} = \frac{a}{i} \left\{ \oint \log \left( \frac{1}{ce^{i\gamma}} \right) d\zeta - \oint \log \left( \frac{1}{ce^{-i\gamma}} \right) d\zeta \right\} = \frac{a}{i} \left( -\frac{1}{ce^{i\gamma}} + \frac{1}{ce^{-i\gamma}} \right) \times 2\pi i = 2\pi a c \sin \gamma. \]  

(33)

Thus, putting these values of \( I_i^{(1)} \) and \( I_i^{(2)} \) in the right-hand side of (31), we have

\[ I_i = \frac{2\pi a}{c} \sin \gamma. \]  

(34)

In a similar manner, the integral \( I_i \) can be evaluated. Writing

\[ I_i^{(3)} = \int_{-\pi}^{\pi} \log \left( \frac{ae^{i\theta} - \frac{a^2 e^{i\gamma}}{c}}{ae^{i\theta} - \frac{a^2 e^{-i\gamma}}{c}} \right) e^{i\theta} d\theta, \]  

\[ I_i^{(4)} = \int_{-\pi}^{\pi} \log \left( \frac{ae^{i\theta} - \frac{a^2 e^{i\gamma}}{c}}{ae^{i\theta} - \frac{a^2 e^{-i\gamma}}{c}} \right) e^{-i\theta} d\theta, \]  

we have

\[ I_i = \frac{1}{2} \Im \left[ I_i^{(3)} + I_i^{(4)} \right]. \]  

(38)

Putting \( \zeta = ae^{i\theta} \) as before and remembering that \( |\zeta| = a > a^2/c \),
Further, we have, since $|\xi| = a > a'/c$,}

$$I_{\xi}\gamma = \frac{a}{i} \left[ \int \log \left( \xi - \frac{a^2}{c} e^{i \gamma} \right) \frac{d\xi}{\xi^2} - \int \log \left( \xi - \frac{a^2}{c} e^{-i \gamma} \right) \frac{d\xi}{\xi^2} \right] = \frac{4\pi a}{c} \sin \gamma. \quad (37)$$

Therefore, we have, by (36), (37) and (38),

$$I_{\xi} = -\frac{2\pi a}{c} \sin \gamma. \quad (39)$$

The combination of (34) and (39) gives

$$\int_{-\pi}^{\pi} (\alpha - \beta) \cos \theta d\theta = -I_1 - I_3 = \frac{4\pi a}{c} \sin \gamma. \quad (40)$$

Inserting this in (27) and taking (25) into account, we get

$$\int \int \frac{\partial^2 \phi}{\partial x^2} \, dx \, dy = \oint \phi \cos (n, x) \, ds = 2\pi \left( c - \frac{a^2}{c} \right) \sin \gamma. \quad (41)$$

Thus, combining this with (24) and assuming, as before, the strength $\kappa$ of the vortices to be invariable, the expression for the drag $P_x$ experienced by the circular cylinder under discussion becomes

$$P_x = 2\pi \rho \frac{d}{dt} \left[ \frac{1}{c - \left( \frac{a^2}{c} \right) \sin \gamma} \right]. \quad (42)$$

It will be seen that this agrees with the previous result (14), which has been obtained by a quite different procedure. Thus, we have obtained, as we should have expected, one and the same formula for the drag experienced by a circular cylinder, which is accompanied by a vortex pair in its wake, by employing two different methods of calculations. Föppel's expression for the resistance is erroneous and the cause of error in Föppel's result has been found to lie in the incorrect evaluation of the integral $I_3$ in his notation, which is equivalent to $I_3$ in the present paper.
IV. Summary.

§7. The resistance experienced by a circular cylinder moving uniformly through a perfect fluid, with a vortex pair in its wake, is calculated, by employing two different methods of analysis, and one and the same expression for the resistance is obtained, as we should have expected. The result is in accordance with Bickley's one, which was obtained by directly summing up the fluid pressures acting on the cylinder, but not with Förnt's, which was calculated by using the theorem of momentum. It is found that Förnt's result is erroneous and that the cause of error lies in the incorrect evaluation of one of the integrals.

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Physical Institute,
Faculty of Science,
Imperial University of Osaka.

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