A Self-reciprocal function.

By R. V. SHASTRY.

(Read by Minoru Tanaka June 4, 1938).

The self-reciprocity of an even function \( f(x) \) is expressed by the equation

\[
\chi(s) = \frac{2^{\frac{1}{2} - 1} \Gamma\left(\frac{1}{4} + \frac{1}{2} s\right) \Gamma\left(\frac{3}{4} - \frac{1}{2} s\right)}{\Gamma\left(\frac{3}{4} + \frac{1}{2} s\right) \Gamma\left(\frac{5}{4} - \frac{1}{2} s\right) \Gamma\left(\frac{1}{4} + \frac{1}{2} \nu + \frac{1}{2} s\right) \Gamma\left(\frac{3}{4} + \frac{1}{2} \nu - \frac{1}{2} s\right)},
\]

where \( 0 < R(s) \leq \frac{1}{2} \). Then, using (2.6) we get

\[
P(x) = x^{-\nu} H_{\nu - \frac{1}{2}}(x),
\]

and \( g(x) \) becomes the function (2.1)

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The self-reciprocity of an even function \( f(x) \) is expressed by the equation

\[
f(x) = \int_{-\infty}^{\infty} \nu x y J_\nu(xy) f(y) \, dy
\]

where \( \nu \geq -\frac{1}{2} \). This integral may be a Cauchy integral or an integral in some generalised sense. For \( \nu = \frac{1}{2} \) the equation (1) reduces to

\[
f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin xy f(y) \, dy,
\]

where \( f(x) \) is said to be self-reciprocal for Sine-Transforms. In this case we shall call \( f(x) \) as an \( R_c \)-function for brevity.

We know that

\[
\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \nu y J_{\nu+1} \left(\frac{1}{2} y^2\right) \sin xy \, dy = \sqrt{2} J_{\nu+1} \left(\frac{1}{2} x^2\right)
\]

Differentiating (2) with respect to \( x \) and assuming its justification,

(1) See Bateman. See also Hardy and Littlehmarsh. The references are given at the end of the paper.
we get
\[ \sqrt{\frac{2}{\pi}} \int_0^\infty y^{3/2} J_{1/4} \left( \frac{1}{2} y \right) \cos xy \, dy = x^{3/2} J_{3/4} \left( \frac{1}{2} x^2 \right) \]  
(3)

Differentiating once again, we get
\[ \sqrt{\frac{2}{\pi}} \int_0^\infty y^{5/2} J_{1/4} \left( \frac{1}{2} y \right) \sin xy \, dy = x^{5/2} J_{1/4} \left( \frac{1}{2} x^2 \right) \]  
(4)

It appears from this integral equation that the function
\[ x^{3/2} J_{1/4} \left( \frac{1}{2} x^2 \right) \]
is \( R_4 \). But the peculiarity of this function is that it does not make the integral in (4) converge in the usual sense of the term "convergence." We proceed to show in this note that the integral (4) is summable (c, 1) and the differentiations under the integral sign are justified. Hence, if we agree to define a function "\( R_4 \) in the extended sense" as a function satisfying the equation
\[ \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \sin xy \, dy = f(x) \quad (c, 1) \]  
(5)

\[ x^{5/2} J_{1/4} \left( \frac{1}{2} x^2 \right) \] would be \( R_4 \) in this sense.

I should like to express my sincere thanks to Dr. Brij Mohan for having suggested this problem to me and for his kind help in preparing this note.

**Lemma** (1). Let \( f(x, y) \) be a continuous function of \((x, y)\) in the region \( S \) defined by

\[ a \leq x \leq a', \quad b \leq y \leq b'. \]

\( a' \) being arbitrary, and have at each point of the region \( S \), a partial differential coefficient

\[ \frac{\partial}{\partial y} f(x, y), \]

which is continuous throughout \( S \). Then, if the integral

\[ \int_a^b f(x, y) \, du, \]

converges, or is summable \((c, v)\) to \( F(y) \), and if the integral

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(1) See Recurrence formulae for Bessel functions: Watson.
(2) See for proof: Brij Mohan.
A Self-reciprocal function.

\[ \int_a^x \frac{\partial}{\partial y} f(x, y) \, dx, \]

is uniformly summable \((c, r)\) in \((b, b')\), \(F(y)\) has a differential coefficient at every point in \((b, b')\) and

\[ F'(y) = \frac{H}{1-x^2} \int_a^x \left(1 - \frac{x}{X}\right)^r \frac{\partial}{\partial y} f(x, y) \, du. \]

Now, let us consider

\[ I_1 = \sqrt{\frac{2}{\pi}} \int_0^\infty y^p \sin \left(\frac{1}{2} y^2 + \frac{3\pi}{8}\right) \sin xy \, dy, \]

for large values of \(y\) \(I_1\) behaves like

\[ \sqrt{\frac{2}{\pi}} \int_0^\infty y^p \cos \left(\frac{1}{2} y^2 - \frac{3\pi}{8}\right) \sin xy \, dy. \]

\[ = \sqrt{\frac{2}{\pi}} \int_0^\infty y^p \cos \left(\frac{1}{2} y^2 - \frac{3\pi}{8}\right) \sin xy \, dy. \]

\[ - \sqrt{\frac{2}{\pi}} \int_0^\infty y^{-1/2} \sin \left(\frac{1}{2} y^2 - \frac{3\pi}{8}\right) \sin xy \, dy. \]

\[ = \sqrt{\frac{3}{\pi}} J_1 - \sqrt{\frac{2}{\pi}} I_2, \text{ say.} \]

Now

\[ I_1 = \frac{1}{2} \int_0^\infty y^p \sin \left(\frac{1}{2} y^2 + xy - \frac{3\pi}{8}\right) \right. \sin xy \, dy. \]

\[ - \sin \left(\frac{1}{2} y^2 - xy - \frac{3\pi}{8}\right) \right) \, dy. \]

\[ = \frac{1}{2} \int_0^\infty y^p \sin \left(\frac{1}{2} y^2 + xy - \frac{3\pi}{8}\right) \, dy \]

\[ - \frac{1}{2} \int_0^\infty y^p \sin \left(\frac{1}{2} y^2 - xy - \frac{3\pi}{8}\right) \, dy. \]

Let us consider,

\[ I_3 = \int_y^\infty \left(1 - \frac{y}{y+x} \right) y^p \sin \left(\frac{1}{2} y^2 + xy - \frac{3\pi}{8}\right) \, dy. \]

Putting \(\frac{1}{2} y^2 + xy - \frac{3\pi}{8} = \omega\) we get.

(1) See Watson §7.21
where \( U \) is the value of \( \omega \) when \( y \) becomes \( Y \).

As \( \omega \) is large, so long as \( x \) lies in \( 0 < x < \beta \), it would be sufficient to consider

\[
J_3 = \int_0^\infty \left[ 1 - \frac{\sqrt{x^2 + 2\omega + \frac{3\pi}{4} - x}}{\sqrt{x^2 + 2U + \frac{3\pi}{4} - x}} \right] \left( \frac{\sqrt{x^2 + 2\omega + \frac{3\pi}{4} - x}}{\sqrt{x^2 + 2\omega + \frac{3\pi}{4} - x}} \right)^{3/2} \sin \omega d\omega.
\]

The first integral converges and the second term tends to 0 as \( U \to \infty \). Hence \( J_3 \) is summable \((c,1)\).

In the same way it could be shown that the other integral in \( I_1 \) is summable \((c,1)\).

Also

\[
I_4 = \frac{1}{2} \int_a^b \frac{1}{\sqrt{y}} \left[ \cos \left( \frac{1}{2} \sqrt{y^2 - xy - \frac{3\pi}{8}} \right) - \cos \left( \frac{1}{2} \sqrt{y^2 + xy - \frac{3\pi}{8}} \right) \right] dy.
\]

Consider

\[
\int_a^b \frac{1}{\sqrt{y}} \cos \left( \frac{1}{2} \sqrt{y^2 - xy - \frac{3\pi}{8}} \right) dy.
\]

Put \( \frac{1}{2} \sqrt{y^2 - xy - \frac{3\pi}{8}} = \omega \).

The integral becomes
For large values of $\omega$. This integral behaves as $\int^{\infty} \omega^{-\alpha} \cos \omega \, d\omega$.

which is convergent.

In the same way it could be shown that the other integral in $I_2$ is convergent. Hence $I$ is summable $(\epsilon, 1)$. And this is so far all $\varepsilon$ in $0 < \varepsilon \leq \beta$. Therefore $I$ is uniform summable $(\epsilon, 1)$ in $0 < \alpha \leq \varepsilon \leq \beta$.

Similarly we can prove that the integral

$$
\int_{0}^{\infty} y^{\alpha/2} J_{\alpha/4} \left( \frac{1}{2} y \right) \cos \alpha y \, dy
$$

is uniformly summable $(\epsilon, 1)$ in $0 < \alpha \leq \varepsilon \leq \beta$. Hence the differentiation under the integral sign in (2) and (3) is justified by Lemma III and the function $x^{\alpha/2} J_{\alpha/4} \left( \frac{1}{2} x \right)$. is $R$ in the extended sense.

Dr. Brij Mohan Mehrotra in his paper(1) has proved a theorem on functions which are self-reciprocal for cosine transforms in the extended sense. Following the lines of his theorem IV. We can prove the following theorem. It will be then seen that the above function is an example under this theorem.

**Theorem.** If (1) the integrals

$$
\int_{0}^{1} |f(x)| \, du, \quad \int_{1}^{\infty} \frac{|f(x)|}{x^{\alpha/4}} \, du
$$

are finite;

(2) the integral

$$
\phi(s) = \int_{0}^{\infty} x^{s-1} f(x) \, dx
$$

exists as a Cauchy integral at both limits for $0 < \alpha \leq \beta < \frac{1}{2}$ and is summable $(\epsilon, 1)$ for $\left| \sigma - \frac{1}{2} \right| < \alpha \leq \frac{1}{2}$;

(3) $\sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \frac{1}{Y} \int_{0}^{\varepsilon} dy \int_{y}^{\infty} f(y) \sin \alpha y \, dy = f(x)$,

So that $f(x)$ is finite for all positive $x$ and is $R$, “in the extended sense”;
then $f(x)$ is of the form

(1) See Brij Mohan (4).
f(x) = \frac{1}{2\pi} \int_{\gamma} \phi(s) x^{-i} \, ds,

where the integral is summable (e, 1) for \(|e - \frac{1}{2}| < \alpha\) and \(\phi(s)\) is regular and satisfies the condition

\[ \phi(s) = \sqrt{\frac{2}{\pi}} \phi'(s) \sin \frac{s\pi}{2} \phi(1-s) \]

for \(|\sigma - \frac{1}{2}| < \alpha\).

References.


On the Numerical Method of Solution of the Characteristic Equations.

By Moto-saburo MASUYAMA.

(Read Jan. 21, 1939).

§ 1. Introduction.

We meet special types of equations, called characteristic equations, 1° in the problem of small oscillations, or in the linear simultaneous differential equations of constant coefficients,

2° in the numerical method of solution of integral equations by Nystrøm,

3° in searching the principal axes of quadratic surfaces in dynamics of rigid,