The Standard Error of the Mean Vector.

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1. Introduction. The author has published a series of papers concerning the statistics of vector quantities, but it has been done from the pure theoretical point of view. If we want to apply these results to the observational data, we must consider the reliability of thus obtained numerical values. We shall now construct the theory of error concerning the vector quantities based upon the ordinary scalar theory. Let us consider here the standard error (the standard deviation) of the arithmetic mean vector.

2. Method of Approach. Let the observational data be $V_1, V_2, \ldots, V_n$ and its arithmetic mean vector $\overline{V}$, where

$$\overline{V} = \frac{1}{N} \sum_{i=1}^{N} V_i.$$

To maintain the special case, viz. the scalar case, we start considering the projection of $V_i$ on the arbitrary direction $e$ (a unit vector). As the projection $V_i e$ is a scalar, the standard error of its arithmetic mean is obtained by the ordinary formula:

$$m = \frac{\sigma}{\sqrt{N}} = \pm \sqrt{\frac{(V e)^2}{N}} = \pm \sqrt{\frac{\overline{V e} \cdot \overline{V e}}{N}},$$

where $V e = V_i - V$.

3. Standard Error of Mean Vector. If we take a vector from each class of observations $\{e_1\}, \{e_2\}, \ldots, \{e_n\}$ and assume that each observation has been done totally independently, then we have as the variance of the sum $\sum_{a=1}^{n} e_a$:

$$\sigma_s = \sum_{a=1}^{n} \sigma_a^2 = \sum_{a=1}^{n} e_a^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2.$$

Therefore if we observe repeatedly $N$ times one and the same vector quantity $|e|$ and each observation is carried out totally independently one after another, we have as the variance of the sum of these $N$ vectors:

$$\sigma_s^2 = N \sigma^2.$$

Accordingly the variance of the mean vector is equal to


Thus the standard error of the arithmetic mean of the projection on the direction $e$ is equal to:

$$m_e = \pm \sqrt{\Phi e}.$$

As we have shown in the previous paper, we are able to consider an ellipsoid $S$ as a geometrical representation of the $\Phi$, where the equation of $S$ is:

$$x \cdot \Phi \cdot x = 1.$$

Let the radius vector of this ellipsoid parallel to $e$ be $x_e$ and its magnitude $r_e$, then we have from (3.4) and (3.5):

$$m_e = \pm \frac{1}{r_e}.$$

As the direction $e$ is taken arbitrarily, we may consider that the standard error of the arithmetic mean vector is represented by the tensor $\Phi$ or (3.5).

**4. Some Remarks.** Obviously, as $\Phi$ is not scalar, we cannot compare the reliabilities of two or more mean vectors utilizing this tensor. But the formula (3.5) and (3.6) suggest that the volume of the ellipsoid of $S$, which is inversely proportional to the determinant of $\Phi$, can be used as a measure of such a comparison, but it is not sufficient, for if at least one characteristic value of $\Phi$ is equal to zero, it is always $\det \Phi = 0$, though other characteristic values are not equal to zero. In such a case the trace of $\Phi$, or $\frac{1}{N} \cdot e \cdot e$, can be used as a measure of comparison.

Further we want to point out that, in general, it is not sufficient to calculate only the scalar standard deviation of each independent component of the arithmetic mean vector to indicate the order of the reliability of the result, for such an aggregate of standard deviations is not independent of the selection of coordinates.

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