In conclusion, the present author expresses his sincere thanks to Messrs. Terada, Amano and Yamanouchi, research members of the Institute, for their kind guidances throughout the present calculation.

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The Totally Orthonormalised Vector Set and the Normal Form of Correlation Tensor.

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Résumé.

Introducing the totally orthonormalised vector set, we give here a general method of determination of the explicit form of the n-dimensional regression tensor and a geometrical meaning of characteristic values of a correlation tensor in normal form.

§ 1. Introduction. In our first paper\(^{1}\) concerning the measure of correlation of vector quantities, we have shown that the regression tensor between two three-dimensional vector functions \(\{A(t)\}\) and \(\{B(t)\}\) is represented in an orthogonal normalised cartesian coordinate system (O.N.C.S.) by the matrix

\[
T = \begin{pmatrix}
[a_1 b_1] & [a_1 b_2] & [a_1 b_3] \\
[a_2 b_1] & [a_2 b_2] & [a_2 b_3] \\
[a_3 b_1] & [a_3 b_2] & [a_3 b_3]
\end{pmatrix}
\begin{pmatrix}
[b_1 b_1] & [b_1 b_2] & [b_1 b_3] \\
[b_2 b_1] & [b_2 b_2] & [b_2 b_3] \\
[b_3 b_1] & [b_3 b_2] & [b_3 b_3]
\end{pmatrix}^{-1}
\]

This formula is obtained under the condition

\[
\sum_{t=1}^{n} |a(t) - T b(t)|^2 = \text{Min.} \quad (T = \text{constant tensor}).
\]

This explicit form of \(T\) has been at first inferred by intuition, considering the following correspondence:

\[
[a b] \rightarrow [a b], \quad [b b]^{-1} \rightarrow [b b]^{-1}
\]

\(^{1}\) The paper I, this Proceedings, 21 (1939), 638.
and then has been verified step by step by actual calculation in O.N.
C.S. We shall give in the following pages the explicit form of $T$ in the
equation (1.2) for the $n$-dimensional vector functions $\{A(t)\}$ and $\{B(t)\}$
by the direct method introducing the totally orthonormalised vector
set. Utilizing this result we shall define the normal form of correlation
tensor and further give a geometrical interpretation of the character-
eristic values of correlation tensor.

§ 2. Total Normalization. Let $m$ linear totally independent $n$-dimen-
sional vector functions $\{f_1(t)\}, \{f_2(t)\}, \ldots, \{f_m(t)\}$, then none of
the $n$-dimensional scattering of these functions is equal to zero, viz.

$$\det [f_i f_i] \neq 0 \quad (i = 1, 2, \ldots, m).$$

Put at first

$$u_i(t) = f_i(t),$$

then

$$[u_i u_i] = [f_i f_i].$$

As $[f_i f_i]$ and accordingly $[f_i f_i]^{-1}$ are self-conjugate and positive defi-
nite, there exists at least a self-conjugate positive definite tensor $F_1$
which satisfies the following tensor equation:

$$F_1^2 = [f_i f_i]^{-1}.$$

Put secondly

$$v_i(t) = F_1 f_i(t),$$

then we have

$$[v_i v_i] = F_1 [f_i f_i] F_1'.$$

As $F_1$ is self-conjugate and positive definite, we obtain from (2.4) and
(2.6)

$$[v_i v_i] = F_1 [f_i f_i] F_1' = F_1^2 [v_i v_i] F_1' \quad (i = 1, 2, \ldots, m).$$

(2) The paper III, this Proceedings, 22 (1940), 579.
(3) The determinant of the tensor $[f(t) f(t)]$. The paper II, this Pro-
cedings, 21 (1939), 647.
(4) The paper IV, this Proceedings, 22 (1940), 855.
(5) The paper II, loc. cit. We have proved in this second paper that a tensor of
the form $[v v]$ is semi-definite or positive definite in a wide sense. In the case treated
above $[f_i f_i]$ is positive definite in a narrow sense, for $\det [f_i f_i] \neq 0$. 

This process of normalisation may be called the total normalisation to differentiate from the ordinary normalisation of vector function. This process is equivalent to construct an orthonormal set of scalar functions \(|v_1(t)|, |v_2(t)|, \ldots, |v_n(t)|\) from the given linear independent functions \(|f_1(t)|, |f_2(t)|, \ldots, |f_n(t)|\) in O.N.C.S.\(^6\).

Put thirdly
\[
(2.8) \quad u_2(t) = f_2(t) - T_{21}v_1(t),
\]
and determine the constant tensor \(T_{21}\) by the condition
\[
(2.9) \quad [u_2, v_1] = 0,
\]
then we have from (2.8)
\[
(2.10) \quad [u_2, v_1] = [f_2, v_1] - T_{21} [v_1, v_1] = [f_2, v_1] - T_{21} = 0,
\]
or
\[
(2.11) \quad T_{21} = [f_2, v_1].
\]

As the \(n\)-dimensional scattering of the function \(|u_2|\) must not be equal to zero\(^7\) and \([u_2, u_2]^{-1}\) is self-conjugate and positive definite, we can find at least a self-conjugate positive definite tensor \(F_2\) such that
\[
(2.12) \quad F_2 = [u_2, u_2]^{-1}.
\]
If we put
\[
(2.13) \quad v_2(t) = F_2 u_2(t),
\]
it would be easy to verify the following equations
\[
(2.14) \quad [v_2, v_1] = 0 \quad \text{and} \quad [v_2, v_2] = E.
\]

Thus we can construct the totally orthonormal set of vector functions \(|v_1(t)|, |v_2(t)|, \ldots, |v_n(t)|\) from the given set of linear totally independent function \(|f_1(t)|, |f_2(t)|, \ldots, |f_n(t)|\) by the analogous method as the Schmidt' one.

§ 3. Tensorial Fourier Coefficients. We want to determine the tensorial Fourier coefficients \(\Gamma_1, \Gamma_2, \ldots, \Gamma_m\) of the vector function \(|g(t)|\) which satisfy the following condition
\[
(3.1) \quad J = \sum _{i=1}^{m} [g(t) - \sum _{i=1}^{m} \Gamma_i v_i(t)]^2 = \text{Min.} \quad (\Gamma_i = \text{constant tensors}),
\]
where \(|v_1(t)|, |v_2(t)|, \ldots, |v_m(t)|\) are the totally orthonormal set of functions cited in § 2.

As the square of a vector is equal to the trace of its square tensor,

\[\text{(6)} \quad \text{In general this process of normalisation is not equal to the one considered by E. Schmidt.}\]
\[\text{(7)} \quad \text{The paper III, loc. cit.}\]
and the minimum (the smallest value in this case!) of $J$ occurs when and only when

$$V_i \equiv [g v_i] \quad (i = 1, 2, \ldots, m).$$

Substituting (3.3) in (3.2), we have a generalised form of the Bessel's inequality, viz.

$$J_{\text{min}} = \sum_{i=1}^{m} [g v_i] [v_i g] \geq 0,$$

or

$$\sum_{i=1}^{m} [g v_i] [v_i g] = \sum_{i=1}^{m} \sum_{j=1}^{m} [g v_i] [v_j g].$$

Thus we have from (3.3)

$$V_i v_i = [g v_i] v_i$$

for the tensor $F_i$ is self-conjugate. This formula has been deduced in our third paper from actual calculation in O.N.C.S.

If we put

$$g(t) = a(t), \quad u_i(t) = b(t) \quad \text{and} \quad i u_j(t) = 0 \quad (j \geq 2),$$

then we obtain from the formula (3.6)

$$[a b] = [b b]^{-1},$$

as the regression tensor between $\{A(t)\}$ and $\{B(t)\}$ in this order.

§ 4. Correlation Tensor. We shall investigate the correlation tensor between two non-degenerated functions $\{f_i(t)\}$ and $\{f_j(t)\}$, utilizing the totally normalised functions $\{v_i(t)\}$ and $\{v_j(t)\}$ where

$$F_i = [f_i f_i]^{-1}, \quad F_j = [f_j f_j]^{-1};$$

By definition we have as the regression tensor between $\{v_i(t)\}$ and $\{v_j(t)\}$ in this order

$$[v_i v_j].$$
and as the one between \|v_2(t)\| and \|v_1(t)\| in this order

\[
(v_2, v_1).
\]

Thus we have as the correlation tensor between \{v_1(t)\} and \{v_2(t)\}

\[
R(v_1; v_2) = (v_1, v_2) = (v_1, v_2)[v_2, v_1]' = R'(v_1; v_2).
\]

It would be easy to see that none of the characteristic values of the correlation tensor is negative, as we have proved in our eighth paper\(^8\). This fact is also true for \(R(f_1; f_2)\), for

\[
F_1 = [f_1, f_2][f_1, f_2]' = F_2 = [f_2, f_1][f_2, f_1]' = F_1^{-1} = [f_1, f_2][f_2, f_1]^{-1} = F_1^{-1},
\]

that is to say, \(R(f_1; f_2)\) is equivalent to \(R(v_1; v_2)\)\(^9\). We shall call \(R(v_1; v_2)\) the normal form of correlation tensor, if both functions \{v_1(t)\} and \{v_2(t)\} are totally normalised.

§ 5. Characteristic Values of Correlation Tensor in Normal Form.

Let a characteristic vector corresponding to a characteristic value \(\lambda\) of correlation tensor in normal form \(R(v_1; v_2)\) be \(v\) and its unit vector be \(e\), then from

\[
Rv = \lambda v,
\]

we obtain

\[
(e, v)Rv = \lambda v^2 = (e, Rv)
\]

or

\[
e, Rv = \lambda e.
\]

Therefore we obtain from (4.4) and (5.3)

\[
\lambda = e, (v_1, v_2) = (v_1, v_2)' = (v_2, v_1)'
\]

which tells us that the characteristic value of \(R(v_1; v_2)\) is equal to the square of the projection of the regression tensor on the direction of the corresponding characteristic vector.

Since \(R(v_1; v_2)\) is positive definite in a wide sense and self-conjugate, we are able to consider an ellipsoid as its geometrical representation and further to give the geometrical interpretation of \(\lambda\) and \(v\) from the stand point of the extremum problem quite analogously as stated in our second paper.

\(^8\) The paper VIII, this Proceedings, 23 (1941), 199.

\(^9\) For we can interpret \(F_1\) as a real non-singular linear transformation in O.N. C.S.
§ 6. On the Limiting Case \( m \to \infty \). We have treated hitherto the problems under the assumption \( m = \text{finite} \), which is nearly satisfied in the practical statistical problems, but our results in our papers will tell us that the greater part of our theory can be extended to the vector case where the parameter \( t \) is continuous and \( m \) is infinite, under certain restriction of convergency. For example, if we define the weak convergency of \( h_m(t) \) to \( g(t) \) by the equation

\[
\lim_{m \to \infty} (g - h_m)^2 = 0,
\]

and the strong convergency of \( h_m(t) \) to \( g(t) \) by the equation

\[
\lim_{m \to \infty} (g - h_m) = 0,
\]

the majority of the usual scalar or bicalar theory of Fourier series can be extended to the vector theory with tensorial Fourier coefficients.

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The Mean Angle between Two Vector Sets.

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Résumé

It is pointed out that the Lambert' method or the Dietzius' or the Sverdrup's of determination of the mean angle between two vector sets \( \{\mathbf{u}\} \) and \( \{\mathbf{v}\} \) may give an angle \( \theta \) which is quite meaningless in practice. The author gives a stochastic condition for non-appearance of such a ghostly angle by analyzing the tensor of the type \( \mathbf{u}\mathbf{r} \cdot \mathbf{v}\mathbf{r}^{-1} \).

§1. It is often necessary to determine statistically the mean angle between two corresponding vectors; for example, the mean angle between the wind near the earth and the corresponding gradient wind. Hitherto this angle has been determined by means of the Lambert's method\(^{(1)}\), or the Dietzius'\(^{(2)}\) or the Sverdrup's\(^{(3)}\). The Lambert's formula

\[ (1) \text{ Quoted in the Dietzius' paper.} \\
(2) \text{ R. Dietzius, Anwendung der Vektorrechnung in der statistischen Meteorologie, Meteorol. Zeitschr. 32 (1915), 433.} \\