On the Disintegration of Deuterons by Neutrons.

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§ 1. Introduction and Summary.

As one of the simplest three-body-problems of the nuclear particles, the disintegration of deuterons by neutron-collision would be of interest. In this paper the cross-sections for this process have been calculated with Born approximation. The interaction potentials between nuclear particles are assumed to be a linear combination of four different exchange potentials, whose spatial dependencies are identical and expressed by a Gauss's error function. The disintegration is considered as transitions of the deuteron from its ground state to the continuous levels, caused by the interaction with the incident neutron. As the wave function for the initial state of the deuteron we have adopted also an error function and for the final states modified "Morse functions" on account of simplicity of calculations. In spite of these simplifications, however, the differential cross-sections cannot be reduced to simple integrals except for spherically-symmetrical disintegrations of the deuteron, so the numerical calculations have been made only for this case, taking two values 30 mev and 17 mev of the incident neutron energy. The total cross-sections have been shown to be for the two cases about $3 \times 10^{-25}$ cm$^2$ and $2 \times 10^{-24}$ cm$^2$, respectively.

§ 2. Formulations of the Process.

The total Hamiltonian of the system of a neutron and a deuteron is given by

$$H = -\sum_{i=1,2,3} \frac{\hbar^2}{2M} \Delta r_i + \sum_{ij} V_{ij},$$

where $r_1$, $r_2$, and $r_3$ mean the position vectors of the incident neutron, the neutron and the proton constituting the deuteron respectively. It can be transformed to the following form, except the kinetic energy of the centre of gravity of the three particles:

$$H' = -\frac{\hbar^2}{2\mu} \Delta r - \frac{\hbar^2}{M} \Delta r_3 + \sum_{ij} V_{ij}, \quad (\mu = \frac{2}{3} M)$$

when we use the coordinates $\mathbf{R} = \frac{r_1 + r_2 + r_3}{3}$, $r = r_1 - \frac{r_2 + r_3}{2}$ and $r_3=$
$r_2 - r_3$ instead of $r_1, r_2$ and $r_3$ \(1)\).

Now we expand the total wave function of the system, also except
the part which describes the motion of the centre of gravity, in two
alternative series of the eigenfunctions of the deuteron as follows:

$$\mathcal{F}(r', r_{23}) \chi(23) = \sum \rho_0 \varphi_0(r_{23}) \chi(23)$$

$$+ \sum \rho_1 \frac{1}{2} \int \varphi_{23}(r', s) \varphi_{s1}(r_{23}) \chi(23) d\kappa,$$  

$$= \sum \rho_0 \varphi_0(r_{23}) \chi(23)$$

$$+ \sum \rho_1 \frac{1}{2} \int \varphi_{23}(r', s) \varphi_{s1}(r_{23}) \chi(13) d\kappa. \quad (1a)$$

$$\left(r' = r_2 - r_1 + r_3, \quad r_{13} = r_1 - r_3. \right)$$

Here $\varphi_0(r_{23})$ is the wave function of the ground state of the deuteron,
supposed here to be the only real bound state, $\varphi_{s1}(r_{23})$ are those of the
continuous levels with energy $\frac{\hbar^2 k^2}{2M}$ and angular momentum $\ell \hbar$
and $\chi$'s are the spin functions, $s_1$ etc. denoting spin variables respectively.$\varphi_0$ and $\varphi_{s1}$ satisfy the deuteron equations

$$\left( -\frac{\hbar^2}{M} \Delta_{23} + V_{23} \right) \varphi_0(r_{23}) \chi(23) = -\varepsilon \varphi_0(r_{23}) \chi(23), \quad (2a)$$

$$\left( -\frac{\hbar^2}{M} \Delta_{23} + V_{23} \right) \varphi_{s1}(r_{23}) \chi(23) = \frac{\hbar^2}{M} k^2 \varphi_{s1}(r_{23}) \chi(23) \quad (2b)$$

respectively, $\varepsilon$ being the binding energy of the deuteron. The second
expansion is necessary in order to obtain the correct formula of the
cross-sections on account of the antisymmetrization with respect to
the coordinates of two neutrons 1 and 2. We assume that the coefficients
$F_0(r', s), G_0(r', s), F_{s1}(r', s)$ and $G_{s1}(r, s)$ of the expansions (1) take their
asymptotic forms as $r \to \infty$ as follows:

$$F_0(r', s) \sim e^{i k_0 r'} \frac{\hbar^2}{\kappa} f_s(\theta, s) \chi(1), \quad G_0(r', s) \sim e^{i k_0 r'} \frac{\hbar^2}{\kappa} g_s(\theta, s) \chi(1),$$

$$F_{s1}(r', s) \sim e^{i k_0 r'} f_{s1}(\theta, s) \chi(1), \quad G_{s1}(r, s) \sim e^{i k_0 r'} g_{s1}(\theta, s) \chi(1),$$

where $k_0$ and $k$ satisfy the relation:

$$\frac{\hbar^2}{2\mu} k^2 - \varepsilon = \frac{\hbar^2}{2\mu} k_0^2 = \frac{\hbar^2}{M} \kappa^2,$$  

expressing the conservation of energy. Then, in the system which is
moving with the centre of gravity of the deuteron ($D$-system), the
differential cross-sections corresponding to scattering of the incident neutron in the direction of $\theta$ relative to its incidence into the solid angle $d\omega$ with definite spin direction, the disintegrated deuteron having its relative kinetic energy $\hbar^2 c^2/2\mu$ and angular momentum $\ell\hbar$, becomes

$$I_{s_l}(\theta, s_l) d\omega d\kappa = \frac{k}{k_0} |f_{s_l}(\theta, s_l) - g_{s_l}(\theta, s_l)|^2 d\omega d\kappa,$$

(4)

$f_{s_l}$ and $g_{s_l}$ are obtained as usually from the asymptotic formulae:

$$F_{s_l}(r, s_l) \sim -\frac{1}{4\pi} \frac{e^{i\mu r_0}}{r} \int e^{-ikr} dr_0 \sum \chi(23)$$

$$\times \int \varphi_{s_l}(r_23) \sum_{i=1,2} V_{i1} F(r, r_{23}) \chi(123) dr_{23},$$

(5a)

$$G_{s_l}(r, s_l) \sim -\frac{1}{4\pi} \frac{e^{i\mu r_0}}{r} \int e^{-ikr} dr_0 \sum \chi(23)$$

$$\times \int \varphi_{s_l}(r_23) \sum_{i=1,2} V_{i2} F(r', r_{13}) \chi(213) dr_{13}.$$  

(5b)

§ 3. Approximations and Assumptions.

In the preceding section we have made no essential assumptions and approximation and the cross-sections are given by (4), (5a) and (5b) exactly if the true wave function $\varphi(r, r_13)$ is substituted in the integrands of (5a) and (5b). But since the true wave function can not be found we must here introduce a few assumptions and approximations in order to proceed with further calculations.

1. Firstly we have restricted our whole calculations within the range of validity of Born's first approximation, so that the total wave function $\varphi(r, r_13) \chi(123)$ is replaced in the integrands of (5a) and (5b) by the product of the plane wave and the wave function of the ground state of the deuteron, i.e.

$$\varphi(r, r_13) \chi(123) \rightarrow \chi(1) e^{ik_0 r} \varphi_0(r_13) \chi(23).$$

(6)

$\chi(1)$ stands for either of two spin functions of the incident neutron, $\alpha(1)$ and $\beta(1)$, and $\chi(23)$ represents one of three spin functions corresponding to the triplet state of the deuteron, $\alpha(1) \alpha(2)$, $1/\sqrt{2} (\alpha(1) \beta(2) + \beta(1) \alpha(2))$ and $\beta(1) \beta(2)$. The calculations in this paper, therefore, should be valid only for high energies of the incident neutron compared with the binding energy of the deuteron, i.e. for

$$\frac{\hbar^2}{2\mu} k_0 \gg \varepsilon.$$  

(7)

(2) e.g., see, Mott and Massey, Theory of Atomic Collisions, p. 88.
(3) Compound nucleus models are, of course, out of consideration in this paper.
Further we have only calculated the cross-sections for the case $l=0$, which corresponds to the disintegration of the deuteron into the $^3S$ and $^1S$ states. If the kinetic energy of the disintegrated deuteron is small enough to satisfy the condition:

$$\frac{1}{\kappa} \gg a, \quad (8)$$

$a$ being the range of the interaction potentials, so the disintegration probabilities into the states $l \neq 0$ are negligible. Both conditions (7) and (8) are satisfied fairly well for the incident energy 17 mev for a most part of $\kappa$-values, but for 30 mev this is not the case. (See Table 1).

2. The interactions between nuclear particles are assumed to be symmetrical for proton and neutron and of the form

$$V_{ij} = J(r_{ij})(c_u + c_u P_u + c_u P_u Q_u + c_u Q_u)$$

where $P_u$ and $Q_u$ are the exchange operators with respect to space- and spin-coordinates. Inserting this into (5a) and (5b), $f_{\kappa} (\theta)$ and $g_{\kappa,l} (\theta)$ give rise to the following three types of integral:

$$I_n = \int e^{-ikr} dr \int \varphi_{\kappa,l}(r_{12}) J(r_{12}) e^{ikr} \varphi_0 (r_{23}) dr_{23}, \quad (9a)$$

$$I_u = \int e^{-ikr} dr \int \varphi_{\kappa,l}(r_{12}) J(r_{12}) e^{ikr} \varphi_0 (r_{13}) dr_{13}, \quad (9b)$$

$$I_x = \int e^{-ikr} dr \int \varphi_{\kappa,l}(r_{12}) J(r_{12}) e^{ikr} \varphi_0 (r_{13}) dr_{13}, \quad (9c)$$

the last one comes only from the exchange wave $g_{\kappa,l}$.

3. Next we have adopted as the spatial part of the interaction potentials

$$J(r) = -V_0 e^{-r^2/\alpha^2}. \quad (10)$$

For this potential, the deuteron equation (2a) or (2b) can not be integrated analytically. As the wave function of the ground state of the deuteron, therefore, also an error function has been assumed for the convenience of calculations:

$$\varphi_0 (r) = \frac{2}{\sqrt{\pi}} \left( \frac{\alpha}{\kappa} \right)^{\frac{3}{2}} e^{\frac{\alpha^2 r^2}{2}}. \quad (11a)$$

From the same reason, for the continuous levels wave function a modified "Morse function"," given by

$$\varphi_{\kappa,l} (r) = \frac{1}{\sqrt{2 \pi}} \frac{\sin (\kappa r + \delta) - e^{-\alpha^2 r^2} \sin \delta \cos \kappa r}{r} \quad (11b)$$

has been assumed, which is normalized with respect to the momentum.

(4) P. Morse, Phys. Rev. 51 (1937), 1003.
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The force range $a$ and the radius of the deuteron $a/\alpha$ were determined so as to satisfy the variation principle and to give the correct value of the binding energy of the deuteron, namely such as

$$\frac{\partial H_d}{\partial \alpha} = 0,$$

and

$$H_d = \int \phi_0(r) H_d \phi_0(r) dr = -\varepsilon,$$

provided the depth of the potential $V_0$ is given. ($H_d$ represents the Hamiltonian operator in the left side of eq. (2a)). For given values $\varepsilon = 2.17 \text{ meV} = 4.25 \text{ mc}^2$ and $V_0 = 72 \text{ mc}^2$ the above two equations give the values

$$\frac{a}{\alpha} = 1.53 a = 3.64 \times 10^{-13} \text{ cm},$$

$$a = 2.38 \times 10^{-13} \text{ cm}.$$

The phase shift $\delta$ in the wave function (11b) was determined by replacing the Gaussian potential (10) with the rectangular hole potential $J(r) = -V_0'$ for $r < a$, $J(r) = 0$ for $r > a$. Then from the condition of continuity of the wave function at $r = a$, $\delta^\pm$ are calculated as a function of $\kappa$ as follows:

$$\frac{\tan(\kappa \delta^\pm a)}{\kappa \delta^\pm} = \frac{\tan(\kappa a + \delta^\pm)}{\kappa},$$

with

$$\frac{\hbar^2}{M} \kappa^2 \mp V_0' + \frac{\hbar^2}{M} \kappa^2.$$

The $+$ signs correspond to the triplet state and the $-$ signs to the singlet, so that

$$V_0^+ = V_0'(c_w + c_M + c_H + c_B) = V_0',$$

$$V_0^- = V_0'(c_w + c_M - c_H - c_B) = \frac{V_0'}{2},$$

and $V_0'$ is adjusted as $\int_0^a V_0' dr = \int_0^\infty V_0 e^{-\alpha^2} dr$, i.e. $V_0' = (\sqrt{\pi}/2) V_0$. The ratio of interactions has been taken, according to Kemmer-Volz\(\textsuperscript{5}\), as follows:

$$c_w = \frac{1}{12}, \quad c_M = \frac{5}{6}, \quad c_H = \frac{1}{6}, \quad c_B = \frac{5}{12}. \quad (12)$$

As will be mentioned in final remarks our results do not depend upon the choice of this ratio sensitively.


Inserting (10), (11a) and (11b) into (9) and carrying out the integrations we can transform these three integrals in to the following forms;

\[ I_n = A e^{-\kappa''} I(\alpha^2, k'), \]
\[ I_u = A (1 + \alpha^2)^{-1/2} e^{-\kappa''/2(1 + \alpha^2)} I(\alpha^2/1 + \alpha^2, k'''), \]
\[ I_v = A (1 + \alpha^2)^{-1/2} e^{-\kappa''/2(1 + \alpha^2)} K, \]

with

\[ I(\alpha^2, k') = \frac{1}{k'} \{ \cos \delta [I_1(\alpha^2, \kappa - k') - I_2(\alpha^2, \kappa + k')] \]
\[ + \sin \delta [I_1(\alpha^2, \kappa + k') - I_2(\alpha^2, \kappa - k') - I_1(1 + \alpha^2, \kappa + k') - I_2(1 + \alpha^2, \kappa - k')] \}
\[ + I_2(1 + \alpha^2, \kappa - k')] \}, \]
\[ K = \sin \delta \left[ \frac{2}{k''^2 - \kappa^2} \frac{I(k'' + \kappa) + I(k'' - \kappa)}{k''} \right], \]

and

\[ A = -\frac{1}{2} \pi^3 \gamma \alpha^2 (4\pi)^{3/2}. \]

\[ I_1(\alpha^2, x) \] and \[ I_2(\alpha^2, x) \] are two single integrals defined by

\[ I_1(\alpha^2, x) = \int_0^\infty e^{-xt} \sin x dt = \frac{1}{\alpha} I_1(x), \]
\[ I_2(\alpha^2, x) = \int_0^\infty e^{-xt} \cos x dt = \frac{1}{\alpha} I_2(x), \]

\[ k', k'', k''' \] and \[ k^{iv} \] are the magnitudes of the following vectors measured by the unit \( 1/a, \)

\[ K' = \frac{1}{2} (K_0 - K), \]
\[ K'' = \frac{1}{2} K_0 + K, \]
\[ K^{iv} = K_0 + \frac{1}{2} K, \]
\[ K''' = \frac{1}{2(1 + \alpha^2)} \{ (1 + 2\alpha^2)K_0 - (1 - \alpha^2)K \}. \]

The calculations of the differential cross-sections are then reduced to evaluations of \( I_1(x) \) and \( I_2(x) \) for several values of \( x \), which can be obtained from their definitions (14) for given magnitudes and given directions of \( K \). \( I_2(x) \) can be integrated analytically so that

\[ I_2(x) = \frac{1}{2} \pi e^{-x^2}, \]

while \( I_1(x) \) allows only numerical calculation; we have obtained this\(^6\)

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\(^6\) S. Oka has pointed out in his recent paper that this function was already calculated and tabulated by T. Terazawa. See ZS. f. Phys. 116 (1940), 642.
by integration of the differential equation
\[
\frac{dI_i}{dx} = \frac{1}{2} (1-xI_i).
\]
Both \(I_1(x)\) and \(I_2(x)\) are calculated from \(x=0\) to \(x=6.0\) with the interval 0.1 and tabulated up to the fourth differences for the purpose of the interpolations.

§5. Final Form of the Differential Cross-sections in \(D\)-system.

Finally the differential cross-sections \(I_{\phi\theta}(\theta, s)\) in (4) are summed over the two spin directions of the scattered neutron and the averaged
over the two spin directions of the incident neutron as well as the three spin states of the ground state of the deuteron (λ(1) and λ(2)) in (6)). Thus the differential cross-sections in D-system, corresponding to disintegration of the deuteron into the continuous energy levels with momentum lying between $\kappa$ and $\kappa + d\kappa$, the incident neutron being scattered into the solid angle $d\omega$ in the direction of $\theta$ relative to its incidence, are given by the following formula:

$$I_{\kappa}(\theta)\,d\omega\,d\kappa = \frac{k}{k_0} \left( -\frac{1}{4\pi} \frac{2\mu}{\hbar^2} \right)^2 \Phi d\omega\,d\kappa;$$

where $\Phi$ is expressed in terms of the integrals defined by (9a), (9b) and (9c) as:

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$0^\circ$</th>
<th>$30^\circ$</th>
<th>$60^\circ$</th>
<th>$90^\circ$</th>
<th>$120^\circ$</th>
<th>$150^\circ$</th>
<th>$180^\circ$</th>
<th>$\kappa / 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r = 0$</td>
<td>0.023</td>
<td>0.025</td>
<td>0.020</td>
<td>0.023</td>
<td>0.030</td>
<td>0.043</td>
<td>0.065</td>
<td>0.087</td>
</tr>
<tr>
<td>$r = 0.2$</td>
<td>0.0050</td>
<td>0.0055</td>
<td>0.0078</td>
<td>0.0042</td>
<td>0.0037</td>
<td>0.0029</td>
<td>0.0083</td>
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</tr>
<tr>
<td>$r = 0.4$</td>
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<td>0.0077</td>
<td>0.0110</td>
<td>0.0034</td>
<td>0.0717</td>
<td>0.2410</td>
<td>0.4447</td>
<td></td>
</tr>
<tr>
<td>$r = 0.6$</td>
<td>0.0068</td>
<td>0.0100</td>
<td>0.0126</td>
<td>0.0278</td>
<td>0.1172</td>
<td>0.6035</td>
<td>1.6010</td>
<td>0.3837</td>
</tr>
<tr>
<td>$r = 0.8$</td>
<td>0.0161</td>
<td>0.0148</td>
<td>0.0140</td>
<td>0.0270</td>
<td>0.1317</td>
<td>0.1703</td>
<td>5.4581</td>
<td>1.59</td>
</tr>
<tr>
<td>$r = 0.9$</td>
<td>0.0030</td>
<td>0.0261</td>
<td>0.0169</td>
<td>0.0227</td>
<td>0.1235</td>
<td>1.2032</td>
<td>5.2723</td>
<td>0.89</td>
</tr>
<tr>
<td>$r = 1.0$</td>
<td>0.0752</td>
<td>0.0568</td>
<td>0.0237</td>
<td>0.0174</td>
<td>0.0349</td>
<td>1.0730</td>
<td>4.8701</td>
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</tr>
<tr>
<td>$r = 1.8$</td>
<td>0.0254</td>
<td>0.1308</td>
<td>0.0385</td>
<td>0.0131</td>
<td>0.0388</td>
<td>0.7619</td>
<td>3.2723</td>
<td>0.89</td>
</tr>
<tr>
<td>$r = 0.9$</td>
<td>0.0026</td>
<td>0.0703</td>
<td>0.0757</td>
<td>0.0104</td>
<td>0.0255</td>
<td>0.4241</td>
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<td></td>
</tr>
<tr>
<td>$r = 1.0$</td>
<td>4.1500</td>
<td>2.0712</td>
<td>0.2453</td>
<td>0.0099</td>
<td>0.0028</td>
<td>0.1158</td>
<td>0.5745</td>
<td>0.00</td>
</tr>
</tbody>
</table>

$I(\theta) = 0.4673 \times 0.2550 \times 0.0549 \times 0.0200 \times 0.1195 \times 0.0955 \times 3.0438$
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\[ \Phi = \frac{1}{64} \left( 83(I_t)^2 - 56 I_t E + 48(I_s)^2 + \frac{1}{9}(10I_t^2 - I_t + 12I_s^2) \right), \]

where the upper suffices \( t \) and \( s \) correspond to the triplet- and the singlet-disintegration respectively.

§ 6. Practical Calculations.

For the two values 30 mev and 17 mev of the incident neutron energy in laboratory system, the differential cross-sections in \( D \)-system were calculated for seven angles \( \theta = 0^\circ, 30^\circ, 60^\circ, 90^\circ, 120^\circ, 150^\circ, 180^\circ \) and for eleven and six values of \( k \) with equal intervals from \( k = 0 \) to \( k = k_{\max} \) for the two cases respectively. The results are shown in Table 1. and Fig. 1. The corresponding \( \kappa \)-values, denoted in the last column of the Tables, show that in the case 17 mev the condition (8) is fulfilled for almost all values of \( \kappa \), so that in this case contributions to the cross-sections from higher values of \( l \) are negligible. For 30 mev, the condition (8) are not satisfied for the most part of \( \kappa \)-values and \( P \)-disintegration and others would contribute to the cross-sections.

The angular distributions for the scattered neutron are obtained by use of the relation:

\[ I(\theta) = \int_0^{k_{\max}} I_k(\theta) dk. \]

They are shown in the last rows in Table 1. and in Fig. 2. In order to give the total number of neutrons to be observed we have to add those of the disintegrated neutron \( (I'(\theta)) \) to \( I(\theta) \). \( I'(\theta) \) are shown also in Fig. 2., which should be equal to those of the proton and constant for all angles, since only \( S \)-disintegration is considered. The total cross-sections were then obtained from

\[ Q = \int_0^\pi 2\pi I(\theta) \sin \theta d\theta = 4\pi I'(\theta), \]

which give the values \( 3.2 \times 10^{-25} \text{ cm}^2 \) for 30 mev and \( 2.1 \times 10^{-24} \text{ cm}^2 \) for 17 mev respectively. The former value should be raised if the \( P \)-disintegration etc. are taken into consideration.
For comparison with experiment we need to turn from the $D$-system, which is moving with the centre of gravity of the particles 2 and 3, to the laboratory system, in which the deuteron is initially at rest.\(^{(7)}\) Let $k_1$ denote the wave vector of the scattered neutron in this system. Then we have $k_1 = k_0/2 + k$, since the wave function which describes the motion of the centre of gravity is written as $e^{i\frac{1}{2}k_0 R} = e^{ik_0 \frac{r_1 + r_2 + r_3}{2}}$. The scattering angle $\theta$ that means the angle between $k_1$ and $k_0$ is given, therefore, by the relation

$$\tan \theta = \frac{k \sin \theta}{k_0/2 + k \cos \theta},$$

remembering that $\theta$ is the angle between $k$ and $k_0$. $\theta$ is not only a function of $\theta$, but also depends upon $k$ except when $\theta = 0^\circ$ and $\theta = 180^\circ$, so that the angular distribution in this system would be some different one from $I(\theta)$. In general, forward scatterings are increased and especially neutrons with $k < k_0/2$ are not scattered backwards at all. Also the angular distribution of the disintegrated neutron or protons is no more spherically symmetrical as $I'(\theta)$ above. The maximum value of $k_1$ in the direction $\theta = 0^\circ$ is given by

$$\left(\frac{k_1}{k_0}\right)_{\text{max}} = \frac{k_0}{2} + k_{\text{max}} = 2.03 \text{ for } 17 \text{ mev},$$

and in the direction $\theta = 180^\circ$.

\(^{(7)}\) See Appendix in this paper.
(k_1)_{\text{max}} = k_{\text{max}} - \frac{k_0}{2} = 0.581/a \quad \text{(for 17 mev)}.

These values correspond to the energies 14.8 mev and 1.2 mev respectively. The energy of the disintegrated neutron or proton takes also the same maximum value, since

\[(k_2)_{\text{max}} = \frac{k_0}{2} + \left( \frac{k}{2} + \kappa \right)_{\text{max}} = \frac{k_0}{2} + k_{\text{max}}, \quad \text{for } \theta = 0^\circ,\]

\[-\left( \frac{k}{2} + \kappa \right)_{\text{max}} = \frac{k_0}{2} = k_{\text{max}} - \frac{k_0}{2}. \quad \text{for } \theta = 180^\circ.\]


The disintegration of deuteron by neutron-collision has not yet been observed experimentally, but from our results it seems to be expected that the measurement of the cross-sections for this process would be possible. As seen in Fig. 2, the cross-sections for the backward scattering of the incident neutron in D-system are very large compared with others in both cases and this comes mainly from the predominance of the pure exchange waves \(I_e\) in (9c) or (13c), that has a steepest maximum for \(\theta = \angle(K_1K) = 180^\circ\), where \(k'', k''\) and especially \(k'' - \kappa^2\) take the smallest values. It is probably true, therefore, that this fact would be almost unchanged not only for any value of \(k_0\), but also for any choice of the ratio of the four interactions \(c_k, c_M, c_H\) and \(c_B\), since the coefficients of \(I_e\) and \(I_e^*\) are not dependent upon these four constants. Thus we may conclude that the angular distribution in D-system would have probably, for all incident energies, not much different form from above two cases and the total cross-section would be not much dependent on the choice (12).

In conclusion, the author has to mention that the suggestion of this problem as well as the fundamental formulations are entirely due to Mr. M. Kotani, for what the author wishes to express his hearty thanks to him and also to Prof. T. Yamanouchi and Mr. T. Miyazima for their kind advices and encouragement.

Appendix. Differential Cross-sections in Laboratory System.

In order to obtain the expression for the differential cross-sections in laboratory system it is more convenient to use the method of time-dependent perturbation theory. For the purpose of comparison we shall begin with two-body collision.

Suppose that the particle 2 (mass \(M_2\)) is initially at rest and the particle 1 (mass \(M_1\)) is moving with constant velocity \(v_0 = k_0h/M_1\), so
that the wave function of the total system is expressed by a plane wave \( e^{ik_1 r_1} \), assumed to be normalized for unit volume. The wave function at any time can be expanded by the products of two plane waves in such a way as

\[
\Psi(r_1, r_2, t) = \sum_{k_1} \sum_{k_2} a_{k_1, k_2}(t) e^{ik_1 r_1} e^{ik_2 r_2},
\]

and we may put \( a_k(t) = c_k(t) e^{-i\frac{E_k t}{\hbar}} \), denoting a set of \( k_1 \) and \( k_2 \) by \( k \), and energy by \( E_k \). \( c_k(0) = 0 \) for all values of \( k_1 \) and \( k_2 \) except \( c_{k_0}(0) = 1 \).

Now, owing to an interaction, say \( V(r_1 - r_2) \), dependent only on the relative position of the two particles, the total system becomes to have some probabilities of finding itself in other states. According to the general theory the probability amplitude \( c_k(t) \) is given by

\[
c_k(t) = V_{k_0} \frac{1 - e^{-i\frac{E_k t}{\hbar}}}{E_k - E_{k_0}}
\]

in the first approximation, where \( V_{k_0} \) is the matrix element of the interaction \( V(r_1 - r_2) \) taken between the initial and the final state. \( V_{k_0} \) is written in this case

\[
V_{k_0} = \int \int e^{-i(k_1 r_1 + k_2 r_2)} V(r_1) r_1 e^{ik_0 r_1} dr_1 dr_2,
\]

and the energy is

\[
E_k = \frac{k^2}{2M_1} + \frac{\hbar^2}{2M_2} k_2^2.
\]

Taking new variables \( R = \frac{(M_1 r_1 + M_2 r_2)(M_1 + M_2)}{M_1} \) and \( r = r_1 - r_2 \), it follows immediately that \( V_{k_0} \) vanishes unless \( k_2 = k_0 - k_1 \), what shows the conservation of momentum. \( V_{k_0} \) becomes then

\[
\int e^{i(k_1 - k_0) r} V(r) dr.
\]

As the transition probability we have

\[
|c_{k_1}(t)|^2 = |V_{k_0}|^2 \frac{4\sin^2 \frac{E_k - E_{k_0} t}{2\hbar}}{|E_k - E_{k_0}|^2} \quad (1)
\]

To obtain a practically significant result, however, we must integrate this over small range of the final momentum of the particle 1 and then we have, if \( \rho(k_1) \) is the density function of the final state,

\[
\sum |c_{k_1}(t)|^2 = \int |c_{k_1}(t)|^2 \rho(k_1) dk_1.
\]

The value of \( \rho(k_1) \) is given by
\[ \rho(k_1)dk_1 = \frac{k_1^3}{(2\pi)^3} dk_1 d\omega_1 = \frac{k_1^3}{(2\pi)^3} \frac{dE_k}{dE_i} \frac{dE_i}{dk_i} d\omega_1. \]

According to the denominator \(|E_k - E_0|^2\) in (1) we may replace the factor \(|V_{ik}|^2 k_i^3 / dE_k / dk_i\) in the integrand of (2) by its constant value for \(E_k = E_0\) and then using the formula \(\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi\), we obtain

\[ \int |c_{k_1}(t)|^2 \rho(k_1) dk_1 = \frac{t}{(2\pi)^3} \left( |V_{ik}|^2 k_i^3 \right) \frac{dE_k}{dk_1} \bigg|_{E_k = E_0} d\omega_1. \]

The corresponding cross-section is, from its definition, given by

\[ \sigma(\theta) = \int |c_{k_1}(t)|^2 \rho(k_1) dk_1 / t \nu_0. \]

The value of \(k_1\) is to be determined as a function of the angle \(\theta = \angle(k_0k_1)\) from

\[ E_k - E_0 = \frac{k_1^2}{2M_1} k_1^2 + \frac{k_0^2}{2M_2} (k_0 - k_1)^2 - \frac{k_0^2}{2M_1} k_0^2 = 0, \]

showing the conservation of energy.

For the special case \(M_1 = M_2 (=M)\), this equation becomes

\[ \frac{k_0^2}{M} k_1 (k_1 - k_0) = 0, \]

giving \(k_1 = k_0 \cos \theta (0 \leq \theta \leq \pi/2)\), and then

\[ \left( \frac{dE_k}{dk_1} \right)_{k_1 = k_0} = \frac{k_0^2}{M} k_0 \cos \theta. \]

With these values we have

\[ \sigma(\theta) = \left( \frac{M}{2\pi \hbar^2} |V_{ik}| \right)^2 \cos \theta d\omega_1. \]  

(1a)

In the relative coordinates system the wave vectors \(\vec{k}\) and \(\vec{k}_0\) are introduced in the place of \(k_1\) and \(k_0\), where

\[ \vec{k} = k_1 - \frac{M_1}{M_1 + M_2} k_0, \quad \vec{k}_0 = \frac{M_2}{M_1 + M_2} k_0. \]

Since

\[ E_k - E_0 = \frac{\vec{k}^2}{2\mu} (\vec{k}^2 - \vec{k}_0^2) \quad \text{and} \quad \frac{dE_k}{dk} = \frac{\vec{k}^2}{\mu} \]

with \(\mu = M_1 M_2 / (M_1 + M_2)\), we have, as the corresponding formula of the cross-section in that system,
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\[ \sigma(\theta) = \frac{1}{(2\pi)^2 \hbar k_{0}\hbar} \int |e_k(t)|^2 \rho(k) d\vec{k} \]

\[ = -\frac{1}{2\mu} \left( \frac{dE_k}{dk} \right)^2 d\omega \]

\[ = \frac{1}{4\pi} \frac{2\mu}{\hbar^2} V_{10}^2 d\omega = \left| f(\theta) \right|^2 d\omega, \]

writing \[ \theta = \angle(\vec{k}, \vec{K}_0). \]

The last equation shows that this value coincides with the well known formula obtained from the usual collision theory (within the range of Born approximation) and can be easily transformed into (Ia) by the relation \[ \theta = 2\theta \] in the case of \[ M_1 = M_2 = M^*. \]

Let us now proceed in the three-body problems as treated in the text. Here the initial state is represented by

\[ \rho_0(r_1, r_2, r_3, t) = e^{-\frac{i}{\hbar} \int \omega d\omega} e^{i\hbar r_1 \psi_0(r_23)} \]

and the final state by

\[ \rho_\omega(r_1, r_2, r_3, t) = e^{-\frac{i}{\hbar} \int \omega d\omega} e^{i\hbar r_1 \psi_\omega(r_23)}, \]

which is assumed temporarily to belong to a discrete level. The matrix element of any interaction potential written as \[ V(r_1-M_2r_2+M_3r_3/M_2+M_3, r_23) \]

vanishes unless \[ \omega' = \omega_0 - \omega \] as before. Hence we have

\[ E_k - E_0 = \frac{\hbar^2}{2} \left( \frac{1}{M' + 1} \right) k_{0}^2 + \frac{\hbar^2}{2} \left( \frac{1}{M' + 1} \right) k_{0}^2 - \frac{\hbar^2}{M'} (k_{0} k_{0} + \epsilon_\omega - \epsilon_0), \]

and

\[ \frac{dE_k}{dk} = \frac{\hbar^2}{M' M} (Mk - k_{0} \cos \theta). \]

\[ \epsilon_0 \text{ and } \epsilon_\omega \text{ denoting the energy of the eigenstate } \psi_0 \text{ and } \psi_\omega, \text{ and } \theta = \angle (k_{0}, k_{0}). \]

For the special case of \[ M_1 = M_2 = M_3 \] \( M = 3M_1 \text{, } M' = 2M_1 \) these equations become

\[ E_k - E_0 = \frac{\hbar^2}{2M_1} \left[ \frac{3}{2} (k^2 - (k_{0} k_{0} - k_{0}^2) \right] + \epsilon_\omega - \epsilon_0 \]

and

\[ \frac{dE_k}{dk} = \frac{\hbar^2}{2M_1} (3k - k_{0} \cos \theta). \]

The differential cross-section in this case is therefore given by

\[ \sigma(\theta) \approx \frac{1}{(2\pi)^2 \hbar k_{0}\hbar} \int |V_{10}|^2 e^{-\frac{k^2}{2M_1}} \left( \frac{3k - k_{0} \cos \theta}{\hbar^2} \right) d\omega \]

\[ = \frac{9}{2} \frac{k}{2k_0} \frac{k}{3k - k_{0} \cos \theta} \left( \frac{1}{4\pi} \right) \frac{2M_1/3 |V_{10}|}{\hbar^2} \left( -1 \right) \left( \frac{2M_1/3 |V_{10}|}{\hbar^2} \right)^2 d\omega, \]

*See, e.g. Mott & Massey, Theory of Atomic Collisions, p. 114.*
where the value of $k$ must be determined from the equation (3).

In the relative coordinates system corresponding to the above, the momentum vectors $k_0$ and $k$ are replaced by new vectors $k_0$ and $k$, defined by

$$k_0 = \frac{M'}{M} k_0, \quad k = \frac{M}{M} k_0.$$

Then

$$E_k - E_0 = \frac{\hbar^2}{2\mu} (k^2 - k_0^2) + \varepsilon_k - \varepsilon_0, \quad \frac{dE_k}{dk} = \frac{\hbar^2}{M} k,$$

with $\mu = M_1 M'/M$, and consequently

$$\sigma(\theta) = \frac{1}{2\pi} \frac{M_1}{(M/M') k_0} \cdot |V_{ko}| \cdot \frac{k^2 d\omega}{(k^2/\mu) k} = \frac{k}{k_0} \left( \frac{-1}{4\pi} \frac{2\mu}{\hbar^2} |V_{ko}| \right)^2 d\omega.$$  \hspace{1cm} (IIb)

If the final state belongs to a continuous level, so that expressions (IIa) and (IIb) should merely replaced by the following respectively:

$$\sigma(\theta) d\kappa = \frac{9}{2} \frac{k}{k_0} \frac{k}{3k_0 \cos \theta} \left( \frac{-1}{4\pi} \frac{2\mu}{\hbar^2} |V_{ko}| \right)^2 d\omega d\kappa, \hspace{1cm} (IIa')$$

$$\sigma(\theta) d\kappa = \frac{k}{k_0} \left( \frac{-1}{4\pi} \frac{2\mu}{\hbar^2} |V_{ko}| \right)^2 d\omega d\kappa. \hspace{1cm} (IIb')$$

The former is the required formula for the differential cross-section in the laboratory system, while the latter is identical with the expression (15) in the text. The direct transformation from the latter to the former will not be impossible, but it needs much complicated reductions.

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