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1. We shall first suppose the temperature \( u \) depends upon one coordinate \( x \) only. Putting

\[
L(u) \equiv \frac{d^2u}{dx^2} - \lambda u,
\]

where \( \lambda^2 \) denotes a constant, we have, for Green's function, satisfying the condition

\[
\lim_{\epsilon \to 0} \frac{dG}{dx} \bigg|_{x-\xi + \epsilon} - \lim_{\epsilon \to 0} \frac{dG}{dx} \bigg|_{x-\xi - \epsilon} = -1,
\]

and vanishing at the boundaries \( x = \pm \infty \),

\[
G(x, \xi) = \begin{cases} 
\frac{1}{2\lambda} e^{-\lambda(x-\xi)}, & \text{for } x > \xi, \\
\frac{1}{2\lambda} e^{-\lambda(x+\xi)}, & \text{for } x < \xi.
\end{cases}
\]

2. The above expression can be put in the following form in the case of \( \xi = 0 \):

\[
G(x, 0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\cos \mu x}{\lambda^2 + \mu^2} \, d\mu.
\]

In this, taking \( e^{i\mu x} \) for \( \cos \mu x \) and calculating the residue around the pole \( \mu = +i\lambda \) when \( x > 0 \), and around \( \mu = -i\lambda \) when \( x < 0 \), we arrive at (3).

If \( \xi \geq 0 \), substituting \( x - \xi \) for \( x \) in (4)

\[
G(x, \xi) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos \mu(x - \xi)}{h^2 + \mu^2} \, d\mu = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos \mu(x - \xi)}{h^2 + \mu^2} \, d\mu. \tag{5}
\]

If \( G=0 \) at the boundary \( x=0 \), instead of \( x=-\infty \), then using the method of image we can get

\[
G(x, \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos \mu(x - \xi) - \cos \mu(x + \xi)}{h^2 + \mu^2} \, d\mu
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \mu x \sin \mu \xi}{h^2 + \mu^2} \, d\mu. \tag{6}
\]

3. The above result can be also arrived at by making use of the normal function. Put

\[
\mathcal{A}(u) \equiv \mathcal{I}(u) + \lambda u = \frac{d^2 u}{dx^2} + (\lambda - h^2)u, \tag{7}
\]

where \( \lambda \) denotes a parameter, and let the normal function of \( \mathcal{A}(u)=0 \) be \( y_{(n)}^{(m)}(x) \) and the corresponding characteristic number (Eigenwert) be \( \kappa_{(n)}^{(m)} \). Taking

\[
\kappa_{(n)}^{(m)} = \lambda - h^2,
\]

\( \mathcal{A}(u)=0 \) can be put in the form

\[
\frac{d^2 u}{dx^2} + \kappa_{(n)}^{(m)} u = 0, \tag{8}
\]

whose normal function we shall denote by \( u_{(n)}^{(m)} \) and characteristic number by \( \kappa_{(n)}^{(m)} \). Then, we have

\[
\kappa_{(n)}^{(m)} = h^2 + \kappa_{(n)}^{(m)}, \tag{9}
\]

and \( y_{(n)}^{(m)}(x) = u_{(n)}^{(m)} \).

Consequently, \( G(x, \xi) \) can be expressed, by the theory of integral equation, as in the following form:

\[
G(x, \xi) = \sum \frac{y_{(n)}^{(m)}(x)y_{(n)}^{(m)}(\xi)}{\kappa_{(n)}^{(m)}} = \sum \frac{u_{(n)}^{(m)}(x)u_{(n)}^{(m)}(\xi)}{h^2 + \kappa_{(n)}^{(m)}}. \tag{10}
\]

If the source is continuously distributed with the density \( f(x) \), then we have by integration

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, \xi) f(x) \, dx \, d\xi.
\]
and if the doublet is distributed continuously,

\[ G(x, \xi) = \sum_{m} \frac{u_m(x)}{h^2 + \kappa_n^2} \int f(\xi) u_m(\xi) d\xi; \]  

Similar relations as (10), (11), (12) and (13), can be also established for the case, when the temperature \( u \) depends upon two co-ordinates instead of one.

4. We shall then derive the expression of the normal function \( u_m \) corresponding to the interval \(-\infty < x < +\infty\). For this purpose, we shall first suppose the interval to be \(-l < x < +l\), where \( l \) is very large, and afterwards take \( l \) to be \( \infty \). As \( u_m \) vanishes at those boundaries \( x = \pm l \), we have

\[ \kappa_m l = \frac{\pi}{2} + m \pi, \]  

where \( m \) is an integer. From (14) it follows that the difference of the successive values of \( \kappa_m \) is

\[ \frac{\pi}{l}. \]  

And using the condition \( \int_{-l}^{l} u_m dx = 1 \), we get finally, for the normal function,

\[ u_m = \frac{1}{l} \cos (\kappa_m x). \]  

Hence, by (10) we get the expression for \( G(x, \xi) \) in the case of \( \xi = 0 \), i.e.,

\[ G(x, 0) = \sum_{m} \frac{1}{l} \frac{\cos (\kappa_m x)}{h^2 + \kappa_n^2}. \]  

Then, take the limit when \( l = \infty \). Putting \( \kappa_m = \mu \), and noticing that
by (15), the above expression (17) reduces to
\[ G(r, 0) = \frac{1}{\pi} \int_{a}^{r} \frac{\cos \mu \tau}{\mu^{2} + \tau^{2}} d\mu, \]
which is the same as (5).

5. Then, we shall proceed to the case, when the temperature depends upon two co-ordinates, which we shall take as polar co-ordinates \( r, \varphi \). Put
\[ I(u) \equiv \frac{\partial^{2} u}{\partial r^{2}} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^{2}} \frac{\partial^{2} u}{\partial \varphi^{2}} - \beta^{2} u, \]
and
\[ A(u) \equiv I(u) + \lambda u = \frac{\partial^{2} u}{\partial r^{2}} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^{2}} \frac{\partial^{2} u}{\partial \varphi^{2}} + (\lambda - \beta^{2}) u, \]
where \( \lambda \) denotes a parameter.

Supposing first the boundary to be \( r=a \), the solution of the equation
\[ \frac{\partial^{2} u}{\partial r^{2}} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^{2}} \frac{\partial^{2} u}{\partial \varphi^{2}} + \kappa^{2} u = 0, \]
which vanishes at this boundary, is
\[ u_{m, n} = A_{m, n} J_{n}(\kappa_{m, n} r) \frac{\cos n \varphi}{\sin n \varphi}, \]
where \( \kappa_{m, n} \) is a root of the equation
\[ J_{n}(\kappa_{m, n} a) = 0, \]
and \( A_{m, n} \) denotes a constant which we shall choose so as to make
\[ \int_{a}^{b} u_{m, n}^{2} r \, dr \, d\varphi = 1. \]

We shall first derive the expression of \( u_{m, n} \). As \( \kappa_{m} \) is the root of the equation \( J_{n}(\kappa_{m} a) = 0 \),
\[ \kappa_{m} \alpha \text{ tends to } -\frac{\pi}{4} + \pi \times \text{an integer}, \]
as \( m \) becomes larger; hence it follows that the difference of the successive roots tends to \( \frac{\pi}{\alpha} \).
On the other hand,
\[ \int \int u_{m \theta} r \, dr \, d\varphi = A_{m \theta} \int_0^\pi J_n(\kappa_m r) 2\pi r \, dr = \pi \alpha^2 J_n^2(\kappa_m \alpha) A_{m \theta}^2. \]

Since the above tends to
\[ \frac{2\alpha}{\kappa_m} A_{m \theta}, \]
we have
\[ A_{m \theta} = \sqrt{\frac{\kappa_m}{2\alpha}}. \]

Consequently
\[ u_{m \theta} = \sqrt{\frac{\kappa_m}{2\alpha}} J_n(\kappa_m r). \]

If the source is situated at \( r=0 \), we get from (10)
\[ G = \sum \frac{\kappa_m}{2\alpha} \frac{J_n(\kappa_m r)}{\mu + \kappa_m^2}, \]
observing that \( J_n(0) = 0 \) for \( n \neq 0 \) and \( J_0(0) = 1 \).

Now take the limit when \( \alpha = \infty \). Putting \( \kappa_m = \mu \), and observing that \( J_n(\mu) = \mu^\mu \), we get from (26), as Green's function corresponding to infinite boundary,
\[ G = \frac{1}{2\pi} \int_0^{2\pi} \frac{J_n(\mu r) \mu \, d\mu}{\mu^2 + r^2}, \]
which is the same as the expression derived by another process(1), and can be written as(2)
\[ G = \frac{1}{2\pi} K_0(\mu r). \]

Substituting into (27)
\[ J_n(\mu r) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\mu r \cos \theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{i\mu r \cos \theta} d\theta, \]
and introducing \( x, y \) and \( \lambda, \nu \) defined by
\[
\begin{align*}
x &= r \cos \varphi \\
y &= r \sin \varphi \\
\lambda &= \mu \cos \theta \\
\nu &= \mu \sin \theta \end{align*}
\]

(1) This proc. [3], 1 (1919), p. 308.
(2) This proc. [3], 2 (1920), p. 8, Eqn. (V).
we get

\[ G = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{\cos (\lambda x + \nu y)}{\dot{h}^2 + \dot{k}^2 + \nu^2}, \tag{28} \]

which is the extended form of (5), corresponding to two variables \( x, y \).

As the relations (3) and (4) give

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \frac{\cos (\nu y)}{(h^2 + k^2) + \nu^2} = \frac{1}{2\sqrt{h^2 + k^2}} e^{-\sqrt{h^2 + k^2} |y|}, \]

(28) becomes

\[ G = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\sqrt{h^2 + k^2}} \cos (\lambda x) e^{-\sqrt{h^2 + k^2} |y|}, \]

which is also equal to \( \frac{1}{2\pi} K_0(hr) \), as we have proved in the previous paper(1).

In the general case, when the source is at a point \((r_0, \varphi_0)\), starting from the expression

\[ \sum_n \cos n (\varphi_0) \cos n (\varphi - \varphi_0), \]

evaluating \( A_{mn} \) and taking the limit when \( a = \infty \), we get

\[ G = \frac{1}{2\pi} \sum_n \int_0^\infty J_n (\mu r) J_n (\mu r_0) \cos n (\varphi - \varphi_0) \frac{d\mu}{h^2 + \mu^2} \]

\[ = \frac{1}{2\pi} \int_0^\infty \frac{d\mu}{h^2 + \mu^2} \frac{J_0 (\mu r) J_0 (\mu r_0)}{h^2 + \mu^2} + \frac{1}{\pi} \sum_n \int_0^\infty \frac{J_n (\mu r) J_n (\mu r_0)}{h^2 + \mu^2} \cos n (\varphi - \varphi_0) d\mu. \tag{29} \]

6. We shall next consider Green's function corresponding to the region which lies outside of the circle \( r=a \).

If a unit source is situated at \((r_0, \varphi_0)\), the temperature at a point \((r, \varphi)\) will be

\[ u = \frac{1}{2\pi} K_0(hr), \tag{30} \]

(1) This proc. [3], 2 (1920), p. 15, Eqn (A).
where \( R \) denotes the distance between the points \((r_0, \varphi_0)\) and \((r, \varphi)\). By the addition theorem of \( K_0 \), the above expression can be put into

\[
u = \begin{cases} 
\frac{1}{2\pi} \left\{ I_0(hr_0)K_0(hr) + \sum_{s=1}^{\infty} I_s(hr_0)K_s(hr) \cos s(\varphi - \varphi_0) \right\}, & \text{for } r > r_0, \\
\frac{1}{2\pi} \left\{ K_0(hr_0)I_0(hr) + \sum_{s=1}^{\infty} K_s(hr_0)I_s(hr) \cos s(\varphi - \varphi_0) \right\}, & \text{for } r < r_0.
\end{cases}
\]  

(31)

We see therefore this unit source produces the temperature

\[
\frac{1}{2\pi} \left\{ K_0(hr_0)I_0(ha) + \sum_{s=1}^{\infty} K_s(hr_0)I_s(ha) \cos s(\varphi - \varphi_0) \right\},
\]

(32)

upon the boundary \( r=a \).

So, if we wish to keep this boundary at 0°, we must superpose another temperature distribution which just cancels the temperature given by (32). This temperature distribution is easily seen to be

\[
-\frac{1}{2\pi} \left\{ K_0(hr_0)I_0(ha) \frac{K_0(hr)}{K_0(ha)} + \sum_{s=1}^{\infty} K_s(hr_0)I_s(ha) \frac{K_s(hr)}{K_s(ha)} \cos s(\varphi - \varphi_0) \right\}.
\]

(33)

Superposing (33) on (31), we get finally

\[
2\pi G = \begin{cases} 
\frac{K_0(hr_0)}{K_0(ha)} \left\{ I_0(ha)K_0(hr) - I_0(ha)K_0(hr) \cos s(\varphi - \varphi_0) \right\}, & \text{for } r < r_0, \\
2\sum_{s=1}^{\infty} \frac{K_s(hr_0)}{K_s(ha)} \left\{ K_s(ha)I_s(hr) - I_s(ha)K_s(hr) \right\} \cos s(\varphi - \varphi_0), & \text{for } r > r_0.
\end{cases}
\]  

(34)

7. The solution can be also derived by making use of the normal function. Suppose first the region to be bounded by the two circles \( r=a \) and \( r=b \), where \( b \) is very large, and afterwards take the limit when \( b=\infty \). Put

\[
u_0(\kappa r) \equiv A_n \frac{J_n(\kappa a)N_n(\kappa r) - N_n(\kappa a)J_n(\kappa r)}{VJ_n^2(\kappa a) + N_n^2(\kappa a)} \quad (1),
\]

(35)

\(^1\) The relation to Hankel's function is

\[2J_n(x) = H_n(\alpha)(x) + H_n(\alpha)(x), \quad 2N_n(x) = i[H_n(\alpha)(x) - H_n(\alpha)(x)].\]
which, after multiplication by $\cos s\varphi$ or $\sin s\varphi$, satisfies the differential equation (21), and vanishes at $r=a$.

The condition $u_n=0$ at $r=b$ determines $\kappa$ to be a root of the equation

$$J_n(\kappa a) N_n(\kappa b) - N_n(\kappa a) J_n(\kappa b) = 0. \quad (36)$$

Making use of the semi-convergent expansion of $J_n(x)$ and $N_n(x)$, we observe that the root $\kappa_m b$ tends to $\kappa a + m\pi$, where $m$ denotes a large integer; hence it follows that the difference of the successive values of $\kappa$ is

$$A\kappa_m = \frac{\pi}{b}. \quad (37)$$

For the constant $A_n$, we have to determine from the condition

$$2\pi \int_a^b u_n^2(\kappa r) r \, dr = 1. \quad \text{In}

$$

$$2\pi \int_a^b u_n^2(\kappa r) r \, dr = \frac{\pi A_n^2}{\sqrt{J_n^2(\kappa a) + N_n^2(\kappa a)}} \left| r^2 \left[ J_n(\kappa a) J_n'(\kappa r) - N_n(\kappa a) N_n'(\kappa r) \right] \right|_{r=a}^{r=b},$$

considering $u_n(\kappa a) = 0$ and $J_n'(x) \xrightarrow{x \to \infty} J_{n-1}(x)$, $N_n'(x) \xrightarrow{x \to \infty} N_{n+1}(x)$ for large value of $x$, and using semi-convergent expansion for $J_n(x)$, $N_n(x)$, this integral is seen to be equal to $A_n^2 (2b)^{-1}$. Therefore,

$$A_n^2 = \kappa (2b)^{-1}.$$ 

Hence, we get by (10)

$$G = \sum_n \sum_m \frac{\kappa_m}{2b} \frac{u_n(\kappa_m r) u_n(\kappa_m r_0)}{J_n^2(\kappa_m a) + N_n^2(\kappa_m a)} \cos n(\varphi - \varphi_0). \quad (38)$$

Now consider the limit when $b = \infty$. Putting $\kappa_m = \mu$, and observing that $d\mu = \frac{\pi}{b}$, (38) reduces to

$$G = \frac{1}{2\pi} \sum_n \int_0^\infty \frac{\mu d\mu}{h^2 + \mu^2} \frac{u_n(\mu r) u_n(\mu r_0)}{J_n^2(\mu a) + N_n^2(\mu a)} \cos n(\varphi - \varphi_0). \quad (39)$$

The identity of (39) and (34) can be proved by the theorem of residaea, as in the case of (3) and (4).