On the Field in Confined Spaces due to Plane Electromagnetic Waves.

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[Read Dec. 19, 1925]

1. Owing to the rapid development of wireless telegraphy and telephony in recent years, our experiments in the laboratory sometimes suffer from disturbances caused by electric waves. A means of keeping a space free from these disturbances is to cover it with metal plates. The object of the present paper is to study mathematically the field in such a shielded space induced by external electromagnetic waves.

Since it is almost impossible to deal with the actual form of a laboratory, we make a simple assumption that a train of simple harmonic electromagnetic waves strikes a spherical shell placed in vacuum. Thus the problem is connected with that of the scattering of electric waves by a sphere, which has been fully discussed by many mathematicians and physicists, of whom Lord Rayleigh was the first. Accordingly, the methods there used to solve the problem will be also available here with some modifications.

We shall give here some of our results. In addition to the relation between the intensity of the field in the spherical cavity and the thickness of the metal plates composing the spherical shell, another interesting result is obtained; that the space bounded by the spherical shell will be resonant with the incident waves if it has a certain dimension. This is in accordance with the conclusions made by many authors that a confined space bounded by a perfectly reflecting surface has proper periods of its electrical oscillations just as a resonance box in Acoustics. This phenomenon of resonance may be taken into account when a laboratory in question is designed, since wireless waves of short wave-lengths will be more frequently used in future.

As appendixes, two cases are added: A train of simple harmonic electromagnetic waves is incident on a space bounded by an infinitely long cylindrical shell or a space bounded by two thin plane plates extending

(1) Full references are given in Bateman's "Electrical and Optical Wave-Motion," 1915, pp. 44, 45, 54, 78.
(2) Ditto, pp. 43, 79.
to infinity. Though these two problems have not so much practical interest as the former, yet they may, on the one hand, present convenient forms for the experimental study of the field and have, on the other hand, interesting physical meanings; in the case of the cylindrical space, there occurs, for instance, a change of the plane of polarization besides the phenomenon of resonance.

Field in a Hollow Sphere.

2. Some remarks will be made on the solution of the fundamental equations of electromagnetic waves. They may be written in spherical polar coordinates \((r, \theta, \phi)\) in the forms

\[
\begin{align*}
\frac{4\pi}{c} \frac{\partial E_r}{\partial t} + \frac{\partial}{\partial t} \left( r \sin \theta \frac{\partial E_r}{\partial r} \right) &= \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( r \sin \theta \frac{\partial E_r}{\partial \theta} \right) - \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \phi} \left( r \sin \theta \frac{\partial E_r}{\partial \phi} \right), \\
\frac{4\pi}{c} \frac{\partial E_\theta}{\partial t} + \frac{\partial}{\partial t} \left( r \sin \theta \frac{\partial E_\theta}{\partial r} \right) &= \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( r \sin \theta \frac{\partial E_\theta}{\partial \theta} \right) - \frac{\partial}{\partial \phi} \left( r \sin \theta \frac{\partial E_\theta}{\partial \phi} \right), \\
\frac{4\pi}{c} \frac{\partial E_\phi}{\partial t} + \frac{\partial}{\partial t} \left( r \sin \theta \frac{\partial E_\phi}{\partial r} \right) &= \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( r \sin \theta \frac{\partial E_\phi}{\partial \theta} \right) - \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \phi} \left( r \sin \theta \frac{\partial E_\phi}{\partial \phi} \right),
\end{align*}
\]

(1)

Here \(E\) and \(\Phi\) denote the vectors of electric and magnetic forces respectively and their components in any directions are distinguished from one another by suffixes which represent the directions in question, \(c\) is the velocity of light in free space, \(\varepsilon\) the dielectric constant of a semi-conducting material, \(\mu\) its magnetic permeability being supposed to be independent of the intensity of the field and \(\sigma\) its electric conductivity.

In order to find the solutions of the fundamental equations convenient for our present purposes, we shall follow the method devised by Bromwich,\(^1\) modifying it in some respects. This simple method is also due to G. Mie.\(^2\)

At first, assuming \(E_r = 0\) in (1) and (2), comparing various relations among thus simplified fundamental equations and introducing a function

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it can be shown that the field will be given by the following set of solutions:

\[\mathcal{E}_r = 0, \quad \mathcal{D}_r = \frac{\partial^2 V}{\partial r^2} - \frac{\mu}{c^2} \frac{\partial^2 V}{\partial t^2} - \frac{4\pi \mu \sigma}{c^2} \frac{\partial V}{\partial t},\]

\[\mathcal{E}_\theta = -\frac{\mu}{c^2} \frac{1}{r \sin \theta} \frac{\partial V}{\partial \theta}, \quad \mathcal{D}_\theta = \frac{1}{r} \frac{\partial V}{\partial \theta},\]

\[\mathcal{E}_\phi = -\frac{\mu}{c^2} \frac{1}{r} \frac{\partial V}{\partial \phi}, \quad \mathcal{D}_\phi = \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi},\]

where \(V\) is a solution of the equation

\[
\frac{\mu}{c^2} \frac{\partial^2 V}{\partial t^2} + \frac{4\pi \mu \sigma}{c^2} \frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}.
\]

Next, in like manner, we obtain another set of solutions by assuming \(\mathcal{D}_r = 0\) is the fundamental equations. This gives the field:

\[\mathcal{E}_r = \frac{\partial^2 U}{\partial r^2} - \frac{4\pi \mu \sigma}{c^2} \frac{\partial U}{\partial t}, \quad \mathcal{D}_r = 0,\]

\[\mathcal{E}_\theta = \frac{1}{r} \frac{\partial V}{\partial \theta}, \quad \mathcal{D}_\theta = \frac{1}{r \sin \theta} \left( \frac{z}{c^2} \frac{\partial U}{\partial t} + \frac{4\pi \mu \sigma}{c^2} U \right),\]

\[\mathcal{E}_\phi = \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi}, \quad \mathcal{D}_\phi = -\frac{1}{r} \left( \frac{z}{c} \frac{\partial U}{\partial t} + \frac{4\pi \mu \sigma}{c} U \right),\]

where

\[
\frac{\mu}{c^2} \frac{\partial^2 U}{\partial t^2} + \frac{4\pi \mu \sigma}{c^2} \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2}.
\]

We assume that the waves are simple harmonic, of wave-length \(\frac{2\pi c}{m}\) in free space; we can then suppose the time to occur only in the form of a time factor \(e^{i \omega t}\). Omitting this time factor for the sake of simplicity and superposing (3) and (4), we have the general solution

\[\mathcal{E}_r = \frac{\partial^2 U}{\partial r^2} + \frac{i}{c} \frac{\partial U}{\partial \phi}, \quad \mathcal{D}_r = \frac{\partial^2 V}{\partial r^2} + \frac{i}{c} \frac{\partial V}{\partial \phi},\]

\[\mathcal{E}_\theta = \frac{1}{r} \frac{\partial V}{\partial \theta}, \quad \mathcal{D}_\theta = \frac{1}{r} \frac{\partial V}{\partial \theta} + \frac{i}{m} \frac{\partial U}{\partial \phi},\]

\[\mathcal{E}_\phi = -\frac{1}{c} \frac{\partial V}{\partial \phi}, \quad \mathcal{D}_\phi = \frac{1}{c} \frac{\partial V}{\partial \phi}.
\]
where $U$ and $V$ are particular solutions of the differential equation

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2} + k_m^2 U = 0, \quad (6)$$

$$k_m^2 = \frac{n^2}{c^2} \left( m^2 - \frac{4\pi^2 \sigma}{c^2} \right) m.$$ 

The general solution of (6) is of the form

$$U = \sum_n Y_n(\theta, \phi) \left[ \phi_n(k_m r) \right] f_n(k_m r), \quad (7)$$

where $Y_n(\theta, \phi)$ is a surface-harmonic of degree $n$, $n$ being an integer, since a surface harmonic of integral degree is the only one which is finite everywhere, and the functions $\phi_n$ and $f_n$ are defined by

$$\phi_n(z) = -\frac{n^2}{2} z J_{n+\frac{1}{2}}(z)$$

$$= -\frac{z^{n+1}}{1 \cdot 3 \cdot \cdots \cdot (2n+1)} \left[ 1 - \frac{z^2}{2(2n+3)} + \frac{z^4}{2 \cdot 4 \cdot (2n+3)(2n+5)} - \cdots \right], \quad (8)$$
or

$$\cos \left( z - n + 1 \frac{\pi}{2} \right)$$

$$\sqrt{1 + \sum_{r=1} \frac{(-1)^r (2n+1)^2 - 1}{(2r-1) \cdot 2^{2r-1}}} \right)$$

$$+ \sin \left( z - n + 1 \frac{\pi}{2} \right)$$

$$\sum_{r=1} \frac{(-1)^r (2n+1)^2 - 1}{(2r-1) \cdot 2^{2r-1}} \cdot \frac{(2n+1)^2 - 4r+1}{(2r-1) \cdot 2^{2r-1}} \right), \quad (9)$$

$$f_n(z) = \psi_n(z) - i \phi_n(z),$$

where

$$\psi_n(z) = -j^n \frac{n}{2} z J_{n+\frac{1}{2}}(z)$$

$$= \frac{1 \cdot 3 \cdots (2n-1)}{z^n} \left[ 1 - \frac{z^2}{2(1-2n)} + \frac{z^4}{2 \cdot 4 \cdot (1-2n)(3-2n)} - \cdots \right]. \quad (10)$$
Therefore \( f_n(z) \) may be written in the form

\[
f_n(z) = \frac{1, 3, \cdots (2n-1)}{z^n} \left\{ 1 - \frac{z^2}{2 (1-2n)} + \cdots \right\}
\]

or

\[
= i^n e^{-i\phi_n} \left\{ 1 + \frac{n(n+1)}{2iz} + \frac{(n-1) n (n+1) (n+2)}{2 \cdot 4 \cdot (iz)^3} + \cdots + \frac{1, 3, \cdots (2n)}{2 \cdot 4 \cdot 6 \cdot 2n \cdot (iz)^{2n-1}} \right\}.
\]

\( \phi_n(z) \) is finite and \( f_n(z) \) infinite at \( z=0 \), so that the former is used for convergent waves and the latter for divergent waves. If we denote \( \frac{df}{dz}, \frac{d\phi}{dz} \) and \( \frac{d\phi}{dz} \) by \( f', \phi' \) and \( \phi'' \) respectively, then the following relation holds generally.

\[
\phi_n' (z) f_n (z) - \phi_n (z) f_n' (z) = 1. \tag{13}
\]

3. Suppose that a train of simple harmonic waves with unit amplitude and polarized in the plane of \((yz)\) is travelling in the negative direction of the axis of \(z\), and strikes a semi-conducting spherical shell of uniform thickness placed in vacuum, the inner and outer radii of the shell being \(r_i\) and \(r_o\) respectively. The electric and magnetic forces of the incident waves are given by

\[
\mathbb{E} = e^{i\mu (\tau + \frac{z}{c})} \quad \text{and} \quad \mathbb{H} = -e^{i\mu (\tau + \frac{z}{c})}.
\]

The following notations are used; the quantities outside the sphere have no suffix, those within the semi-conducting medium suffix 1 and those in the interior of the spherical cavity suffix 2. \( U \) or \( V \) outside the sphere consists of two parts; the one is due to the incident waves (say \( U' \) or \( V' \)) and the another to the reflected waves (say \( U'' \) or \( V'' \)). Therefore the former is finite and the latter infinite at \( r=0 \). \( U_i \) or \( V_i \) within the semi-conducting medium also consists of two parts, the one (say \( U'_i \) or \( V'_i \)) being finite and the another (say \( U''_i \) or \( V''_i \)) infinite at \( r=0 \). In the interior of the spherical cavity, \( U_2 \) or \( V_2 \) consists of only one part which is finite at \( r=0 \).

The boundary conditions are given by the continuity of the tangential

\(\text{footnote} \) In his paper already cit 3, Bromwich proved this relation in the limiting case when \( z \) is very small, but it holds good for all values of \( z \).
electric and magnetic forces on the inner and outer surfaces of the spherical shell. It can be seen from (5) that these conditions will be satisfied if we take

\[
\begin{align*}
\frac{\partial U_i}{\partial r} &= \frac{\partial U}{\partial r}, \\
\frac{\partial V_i}{\partial r} &= \frac{\partial V}{\partial r}, \\
\frac{k_m z}{\mu} U_i &= \left( \frac{m}{c} \right)^2 U, \\
\mu V_i &= V
\end{align*}
\]

(14)

at \( r = r_i \), and

\[
\begin{align*}
\frac{\partial U_i}{\partial r} &= \frac{\partial U}{\partial r}, \\
\frac{\partial V_i}{\partial r} &= \frac{\partial V}{\partial r}, \\
\frac{k_m z}{\mu} U_i &= \left( \frac{m}{c} \right)^2 U, \\
\mu V_i &= V
\end{align*}
\]

(15)

at \( r = r_i \).

Our next step is to find \( U \) and \( V \). They will be found for the incident waves in the following manner: The radial components of the incident waves are found to be

\[
\mathcal{E}_r = \sin \theta \cos \phi e^{i \frac{m}{c} r \cos \theta} \quad \text{and} \quad \mathcal{D}_r = -\sin \theta \sin \phi e^{i \frac{m}{c} r \cos \theta},
\]

the time factor being again omitted. They are transformed into

\[
\begin{align*}
\mathcal{E}_r &= \sin \theta \cos \phi \left( \frac{m}{c} \right)^2 \sum_{n=1}^{\infty} \frac{(2n+1)}{2} \frac{m}{c} P_n' (\cos \theta), \\
\mathcal{D}_r &= -\sin \theta \sin \phi \left( \frac{m}{c} \right)^2 \sum_{n=1}^{\infty} \frac{1}{2} (2n+1) \frac{m}{c} P_n' (\cos \theta),
\end{align*}
\]

(16)

by means of the relation

\[
e^{ik r \cos \theta} = \frac{1}{k r} \sum_{n=0}^{\infty} (2n+1) \frac{m}{c} \phi_n (kr) \frac{d P_n (\cos \theta)}{d \cos \theta},
\]

where \( P_n (\xi) \) is a zonal surface-harmonic of degree \( n \) and \( P_n' = \frac{d P_n (\xi)}{d \xi} \).

On the other hand, these components can be put in the form

\[
\begin{align*}
\mathcal{E}_r \quad \text{or} \quad \mathcal{D}_r &= \sum \frac{n (n+1)}{r^2} Y_n (\theta, \phi) \left[ \frac{\phi_n (k_m r)}{f_n (k_m r)} \right]
\end{align*}
\]

(17)

by means of (5) and (6). Comparing (7), (16), and (17), we find that in the incident waves
In order to obtain the corresponding functions in the scattered waves, within the material and in the interior of the spherical cavity, take

\[
U' = \cos \phi \sin \theta \left( \frac{1}{m^2} \right) \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \phi_n \left( \frac{m}{r} \right) P_n' \left( \cos \theta \right),
\]

\[
V' = -\sin \phi \sin \theta \left( \frac{1}{m^2} \right) \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \phi_n \left( \frac{m}{r} \right) P_n' \left( \cos \theta \right).
\]

As it is evident that all of these functions satisfy (6) and subject to

\[
U'' = \cos \phi \sin \theta \left( \frac{1}{m^2} \right) \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} A_n f_n \left( \frac{m}{r} \right) P_n' \left( \cos \theta \right),
\]

\[
V'' = -\sin \phi \sin \theta \left( \frac{1}{m^2} \right) \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} B_n f_n \left( \frac{m}{r} \right) P_n' \left( \cos \theta \right),
\]

\[
U_i' = \cos \phi \sin \theta \left( \frac{1}{m^2} \right) \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} C_n \phi_n \left( k_m r \right) P_n' \left( \cos \theta \right),
\]

\[
V_i' = -\sin \phi \sin \theta \left( \frac{1}{m^2} \right) \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} D_n \phi_n \left( k_m r \right) P_n' \left( \cos \theta \right),
\]

\[
U_i'' = \cos \phi \sin \theta \left( \frac{1}{m^2} \right) \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} E_n f_n \left( k_m r \right) P_n' \left( \cos \theta \right),
\]

\[
V_i'' = -\sin \phi \sin \theta \left( \frac{1}{m^2} \right) \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} F_n f_n \left( k_m r \right) P_n' \left( \cos \theta \right),
\]

\[
U_z = \cos \phi \sin \theta \left( \frac{1}{m^2} \right) \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} G_n \phi_n \left( \frac{m}{r} \right) P_n' \left( \cos \theta \right),
\]

\[
V_z = -\sin \phi \sin \theta \left( \frac{1}{m^2} \right) \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} H_n \phi_n \left( \frac{m}{r} \right) P_n' \left( \cos \theta \right),
\]

where \( A_n, \ldots, H_n \) are arbitrary constants.

As it is evident that all of these functions satisfy (6) and subject to
the remarks made in the page 5, we have only to determine the coefficients
$A_n\cdots H_n$ so that they may satisfy the boundary conditions. Substituting
above expressions into (14) and (15), and solving eight linear equation
obtained thereby taking into account the relation (13), we get

$$G_n = \frac{1}{c k_m} \frac{1}{\mu m} \frac{1}{S_n^{(e)} S_n^{(e)}} - S_n^{(e)} S_n^{(e)} , \quad H_n = \frac{1}{c k_m} \frac{1}{\mu m} \frac{1}{S_n^{(e)} S_n^{(e)}} - S_n^{(e)} S_n^{(e)} , \quad (19)$$

where

$$S_n^{(e)} = \frac{\phi_n^*(\beta_n) f_n^*(\alpha_n) - \frac{1}{c k_m} \phi_n^*(\beta_n) f_n^*(\alpha_n)}{\mu m} ,$$

$$S_n^{(e)} = \frac{\phi_n^*(\beta_n) f_n^*(\alpha_n)}{\mu m} ,$$

$$S_n^{(a)} = \frac{\phi_n^*(\beta_n) f_n^*(\alpha_n)}{\mu m} ,$$

$$S_n^{(e)} = \frac{\phi_n^*(\alpha_n) f_n^* (\beta_n) - \frac{1}{c k_m} \phi_n^*(\alpha_n) f_n^* (\beta_n)}{\mu m} ,$$

$$S_n^{(a)} = \frac{\phi_n^*(\alpha_n) f_n^* (\beta_n)}{\mu m} ,$$

$$S_n^{(e)} = \frac{\phi_n^*(\alpha_n) f_n^* (\beta_n)}{\mu m} ,$$

$$S_n^{(e)} = \frac{\phi_n^*(\alpha_n) f_n^* (\beta_n)}{\mu m} ,$$

$$S_n^{(e)} = \frac{\phi_n^*(\alpha_n) f_n^* (\beta_n)}{\mu m} ,$$

and

$$\alpha_n = \frac{m}{c} r_0 , \quad \alpha_n = \frac{m}{c} r_4 , \quad \beta_n = k_n r_0 \quad \text{and} \quad \beta_n = k_n r_4 .$$

The field in the spherical cavity is given by (5) and the last two
expressions in (18); namely
\[ C_\theta = \cos \phi \sin \theta \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} G_n \phi_n \left( \frac{m}{c} r \right) + \phi_n'' \left( \frac{m}{c} r \right) P_n' (\cos \theta), \]

\[ \delta_\theta = -\sin \phi \sin \theta \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} H_n \phi_n \left( \frac{m}{c} r \right) + \phi_n'' \left( \frac{m}{c} r \right) P_n' (\cos \theta), \]

\[ C_r = \cos \phi \frac{1}{m} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left\{ G_n \phi_n \left( \frac{m}{c} r \right) \frac{dP_n' (\theta)}{d\theta} \right\}, \]

\[ \sin \phi \frac{1}{m} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left\{ H_n \phi_n \left( \frac{m}{c} r \right) \frac{dP_n' (\theta)}{d\theta} \right\}, \]

\[ \delta_r = -\sin \phi \frac{1}{m} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left\{ H_n \phi_n \left( \frac{m}{c} r \right) \frac{dP_n' (\theta)}{d\theta} \right\}, \]

\[ \sin \phi \frac{1}{m} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left\{ H_n \phi_n \left( \frac{m}{c} r \right) \frac{dP_n' (\theta)}{d\theta} \right\}, \]

where \( P_n' (\theta) \) is a Legendre's associated function of the first kind of degree \( n \) and order 1. We can prove that the expressions (20) for \( C \) and \( \delta \) all converge absolutely and uniformly at about the same rate as a power series of the form \( \left( \frac{r}{2n (n+1)} \right)^n \).

4. As the general discussions of the above obtained solutions is
difficult, some approximations will be made. Since \( \frac{1}{\varepsilon} \), \( \frac{1}{|\beta|} \) and
\( \frac{1}{|\varepsilon|} \) are very small for most metals and the terms which contain them may be neglected, we have, by means of (9) and (12), that

\[
\begin{align*}
(\text{i}) & \text{ if } \phi'_{\alpha'}(\alpha_i) \neq 0, \\
G_n &= -\frac{1}{\varepsilon} \frac{1}{\mu \mu} \sin k_n d \phi'_{\alpha'}(\alpha_i) f'_{\alpha_n}(\alpha_i), \\
H_n &= -\frac{1}{\mu \mu} \sin k_n d \phi_{\alpha_n}(\alpha_i) f_{\alpha_i}(\alpha_i), \\
\end{align*}
\]

where \( d = r_0 - r_i \),

\[
(\text{ii}) \text{ if } \phi'_{\alpha'}(\alpha_i) = 0, \\
G_n^\# &= -\cos k_n d \phi_{\alpha'}(\alpha_i) f_{\alpha_n}(\alpha_i), \\
H_n^\# &= -\cos k_n d \phi'_{\alpha'}(\alpha_i) f_n(\alpha_i), \\
\]

and (\text{iv}) if \( \phi_{\alpha_n}(\alpha_i) = 0 \),

\[
H_n^\# = \cos k_n d \phi_{\alpha_i}(\alpha_i) f_{\alpha_n}(\alpha_i). \\
\]

When the material is supposed to be of copper, \( \frac{1}{\varepsilon} \mu \mu \) is nearly equal to \( 10^{-5} \) if the wave-length of the incident waves is 400 cm. Therefore \( G_n \) and \( H_n \) can be neglected compared with \( G_n^\# \) and \( H_n^\# \); in other words, when \( \phi'_{\alpha'}(\alpha_i) = 0 \) or \( \phi_{\alpha_n}(\alpha_i) = 0 \), there occurs a phenomenon of resonance.

The field will generally be given by the superposition of the terms in the series (20). If \( \phi_{\alpha_n}(\alpha_i) = 0 \) or \( \phi'_{\alpha'}(\alpha_i) = 0 \), the predominating term is the \( k \)-th one, so that it determines the field approximately, since other terms are all of the order \( \frac{1}{\varepsilon} \mu \mu \) and the series converge at about the same rate as a power series of the form \( \left( \frac{c}{2m} \right)^n \).

Let us study the field near the centre of the sphere more in detail. Putting \( r = 0 \), and expressing the forces in Cartesian coordinates, we get

\[
\mathbf{E} = \mathbf{E}_z = G_1 \text{ and } \mathbf{H}_z = -H_1.
\]

Thus near the centre, the electric and magnetic forces are perpendicular to each other and no rotation of the plane of polarization occurs as might have been expected from the symmetrical property of the problem. The resonance occurs when \( \phi'_{\alpha'}(\alpha_i) = 0 \) and \( \phi_{\alpha_i}(\alpha_i) = 0 \), which give
\[ \lambda = \infty, \quad 2.23 \nu_1, \quad 1.03 \nu_1, \quad 0.67 \nu_1, \ldots \]

and

\[ \lambda = \infty, \quad 1.40 \nu_1, \quad 0.81 \nu_1, \quad 0.58 \nu_1, \ldots \]

respectively, if \( \lambda \) denotes the wave-length of the incident waves.

The case where the wave-length is infinitely large must be independently investigated, for this makes \( \sin k_0 d, \beta_0 \) and \( \beta_1 \) equal to zero, which is in conflict with our assumptions made in the above calculation. In this case we get

\[ E = 0 \quad \text{and} \quad H = H_0 = -\frac{9\mu}{(\mu + 2)(2\mu + 1) - 2(\mu - 1)^2 \left( \frac{\nu_1}{\nu_0} \right)^2}, \quad (23) \]

Therefore the electric force is zero and the magnetic force is finite and uniform throughout the interior of the spherical cavity. If we put \( \lambda = \infty \) in all of the equations which determine the present problem uniquely, they reduce to the equations which determine the field in a spherical shell of a uniform thickness placed in a uniform static field of force. The results (2) coincide with this problem in a static field and serves as a verification of our calculations.

Fig. 1.
In order to make the idea clear, numerical examples are given. Taking copper plates and putting

\[ r_j = 100 \text{ cm. and } d = 0.0001 \text{ cm.}, \]

we have the above resonance curves (Fig. 1). The magnitude of the intensity of the field, when there occurs a resonance, will be seen from Fig. 2, for the factor of \( \frac{1}{\cos \kappa m d} \) in \( G_n^\omega \) or \( H_n^\omega \) has the value not so much different from 1. The numerical values in Fig. 2 are multiplied by suitable quantities in order to bring all curves in one diagram.

5. The technique of sending out short waves has recently been developed; we have, for instance, a so-called beam system, in which waves of the order of ten metre wave-length are transmitted. The preceding studies are mainly applicable to such Hertzian waves. In this article, let us consider wireless waves with the wave-length from 100 m. to 10 k.m.

Here \( \alpha = 2\pi \frac{T}{\lambda} \) may practically be taken to be very small but \( \frac{ck_m}{\mu m} \) and \( |k_m r| \) to be very large as before. By means of (8) and (11), and neglecting \( \alpha^2 \) compared with unity, we get near the centre of the sphere,

\[ \mathcal{E}_m = \mathcal{E}_d = G_1 \quad \text{and} \quad \mathcal{H}_m = \mathcal{H}_d = -H_i, \]
where \( G_1 = -\frac{1}{2} \alpha^2 H_1 \) and \( H_1 = -3 \frac{\mu}{k_\nu r} \frac{1}{\sin k_\nu d} \).

Therefore the intensity of the electric force is proportional and that of the magnetic force inversely proportional to \( r \). The following table gives how much the intensity of the field inside the spherical cavity is diminished for different values of the thickness of the spherical shell and of the wave-length of the incident waves. The material composing the spherical shell is supposed to be copper as before.

<table>
<thead>
<tr>
<th>( \lambda ) (m)</th>
<th>0.01 cm</th>
<th>0.05 cm</th>
<th>0.1 cm</th>
<th>0.5 cm</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>2.6 x 10^{-3}</td>
<td>10^{-11}</td>
<td>10^{-10}</td>
<td>1.56 x 10^{-2}</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>2.3 x 10^{-3}</td>
<td>2.19 x 10^{-7}</td>
<td>10^{-10}</td>
<td>6.33 x 10^{-1}</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>4.49 x 10^{-3}</td>
<td>1.76 x 10^{-6}</td>
<td>10^{-9}</td>
<td>1.56 x 10^{-4}</td>
<td></td>
</tr>
<tr>
<td>5000</td>
<td>2.46 x 10^{-1}</td>
<td>3.38 x 10^{-5}</td>
<td>6.1 x 10^{-6}</td>
<td>6.33 x 10^{-6}</td>
<td></td>
</tr>
<tr>
<td>10000</td>
<td>4.97 x 10^{-1}</td>
<td>9.38 x 10^{-5}</td>
<td>2.61 x 10^{-5}</td>
<td>1.56 x 10^{-6}</td>
<td></td>
</tr>
</tbody>
</table>

As one can see from the table, the field is very weak, especially that of the electric force.

When we design an observatory which must be kept free from electromagnetic disturbances, the above results will give some standards as to the dimension of the observatory and the thickness of the metal plates to be used.

**Field in a Hollow Cylinder.**

6. If we use the cylindrical coordinates \((r, \theta, z)\), exactly similar calculations with the above can be carried out. Therefore only a brief account of them will be given.

Suppose that a train of simple harmonic electromagnetic waves is incident in the negative direction of the \( x \)-axis on an infinitely long right circular cylindrical shell of an uniform thickness, its inner and outer radii being \( r_i \) and \( r_o \) respectively and its axis coinciding with the \( z \)-axis. The direction of the electric force of the incident waves is supposed to lie in the \((yz)\) plane and makes an angle \( \alpha \) with the axis of \( y \). We can show
that the field in the interior of the cylindrical cavity is given by the
following set of expressions:

\[
\begin{align*}
\mathcal{E}_r &= -\frac{2i}{m} \cos \alpha \sum_{n=1}^{\infty} H_n \frac{v}{c} J_n \left( \frac{m}{c} r \right) \sin n\theta, \\
\mathcal{E}_\theta &= -\frac{2i}{m} \sin \alpha \sum_{n=1}^{\infty} G_n \frac{v}{c} J_n \left( \frac{m}{c} r \right) \sin n\theta, \\
\mathcal{E}_\varphi &= -i \cos \alpha \sum_{n=1}^{\infty} H_n \frac{v}{c} J_n \left( \frac{m}{c} r \right) \cos n\theta, \\
\mathcal{B}_r &= -i \sin \alpha \sum_{n=1}^{\infty} G_n \frac{v}{c} J_n \left( \frac{m}{c} r \right) \cos n\theta, \\
\mathcal{B}_\theta &= \sin \alpha \sum_{n=1}^{\infty} G_n \frac{v}{c} J_n \left( \frac{m}{c} r \right) \cos n\theta, \\
\mathcal{B}_\varphi &= -\cos \alpha \sum_{n=1}^{\infty} H_n \frac{v}{c} J_n \left( \frac{m}{c} r \right) \cos n\theta,
\end{align*}
\]

where \( \varepsilon_0 = 1, \varepsilon_1 = \varepsilon_2 = \cdots = 2 \).

\[
\begin{align*}
G_n &= \left( \frac{2}{\pi} \right)^2 \frac{\beta_0 \beta_1}{\beta_0 \beta_1} \frac{1}{S_n^{(1)} S_n^{(2)} - S_n^{(1)} S_n^{(2)}}, \\
H_n &= \left( \frac{2}{\pi} \right)^2 \frac{1}{\beta_0 \beta_1} \frac{1}{S_n^{(1)} S_n^{(2)} - S_n^{(1)} S_n^{(2)}}, \\
S_n^{(1)} &= H_n^{(1)}(\beta_0) H_n^{(2)}(\alpha_0) - \frac{1}{\mu m} J_n(\beta_0) H_n^{(1)}(\alpha_0), \\
S_n^{(2)} &= J_n^{(1)}(\beta_0) H_n^{(2)}(\alpha_0) - \frac{1}{\mu m} J_n(\beta_0) J_n^{(1)}(\alpha_0), \\
S_n^{(3)} &= J_n^{(1)}(\beta_0) J_n(\alpha_0) - \frac{1}{\mu m} J_n(\beta_0) J_n^{(1)}(\alpha_0), \\
S_n^{(4)} &= H_n^{(2)}(\beta_0) J_n(\alpha_0) - \frac{1}{\mu m} H_n^{(2)}(\beta_0) J_n^{(1)}(\alpha_0),
\end{align*}
\]
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and

The series (24) converge at about the same rate as a power series of the form

\[ \frac{1}{n} \left( \frac{e}{c} \mu \right)^n \]

7. Supposing \( |\frac{ck_m}{\mu m}|, |\beta_0| \) and \( |\beta_1| \) to be very large as before, we have,

(i) if \( j_n(\alpha_i) \neq 0 \),

\[ G_n = 2 \frac{\mu}{\pi} \sqrt{\beta_0 \beta_1} \frac{1}{i \sin k_m d H_n^{(2)}(\alpha_i) J_n(\alpha_i)} \]

(ii) if \( j_n(\alpha_i) = 0 \),

\[ G_n = -2 \frac{\mu}{\pi} \sqrt{\beta_0 \beta_1} \frac{ck_m}{\mu m} \frac{1}{i \cos k_m d H_n^{(2)}(\alpha_i) J_n(\alpha_i)} \]
(iii) if \( J_n'(\alpha_i) \neq 0 \),
\[
\bar{H}_n = \frac{2}{\pi} \sqrt{\frac{1}{\beta_n \beta_i}} \frac{1}{i \sin k_m d \bar{H}_n''(\alpha_i) J_n'(\alpha_i)},
\]
and (iv) if \( J_n'(\alpha_i) = 0 \),
\[
\bar{H}_n = \frac{2}{\pi} \sqrt{\frac{1}{\beta_n \beta_i}} \frac{e k_{n}}{m} \frac{1}{i \cos k_m d \bar{H}_n''(\alpha_i) J_n(\alpha_i)}.
\]
As \( \bar{H}_n''(\alpha_i) = 0 \) and \( \bar{H}_n''(\alpha_i) = 0 \) have no real roots, the resonance occurs when \( J_n(\alpha_i) = 0 \) and \( J_n'(\alpha_i) = 0 \). When \( \alpha = 0 \) (or \( \theta = 0 \)), \( J_n'(\alpha_i) = 0 \) and when \( \alpha = \frac{\pi}{2} \) (or \( \theta = \pi \)), \( J_n(\alpha_i) = 0 \) are in question. The periods determined by \( J_n(\alpha_i) = 0 \) and \( J_n'(\alpha_i) = 0 \) coincide with the proper periods of electrical oscillations in a cylindrical cavity studied by many authors.

Another interesting result is a change of the plane of polarization. Separating the real and imaginary parts, we get
\[
G_n = g_n e^{i(\phi + \pi/4)} \quad \text{and} \quad \bar{H}_n = h_n e^{i(\theta + \pi/4)},
\]
if there occurs no resonance.

Here
\[
g_n = \frac{\sqrt{2}}{\pi} \sqrt{\frac{\mu}{\nu_0 \nu_1}} \frac{1}{|k_m \sin k_m d \bar{H}_n''(\alpha_i) J_n'(\alpha_i)|},
\]
\[h_n = \frac{\sqrt{2}}{\pi} \sqrt{\frac{\nu_0 \nu_1}{k_m \sin k_m d \bar{H}_n''(\alpha_i) J_n'(\alpha_i)}}\]
\[\phi_n = \arctan \frac{Y_n(\alpha_i) \cos \nu_m d \sinh \nu_m d - J_n(\alpha_i) \sin \nu_m d \cosh \nu_m d}{Y_n(\alpha_i) \sin \nu_m d \cosh \nu_m d + J_n(\alpha_i) \cos \nu_m d \sinh \nu_m d},
\]
\[\theta_n = \arctan \frac{Y_n'(\alpha_i) \cos \nu_m d \sinh \nu_m d - J_n'(\alpha_i) \sin \nu_m d \cosh \nu_m d}{Y_n'(\alpha_i) \sin \nu_m d \cosh \nu_m d + J_n'(\alpha_i) \cos \nu_m d \sinh \nu_m d},
\]
and
\[\nu_m = \sqrt{\frac{2\pi \mu \sigma m}{c^2}}.
\]
When \( \mu = 1 \), we can show that \( \phi_1 = \theta_1 \) and \( g_1 = h_1 \).

If there occurs a resonance, we get
\[
G_n^* = g_n^* e^{i\phi_n^*} \quad \text{and} \quad H_n^* = h_n^* e^{i\theta_n^*},
\]
where
\[
g_n^* = -\frac{1}{\pi^2} \frac{\lambda}{r_1} \frac{1}{|\cos k_m d|} \frac{1}{|Y_n(\alpha_i) J_n'(\alpha_i)|}, \quad (J_n(\alpha_i) = 0),
\]
and...
Here we have supposed that \( r_i \) and \( r_o \) are nearly equal.

If we confine our considerations to the field near the axis of the cylinder, we get

\[
\begin{align*}
\mathcal{E}_r &= 0, \\
\mathcal{E}_\phi &= H_e \cos \alpha e^{i\phi}, \\
\mathcal{E}_z &= G_e \sin \alpha e^{i\phi}, \\
\mathcal{D}_r &= 0, \\
\mathcal{D}_\phi &= G_e \sin \alpha e^{i\phi}, \\
\mathcal{D}_z &= -H_e \cos \alpha e^{i\phi},
\end{align*}
\]

recovering the time factor.

Following cases are in question:

(I) Electric force.

(i) If \( J_0 (\alpha_i) \neq 0 \) and \( J'_0 (\alpha_i) \neq 0 \),

\[
\begin{align*}
\mathcal{E}_r &= h_1 \cos \alpha \cos \left( mt + \phi_1 + \frac{\pi}{4} \right), \\
\mathcal{E}_\phi &= y_1 \sin \alpha \cos \left( mt + \phi_1 + \frac{\pi}{4} \right), \\
\mathcal{E}_z &= g_1 \sin \alpha \cos \left( mt + \phi_1 + \frac{\pi}{4} \right)
\end{align*}
\]

(ii) If \( J_0 (\alpha_i) = 0 \) and \( J'_0 (\alpha_i) \neq 0 \),

\[
\begin{align*}
\mathcal{E}_r &= h_1 \cos \alpha \cos \left( mt + \phi_1 + \frac{\pi}{4} \right), \\
\mathcal{E}_\phi &= g_1 \sin \alpha \cos \left( mt + \phi_1 + \frac{\pi}{4} \right), \\
\mathcal{E}_z &= h_1 \cos \alpha \cos \left( mt + \phi_1 + \frac{\pi}{4} \right)
\end{align*}
\]

(iii) If \( J_0 (\alpha_i) \neq 0 \) and \( J'_0 (\alpha_i) = 0 \),

\[
\begin{align*}
\mathcal{E}_r &= h_1 \cos \alpha \cos \left( mt + \phi_1 + \frac{\pi}{4} \right), \\
\mathcal{E}_\phi &= g_1 \sin \alpha \cos \left( mt + \phi_1 + \frac{\pi}{4} \right), \\
\mathcal{E}_z &= g_1 \sin \alpha \cos \left( mt + \phi_1 + \frac{\pi}{4} \right)
\end{align*}
\]

We conclude from (27) that the electric force is generally elliptically polarized, and further that, in the case (ii), as the \( z \)-component is very large compared with the \( y \)-component, it is polarized nearly in the direction parallel to the axis of the cylinder whatever direction that of the incident waves may take, so long as the latter do not lie in the direction perpendicular to the axis, while in the case (iii), exactly inverse circumstances as the above are true. For example, when the material of the cylindrical shell is of copper and \( r_i = 100 \text{ cm} \), \( d = 0.001 \text{ cm} \), \( \lambda = 356 \text{ cm} \), \( h \) is of the order of 1 while \( g \) of the order of \( 10^{-5} \) which is negligibly
small compared with the former. We remark here that the above mentioned change of the plane of polarization is not due to the phase difference as in usual cases but due to the peculiarity of a confined space concerning its electrical oscillation. In the present case, the oscillations in the direction parallel or perpendicular to the axis characterize the cylindrical cavity, so that external disturbances tend to excite the space to vibrate in these directions. We remark also that as the roots of \( J_0(\alpha_i) = 0 \) and \( J_1'(\alpha_i) = 0 \) are nearly equal except the first few roots, the above mentioned phenomena occur only for the wave-length corresponding to these few roots.

(II) Magnetic force.

(i) If \( J_0'(\alpha_i) = -J_1(\alpha_i) \neq 0 \),
\[
\begin{align*}
\mathcal{E}_x &= g_1 \sin \alpha \cos \left( mt + \phi_1 + \frac{\pi}{4} \right), \\
\mathcal{E}_y &= g_1 \cos \alpha \cos \left( mt + \phi_1 + \frac{\pi}{4} \right).
\end{align*}
\]

(ii) If \( J_0'(\alpha_i) = -J_1(\alpha_i) = 0 \),
\[
\begin{align*}
\mathcal{E}_x &= g_1 \sin \alpha \cos \left( mt + \phi_1^\alpha \right), \\
\mathcal{E}_y &= -h_1 \sin \alpha \cos \left( mt + \phi_1^\alpha \right).
\end{align*}
\]

The magnetic force are always linearly polarized, nearly in the same direction as that of the incident waves. In the case (ii), there occurs a resonance.

The roots of \( J_0(\alpha_i) = 0 \), \( J_1'(\alpha_i) = 0 \) and \( J_0'(\alpha_i) = 0 \) show that \( \mathcal{E}_z \) becomes very large when
\[
\lambda = 2.62 \ r_i, \quad 114 \ r_i, \quad 0.73 \ r_i, \ldots
\]
\( \mathcal{E}_y \) when
\[
\lambda = 3.36 \ r_i, \quad 118 \ r_i, \quad 0.74 \ r_i, \ldots
\]
and \( \mathcal{E}_y \) or \( \mathcal{E}_x \) when
\[
\lambda = \infty, \quad 164 \ r_i, \quad 0.90 \ r_i, \ldots
\]
\( \lambda = \infty \) must be independently considered for the same reason as in Art. 4. In this case we obtain
\[
\begin{align*}
\mathcal{E}_x &= 0, & \mathcal{E}_y &= 0, \\
\mathcal{E}_y &= 0, & \mathcal{E}_z &= \sin \alpha, \\
\mathcal{E}_z &= \sin \alpha, & \mathcal{E}_z &= -\cos \alpha.
\end{align*}
\]
throughout the interior of the cylindrical cavity, this result being equivalent to a problem in a static field.

Field between Two Parallel Plates.

8. Suppose that a space is bounded by two parallel semi-conducting plane plates with a thickness \( d \) and extending to infinity, and that a train of simple harmonic electromagnetic waves is incident perpendicular on the first plate. If the incident waves is expressed by

\[
\mathcal{E}_i = e^{i\alpha(t + \frac{y}{c})} \quad \text{and} \quad \mathcal{D}_i = e^{i\beta(t + \frac{y}{c})},
\]

the field in the space between the two plates is given by

\[
\begin{align*}
\mathcal{E}_r &= \frac{4}{D^2} \frac{\varepsilon_{\mu m}}{\varepsilon_{\mu m}} e^{-\frac{i\lambda}{c} y} \left( F e^{\frac{i\lambda}{c} D} + f e^{-\frac{i\lambda}{c} D} \right), \\
\mathcal{D}_r &= \frac{4}{D^2} \frac{\varepsilon_{\mu m}}{\varepsilon_{\mu m}} e^{-\frac{i\lambda}{c} y} \left( F e^{\frac{i\lambda}{c} D} - f e^{-\frac{i\lambda}{c} D} \right),
\end{align*}
\]

\( \text{time factor } e^{\frac{\lambda t}{c}} \) being omitted. Here \( D \) is the distance between the two plates, \( y \) the distance of the outer surface of the first plate from the coordinate origin, \( Y \) the distance of a point in the space between the two plates from the inner surface of the second plate and

\[
f = 1 - \left( \frac{c k_m}{\varepsilon_{\mu m}} \right)^2 \left( e^{ik_m y} - e^{-ik_m y} \right),
\]

\[
F = e^{ik_m y} \left( 1 + \frac{c k_m}{\varepsilon_{\mu m}} \right)^2 e^{-ik_m y} \left( 1 - \frac{c k_m}{\varepsilon_{\mu m}} \right)^2.
\]

By the same approximation as before, the field at \( Y = \frac{D}{2} \) is given by

(i) if \( \cos \frac{m}{c} \frac{D}{2} = 0 \),

\[
\mathcal{E}_r = \frac{1}{c k_m \sin k_m d \cos \frac{m}{c} \frac{D}{2}} e^{i\frac{m}{c} y - \frac{m}{2}},
\]

(ii) if \( \cos \frac{m}{c} \frac{D}{2} = 0 \),

\[
\mathcal{E}_r = \frac{1}{\cos k_m d \sin \frac{m}{c} \frac{D}{2}} e^{i\frac{m}{c} y - \frac{m}{2}},
\]

(iii) if \( \sin \frac{m}{c} \frac{D}{2} = 0 \),

\[
\mathcal{D}_r = \frac{1}{c k_m \sin k_m d \sin \frac{m}{c} \frac{D}{2}} e^{i\frac{m}{c} y - \frac{m}{2}}.
\]
and (iv) if \( \sin \frac{m}{c} \frac{D}{2} = 0 \), \( \delta_2 = \frac{1}{\cos k_m d \cos \frac{m}{c} \frac{D}{2}} \).

Therefore the resonance of the electric force occurs when
\[
\lambda = 2D, \quad 0.67D, \quad 0.4D, \ldots
\]
and that of the magnetic force when
\[
\lambda = \infty, \quad D, \quad 0.5D, \quad 0.33D, \ldots.
\]

When \( \lambda = \infty \), we get
\[
\xi = \frac{1}{1 + \frac{4\pi \sigma d}{c}} \quad \text{and} \quad \delta_2 = 1.
\]

Finally, the author is indebted to the late Prof. S. Sano and Prof. D. Nukiyama for valuable suggestions.