Short Reports

THE APPLICATION OF THE WEIBULL DISTRIBUTION TO THE ANALYSIS OF THE REACTION TIME DATA

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The Weibull distribution was applied to the data of simple RT experiments. The basic experimental paradigm was a simple RT design with response terminated signals and exponentially distributed random foreperiods. The stimulus was a signal of white light. The task of the subjects was to detect the change of the signal intensity. In nine cases out of eleven, the composite Weibull was of good fit for the obtained data. The remaining two cases showed the simple Weibull. The detection mechanisms were inferred by the use of the hazard ("the instantaneous response rate") functions. The theoretical basis for the application of the Weibull distribution to the RT data was also discussed.

The Weibull distribution has been widely used in reliability engineering. In the psychological literature, it has been used in analysis of distribution of human errors, a theory of mental test score (Kashiwagi, 1969) and social phenomena (Horvath, 1968; Indow, 1971). These articles reveal the usefulness of the Weibull distribution in the respective fields. In this paper, the Weibull distribution is applied to analysis of the distribution of simple reaction time (RT) data, which tells how the sensory information is processed. This topic, especially concerning the "neural counter" mechanism and its decision rule of counting vs. timing, has been studied (McGill, 1963; Green & Luce, 1973; Wandell, 1977).

Recently, a model for the detection mechanism of disappearance or "offset" of faint pure tones against background white noise was proposed on the basis of distributions of simple RT (Rubuck, 1979). The distribution of simple RT for detecting the offset is analyzed by using the "hazard rate" or "instantaneous failure rate" function. In general, it is troublesome to obtain a hazard rate function when the form of a distribution function is not known. Although Rubuck did not adopt analysis of the Weibull distribution, one can easily obtain the hazard rate function by means of the Weibull Graph Paper.

THE WEIBULL DISTRIBUTION

Let \( F(t) \) be the cumulative probability up to time \( t \) that a subject will detect an onset (or offset) of the target stimulus. Time \( t \) is measured from the stimulus onset (or offset).

The distribution function is defined by,

\[
F(t) = 1 - e^{-\frac{t-L}{m}}
\]

where \( L \) and \( m \) are the location parameter and the shape parameter, respectively. In the case of stimulus detection, \( L \)

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represents the "irreducible minimum" time.

Let \( \lambda(t) \) be the conditional probability density at \( t \) that a subject who have not yet detected the stimulus change will come to detect it. By definition,

\[
\lambda(t) = -\frac{d(1-F(t))}{dt} / (1-F(t)) = \frac{d \ln (1-F(t))}{dt}.
\]

Equation (2) is treated in reliability engineering as the instantaneous failure rate function. In the case of stimulus detection, \( \lambda(t) \) is the "instantaneous detecting" rate function.

From (1) and (2),

\[
\lambda(t) = \lambda m(t-L)^{m-1}.
\]

From (1), we have,

\[
1-F(t) = e^{-\lambda(t-L)^m}.
\]

Denoting \( T \) by \( 1/\lambda \), and taking natural logarithm of both sides of (4), we have,

\[
\ln \ln (1/1-F(t)) = m \ln \ln (t-L) - \ln T.
\]

Although it is difficult to estimate these parameters exactly, the Weibull Graph Paper (WGP) comes in handy to estimate the parameters practically (Horvath, 1968; Kashiwagi, 1969). The abscissa and the ordinate of the WGP represent \( (t-L) \) in \( \ln (t-L) \) and \( F(t) \) in \( x(t) = \ln \ln (1/1-F(t)) \) corresponding to the \( F(t) \), respectively. We have a straight line on the WGP through the whole range of \( t \), when \( F(t) \) obeys the Weibull distribution (simple Weibull). There are more complicated cases, the mixed Weibull and the composite Weibull which will be seen later. Fitting a straight line by an appropriate method, we can determine the values of the parameters. In this paper, the values of parameters were determined by fitting a straight line on the WGP by rule of thumb.

**Examples**

The basic paradigm was a simple RT design with response terminated signals and exponentially distributed random foreperiods. On each trial, after a warning tone was presented, an exponential random foreperiod followed, and then the target stimulus was presented. The warning tone was presented through earphones. The target stimuli were lights projected on the translucent screen and the subjects observed them from the opposite side. The task of the subject was to detect the target and to respond by pressing the response button as quickly as possible. The response of the subject terminated the target, so, the condition remained constant until the subject responded. There were two experiments. The basic paradigms were the same.

**Experiment 1**

Both of the target and the fixation disk were of very dim red light (Kodak Wratten Gelatin filter No. 25) and of the same size. The background was dark. The intensities of the Target and the fixation disk on the screen were 2.8 \( \times \) \( 10^{-2} \) cd/m\(^2\) and 6.2 \( \times \) \( 10^{-3} \) cd/m\(^2\), respectively. The size was 0.13 cm in diameter. The natural viewing condition was adopted. The viewing distance was 30 cm. After the 10 min dark adaptation, the experiment began. The number of trials of each subject was 100. The mean of the exponential foreperiod was 4 s, truncated at 15 s.

**Experiment 2**

Both of the target and the surround were of white light. The intensities were far more intense than those of Exp. 1. The intensities of the target and the surround were held constant throughout one experimental session. One daily session consisted of 120 trials. Three kinds of intensities were used as the targets. And there were three conditions B, M, and D, the
The intensities of the targets were \(3.0 \times 10^2\) cd/m\(^2\), \(2.0 \times 10^2\) cd/m\(^2\), and \(1.2 \times 10^2\) cd/m\(^2\), respectively. In each condition the target was superimposed on the surround in intensity of \(2.2 \times 10^3\) cd/m\(^2\). The surround remained presented.

The mean of the exponential foreperiods was 4 s. The viewing condition was the same as in Exp. 1 except for the viewing distance, 85 cm. The sizes of the target and the surround were 0.25 cm, 14.4 cm in diameter, respectively. The arrangement of the stimuli and the time schedule in each trial are presented in Fig. 1.

**Results**

The range of each subject's RT distribution was fairly small. The form of the distributions was positively skewed. The data obtained were plotted on the WGP as follows.

1. Let \(x(t)\) be the cumulative probability up to time \(t\). \(x(t)\) is treated as an estimate of \(F(t)\) given in the form of Eq. (1).

2. These pairs, \(t\) and \(x(t)\), were plotted on the WGP.

3. The Weibull plots exhibited a trend of upward convex. Then, such values of parameter \(L\) were estimated and each Weibull plot was linearized. Now, newly denoting \(t - L\) by \(t\), these \(t\) and \(x(t)\) are replotted on the WGP.

There were two cases out of eleven that showed the simple Weibull. The others showed a radical change in slope at \(d\) in terms of \(t\). This suggests that the response process is a composite Weibull. In the two cases of the simple Weibull, the distribution of the RTs had no slower components than \(d\).

The typical results of Exp. 1 and Exp. 2 are depicted in Figs. 2 and 3, respectively.

The composite Weibull consists of \(k\) simple Weibull segments in succession, which is represented on the WGP by a series of segments, which have radical changes at \(d_i\).

In the following, the data were treated as composite Weibull of two-segment, because the Weibull plots showed clearly a radical change at the time of \(d\), the composite Weibull is easier to handle than the mixed-Weibull which consists of \(k\) simple Weibull mixed in some ratio.

\[
x_i(t) = 1 - \exp \left[-\left((t - L)/T_i\right)^m\right],
\]
\[
d_{i-1} \leq t \leq d_i,
\]
\[
(i = 1, 2, \ldots, k, d_0 = L, d_k = +\infty).
\]

The parameters of each hazard function for \(x_i(t)\) in Eq. (6) are determined by each slope and intercept, because the definition of hazard function upon which the previous period has no effect.

**Discussion**

By using the hazard function (not obtained by Weibull distribution), Rubuck has shown that the hazard function of the RT in detecting the offset exhibit a characteristic form that is positively skewed, long- and high-tailed. He split the hazard function into two components, one of the “change detector” and the
other of the "level detector", because this type of hazard function cannot be obtained from a single distribution function.

The level detector indicates the amount of the sensory information in some proportional way to the intensity of the stimulus (Fig. 5-a). The change detector which are assumed to represent the operation of the possible physiological substance (the Post-inhibitory Rebound, Perkel & Mulloney, 1974) indicates the offset of the target.

Apart from the substance model, taking the structural view, the model above may be so generalized that it is able to treat the detection of the stimulus onset as well.
The change of the sensory information occurs in the two cases. The one is accompanied with the change from no-signal to signal (onset), and the other is the reverse case (offset). The change detector is now assumed to respond to both cases. When the change is detected, these two mechanisms work independently or in another possible combination.

As shown in Fig. 5-c, composing the \( h_1(t) \), \( h_2(t) \), which represent the operation of each detector, respectively, we have \( h(t) \) which resembles the obtained hazard function. We should take care in interpreting the hazard function when the "picture of the mechanism in operation" is not clear, and although the time function \( \lambda(t) \) can be affected only by delay of responses (McGill, 1963), the two-component model may illustrate the obtained composite Weibull.

As shown above, because of its flexibility, the Weibull distribution can be used as a tool for obtaining hazard functions without supposing any specific distribution function beforehand.

The validity of the application of the Weibull distribution consists in the fact that the data from RT experiments can be taken for the extreme statistics, in this case, the smallest values. The Weibull distribution is an asymptotic distribution for the smallest values, which are satisfied under very weak conditions (Horvath, 1968). The stochastic process of the Weibull distribution is full of suggestions to the question why RT data obey the Weibull distribution. From this point of view, another model of the detection can be made.

**The Spark Discharge (SD) Model**

This is modeled after the phenomenon of the delay of the occurrence of spark discharge which is observed when voltage is applied between electrodes (Taki, Koya, Miyagawa, & Sekine, 1978). The same

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*Fig. 4. The hazard functions of Exp. 2.*

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*Fig. 5. Schematic Diagram of the two-component model; dotted lines indicate the false alarm rate.*
structure seems to be inferred in psychological phenomena. The assumption are as follows.

(i) The delay of neural information from the time of onset of a target stimulus above threshold ($I_0$) obeys the exponential distribution,

$$f_1(t) = \lambda \exp(-\lambda t) \quad (8)$$

where $t \geq 0$, and $1/\lambda$ is the mean delay.

(ii) $\lambda$ increases as a function of intensity of the stimulus ($i = I - I_0$) and in proportion to $i$. That is,

$$\lambda = ai \quad (9)$$

where $a$ is a positive constant.

(iii) Suppose the experiment in which the intensity of the target stimulus increases with $t$ at a constant rate, until a subject detects the target.

$$i = bt \quad (10)$$

where $b$ is a positive constant. What kind of distribution is the latency?

From (9) and (10), we have a time function,

$$\lambda(t) = abt. \quad (11)$$

When a time function is given, we can obtain its distribution function by using the following relationship,

$$1 - F(t) = \exp\left( - \int_0^t \lambda(x)dx \right). \quad (12)$$

From (11) and (12), and by transposition, we have,

$$F(t) = 1 - \exp\left( - \frac{ab}{2} t^2 \right). \quad (13)$$

This is the distribution function of Weibull distribution with $\lambda = ab/2$, $m = 2$ in (1).

It was shown that the latency mechanism of the neural response of the optic nerve of the limulus (Mueller, cited in McGill, 1963) is represented by,

$$f(t) = \lambda t \exp(-\lambda t^2/2) \quad (14)$$

when the flash is kept at the same intensity and lengthened into a steady light as the target stimulus in this paper. In this case, the number of active molecules in the photoreceptor increases steadily with time, because quanta are being absorbed at a steady rate. Thus the time function of the response is,

$$\lambda(t) = \lambda t. \quad (15)$$

Although McGill did not point out, (14) is the p.d.f. of the Weibull distribution with $m = 2$ in (1). (14) is equivalent to (13).

Although the SD model can explain the form of distribution, it cannot explain the composite Weibull. But it is possible to analyze the parameters $a$, $b$ by the experiment which incorporates the operation of $b$ in (10).

References


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