AN EXPERIMENTAL STUDY OF THE LUNEBURG THEORY OF BINOCULAR SPACE PERCEPTION (1). THE 3- AND 4-POINT EXPERIMENTS¹

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It was in 1947 that Luneburg (1947, 1948, 1950) gave utterance for the first time in the form of publication to the following idea which had come to him two years before while visiting the Dartmouth Eye Institute (Blank, 1959). The elusive world of visual appearance binocularly observed is described in terms of Riemannian geometry of constant curvature, the numerical parameters of which depend upon the individual observer and further certain well-known phenomena, such as the horopter, the discrepancy between the parallel alley and the distance alley are explained on that basis. Psychologists had had to wait nine years, however, until the report of an attempt to experimentally verify the theory appeared for the first time in journals familiar to them. Zajaczkowska (1956 a, b) determined the two personal constants involved in the theory, σ and K , for each of 36 Ss by the method suggested by Luneburg himself (1950) and then compared the shapes of horopters and of alleys predicted from these constants with the results experimentally obtained. The present study followed this pattern of approach to the experimental verification of the Luneburg theory. It is to be noted, however, that a thorough study

on the Luneburg theory, theoretical as well as experimental, was reported three years earlier than Zajaczkowska by a group of investigators in the Knapp Memorial Laboratories, Institute of Ophthalmology, Columbia University (Hardy, Rand, Rittler, Blank, & Boeder, 1953). Because of the form of its publication, unfortunately it had only a limited circulation among psychologists. It was while preparing this manuscript that the authors came to know their study.

THE LUNEBURG THEORY

It would take up too much space to go into details of the theory, but the minimum amount of the exposition might be indispensable to understand the experiments to be described below. This theory is applicable to the binocular vision solely based on the convergence of the optic axes in the absence of monocular and experiential factors and hence it refers to shapes, distance between, or relative localization of small light points presented to the S in darkness. The *binocular visual space* in this sense is assumed to be a homogeneous metric space, that is, a Riemannian space of constant curvature, which comes from the following observational facts. We have an immediate feeling of straightness, of plane surfaces, and if any points in a plane are connected by a visually straight line it would not leave the plane. Let a stimulus point in the physical space be denoted by $Q(x, y, z)$ where x, y, z indicates the Cartesian coordinates as shown in Fig. 1. It is more

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FIG. 1. Bipolar coordinates of a stimulus point Q in the physical space where x' represents x -axis projected to a plane of elevation θ . The circle is a Vieth-Müller circle passing through Q .

FIG. 2. Polar coordinates of a perceived point P in the Euclidean map of the binocular visual space where ξ' represents ξ axis projected to a plane of elevation θ .

FIG. 3. Parallel and distance alleys in the Euclidean map of the binocular visual space (horizontal plane).

advantageous, however, to use the bipolar coordinates $Q(\gamma, \phi, \theta)$ taking the right eye (R) and the left eye (L) as the origins and γ representing the bipolar parallax, ϕ the bipolar latitude, and θ the angle of elevation. In the latter each variable possesses more direct physical meaning

² A generalization of Gaussian curvature to the case of manifolds of these dimensions.

of its own. Corresponding to Q , the S perceives in his visual space a point P which is adequately specified in terms of the Cartesian coordinate $P(\xi, \eta, \zeta)$ or of the polar coordinates $P(\rho, \varphi, \theta)$.

The meaning of the variables might be clear from Fig. 2 which represents the Euclidean map of the visual space. It is to be noticed that there is only a single origin because we have no phenomenal counterpart to the fact that two eyes are involved in giving rise to P . With respect to any pair of points, P_i and P_j , the corresponding distance on that map is not necessarily equivalent to the psychometric distance function D_{ij} which is to be defined in the visual space itself. In so far as the visual space is describable in terms of Riemannian geometry of constant curvature, K ,² the line element in the space is given by

$$ds = \left[\frac{d\xi^2 + d\eta^2 + d\zeta^2}{\left(1 + \frac{K}{4}\rho^2\right)^2} \right]^{\frac{1}{2}} \quad (1)$$

and if D_{ij} is assumed to be the geodesic, then the following formulism holds;

$$\begin{aligned} & \frac{2}{\sqrt{-K}} \sinh \left(\frac{\sqrt{-K}}{2} D_{ij} \right) \\ &= \left[\frac{(\xi_i - \xi_j)^2 + (\eta_i - \eta_j)^2 + (\zeta_i - \zeta_j)^2}{\left(1 + \frac{K}{4}\rho_i^2\right)\left(1 + \frac{K}{4}\rho_j^2\right)} \right]^{\frac{1}{2}} \quad (2) \end{aligned}$$

For $K=0$ we obtain the Euclidean geometry, for $K>0$ the elliptic geometry, and for $K<0$ the hyperbolic geometry.

The Luneburg theory is based on the assumption that the binocular visual space as defined above be hyperbolic in nature, and further Luneburg had evidence to suppose the following functional relationships between the physical space and the visual space:

$$\left. \begin{aligned} \theta &= \theta \\ \varphi &= \phi \\ \rho &= 2e^{-\sigma\gamma} \end{aligned} \right\} \quad (3)$$

Under the set of assumptions, it is shown that the total visual space is represented

on the Euclidean map by the interior of the sphere $\rho=2/\sqrt{-K}$ (Fig. 3). For infinitely distant points of the visual space, we have $\gamma=0$ and $\rho=2$ according to (3). This is the definition of the metrics given to the Euclidean map, which as a consequence implies that the value of K must be within the limit:

$$-1 < K < 0 \quad (4)$$

otherwise the assumptions might not be consistent to each other. It is also evident that the alleys and the horopter mentioned above must be represented in the interior of the basic circle ($\rho \leq 2/\sqrt{-K}$) as geodesics of special properties respectively; each reflects its observational condition.

In the *parallel alley* experiment, the S is confronted with two rows of lights lying in the horizontal plane of his eyes ($\theta=0$). The two end lights $Q_0(x_0, \pm y_0)$, the farthest from the eyes, are fixed. Pairs of lights $Q_i(x_i, \pm y_i)$ are movable sideways, i denoting the number counted in order from Q_0 . A pair of lights Q_i are presented

FIG. 4. Parallel and distance alleys in the physical space (horizontal plane).

to the S one by one in addition to Q_{i-1} , and the S is required to adjust their positions so as to obtain an impression that the two rows of lights form parallel walls originating from Q_0 . Hence, to define parallel alleys, out of an infinite number of geodesics passing through $P_0(\xi_0, \pm\eta_0)$ without intersecting, Luneburg selected those which are normal to the basic circle and also to the η axis. This is Luneburg's interpretation of being phenomenally parallel. The equation of these geodesics is

FIG. 5. The configuration of light points in the physical space (the 3-point experiment).

FIG. 6. The configuration of light points in the physical space (the 4-point experiment).

$$\frac{K}{4}(\xi^2 + \eta^2) - 1 = -C\eta \quad (5)$$

where C is a constant. In the *distance alley* experiment, on the other hand, in addition to the pair Q_0 , only one more pair Q_i are presented at a time. The S is required to adjust Q_i so that the interval between the two lights appears equal to that between the fixed end lights. When a pair Q_i are presented, all the other pairs except Q_0 are being put off and the S makes a comparison of intervals only between Q_0 and Q_i . Hence the distance alleys are given by the equation of constant geodesic distance, which is of form:

$$\frac{K}{4}(\xi^2 + \eta^2) + 1 = C\eta \quad (6)$$

The curves representing both kinds of alleys in the Euclidean map of the visual space are shown in Fig. 3 and those transformed to the physical space in Fig. 4. The distance alley lies outside the parallel alley and this is regarded by Luneburg as a conclusive evidence for that the visual space is hyperbolic. The experimental results since Hillebrand (1902) and Blumenfeld (1913) have shown it to be the case at least qualitatively. (Hardy, Rand and Rittler 1951, Zajackowska 1956 b, Hardy et al., 1953). The final test of the theory must, however, be based upon the quantitative coincidence of the observed alleys with the curves predicted by the use of individual values of K and σ . Early as in 1951, an attempt was made to estimate the values of personal constants, K and σ , for 15 S s from the data of the alley experiments with the conclusion that these provide a convincing demonstration of the hyperbolic character of the visual space but it became also clear that the procedure is not too well suited for measuring the personal constants (Hardy, et al., 1951). Following Luneburg's scheme Zajackowska (1956a, b), obtained the personal constants by entirely different procedures and then applied those to predict results of the alley experiments with the same S s. The

authors adopted the policy of the latter. The report is divided into two parts, the experiments to measure K and σ will be fully described in the present article whereas the alley experiments will constitute the subject of the article to follow.

METHODS OF MEASURING K AND σ

Luneburg (1950) suggested the following method as the one that so far seems to promise the best results and the method was experimentally tested by Zajackowska (1956a). It consists of two experiments.

The 3-point experiment. Three light points Q_0 , Q_1 , and Q_2 are presented simultaneously in a manner as shown in Fig. 5. The points Q_0 and Q_1 are fixed and the S is required to place the movable point Q_2 , by the method of adjustment, in such a position that D_{01} and D_{02} appear to be equal. The distances $Q_0 Q_1$ are varied by setting Q_1 in different positions on the circle $r=r_0$ and a number of determinations are obtained as to Q_2 under the given condition of Q_1 . As Q_0 , Q_1 and Q_2 are in the horizontal plane of the S 's eyes and $\theta=0$, the equation (2) is written in the form:

$$\frac{2}{\sqrt{-K}} \sinh\left(\frac{\sqrt{-K}}{2} D_{ij}\right) = \left[\frac{\rho_i^2 + \rho_j^2 - 2\rho_i\rho_j \cos(\varphi_i - \varphi_j)}{\left(1 + \frac{K}{4}\rho_i^2\right)\left(1 + \frac{K}{4}\rho_j^2\right)} \right]^{\frac{1}{2}} \quad (7)$$

Then from (3) and (7) it can be easily shown

$$Y = AX - B \quad (8)$$

where

$$\left. \begin{aligned} X &= 4 \sin^2 \frac{1}{2} \phi_1 \\ Y &= 4 \sin^2 \frac{1}{2} \phi_2 \end{aligned} \right\} \quad (9)$$

$$A = \frac{\rho_0}{\rho_1} \frac{1 + \frac{K}{4}\rho_1^2}{1 + \frac{K}{4}\rho_0^2} \quad (10)$$

$$B = \left[\left(\frac{\rho_0}{\rho_1} \right)^{\frac{1}{2}} - \left(\frac{\rho_1}{\rho_0} \right)^{\frac{1}{2}} \right]^2 \quad (11)$$

Hence a plot of a number of experimental values of Y against the given values of X must be linear if the theory holds. The experimental value for B , obtainable from the straight line best fitting the plot, serves for the estimation of σ because it follows from (3) and (11) in sufficient approximation

$$\sigma = \frac{\sqrt{B}}{(\gamma_1 - \gamma_0)} \quad (12)$$

In theory it should be also possible to obtain the value of K by the equation

$$K = \frac{2e^{2\sigma\gamma_0} \sinh [\sigma(\gamma_1 - \gamma_0)]}{A - e^{-\sigma(\gamma_1 - \gamma_0)}} - e^{2\sigma\gamma_0} \quad (13)$$

which follows from (3) and (10). In order for the matching $D_{01} = D_{02}$ to be possible, $(\gamma_1 - \gamma_0)$ and hence $(\rho_0 - \rho_1)$ can not be too large relative to $(\phi_1 - \phi_0)$ and the restriction presents some difficulties to the use of (13) because it becomes extremely unstable. Hence the alternative procedure has been devised as to the measurement of K .

The 4-point experiment. Four points Q_0 , Q_1 , Q_2 and Q_3 are presented simultaneously in the horizontal plane of the S 's eyes as shown in Fig. 6. The S 's task is to equate D_{23} to D_{01} by moving Q_3 by the method of adjustment on the circle $\gamma = \gamma_1$. Whereas Q_0 and Q_2 are fixed, Q_1 is varied on the circle $\gamma = \gamma_0$ and a number of determinations are obtained as to Q_3 under the given condition of Q_1 . Then from (7) we find

$$Y = AX \quad (14)$$

where

$$\left. \begin{aligned} X &= 2 \sin \frac{1}{2} (\phi_1 - \phi_0) \\ Y &= 2 \sin \frac{1}{2} (\phi_2 - \phi_3) \end{aligned} \right\} \quad (15)$$

and A has the same meaning as in (10). In this case no restriction is imposed and $(\gamma_1 - \gamma_0)$ or $(\rho_0 - \rho_1)$ can be freely chosen so that (13) becomes stable. Therefore,

the straight line passing the origin which fits best the experimental values of Y plotted against the given values of X would provide us with necessary data in order to make successful use of (13). Furthermore, it has been proved by Luneburg that the linearity of the relation between X and Y in the 3- and 4-point experiments is a direct indication of the homogeneous character (constant curvature) of the visual space (1950, p. 640).

EXPERIMENTS OF MEASURING K AND σ

In authors' laboratory, experiments with the purpose of obtaining the precise estimates of K and σ for individual S s have been repeated with the same S s for three years since 1958 in order to secure informations concerning stability of the estimation. Results of these experiments will be reported in this article. As mentioned above, these values were used in predicting the curves of the alley experiments having been carried out at the same time with the same S s, of which reports will be made in the next article.

The whole experiments, the ones of measuring the personal constants and others of constructing the alleys, were made in the vertical direction (V) as well as in the horizontal direction (H). Should the anisotropy of visual space exist, the personal constants and hence the form of the alleys will vary according to direction.

Apparatus. The experiment was carried out in a dark room where no other things were visible except small lights presented. The S 's face was held in position by a headrest and by having him bite a bar. When the experiment was of horizontal direction the S was seated and when it was of vertical direction the S laid himself on the floor facing upward. The apparatus consisted of long movable rods (150 cm long) stretched from a single center of rotation. Along the rod there was a small bar containing a tiny lamp which could be fixed at any distance from the terminal of rod at any height (distance from rod to box). The small box had 0.5 mm aperture drilled in metal sheet and the ray through it served as the light point Q . Luminances of the points presented at a time were

adjusted subjectively equal by pasting sheets of neutral filters in the aperture. The whole set of apparatus could be settled either horizontally or vertically. By the use of this apparatus the arrangement of light points with respect to the S 's eyes shown in Fig. 5 could be easily obtained and the rod to which Q_8 was attached was moved freely on the circle $\gamma=\gamma_1$ by the S through a handle. In this case the diameter of the larger circle was 120 cm ($\gamma_0=0.0541$ for the S whose interpupillary distance is 65 mm) and that of the smaller one 100 cm ($\gamma_1=0.0651$ for the S). For the arrangement shown in Fig. 6 where the diameter of the larger circle was 250 cm ($\gamma_0=0.0259$ for the S), Q_0 and Q_1 were fixed to a frame independent from the apparatus mentioned above. Q_2 and Q_8 were attached to two rods and the rod for the latter was movable along the circle of which diameter was 100 cm ($\gamma_1=0.0651$ for the S). If a long needle is put through a hole drilled in the rod, it indicates the position of the movable rod on a board attached to the apparatus. While the experimenter was reading the graduations on the board and making arrangements for the next presentation of lights with the aid of small flashlight, the S was being blindfolded.

Subject. Three S s TI, NM, and MO participated in the preliminary experiment in 1958, five S s, TI, EI, TK, HI, and MS in the experiment in 1959, and four of the five except MS and a new S , KM took the part of S in the experiment in 1960. In 1958 either NM or MO, in 1959 and 1960 almost exclusively MS and KM served as the E respectively. While TK and HI did not know much about the purpose of the experiments, the remaining S s were aware more or less of the results obtained by Zajackowska because they were either the authors or the collaborators.

Procedure. As mentioned before, the task of S was to equate D_{02} to D_{01} by adjusting the position of Q_2 in the 3-point experiment and D_{28} to D_{01} by adjusting the position of Q_8 in the 4-point experiment where D_{ij} denotes a phenomenal distance between percepts P_i and P_j which correspond stimuli; Q_i and Q_j respectively. It is to be noticed that, for example, P_0 , P_1 and P_2 in the 3-point experiment do

not stand in simple relation to each other, P_2 appearing nearer than P_0 and P_1 . The S had to equate to D_{01} , therefore, D_{02} spanning aslant between P_0 in the right and behind and P_2 in the left and front. Hence, it was apparent from the beginning that *attitude* and *criterion* the S assumes while observing D_{ij} may influence the results a great deal. For example, the S was apt to make judgements solely on the basis of angular separation between P_i and P_j , that is to say, to disregard differences of phenomenal distances from the S to the points. For this reason, after several preliminary trials, the authors had decided to instruct the S as follows. In the case of 3-point experiment: "Suppose you are going to strain a thread between P_0 and P_1 , and then between P_0 and P_2 . You may fixate each light point freely in turn to estimate the distance, but without moving your head. Adjust the position of P_2 until you are confident that P_0 and P_1 are tied with a thread of the same length as the one tying P_0 and P_1 ."

The role that eye-movements play in the Luneburg theory is not entirely clear to the authors. It seems obvious at least that the line of regard should not be fixed to a single point but points have to be fixated in turn (Hardy, et al., 1951). Successive fixations involve necessarily movements of the eyes. The transitional phase of the eyes does not have its counterpart in the theory, however.³

Presentations of stimuli were begun after 10 minute dark adaptation. The method of adjustment by the hands of S was used. Standard distance between Q_0 and Q_1 was varied in five ways in both of 3- and 4-point experiments and a given length of the standard distance was presented ordinarily 5 times in succession. The length of the distance to be adjusted was so chosen at the moment of presentation that it looks unmistakably larger or smaller than the standard distance. As to lengths of the standard distance, the order of experimentation was randomized. Ordinarily an experimental sitting consisted of 25 determinations each for the 3- and 4-point experiments

³ Observation with head movement was referred to very briefly. (Luneburg, 1947, p. 25)

in one direction, either horizontal or vertical, and took from an hour and a half to two hours.

Computations. Each determination of ϕ for the movable Q was converted from angular measure to Y according to (9) in the 3-point experiment and to (15) in the 4-point experiment. Then results of repeated determinations under the given value of X were averaged and hereafter the average will be denoted as Y . The number of repetition was 10 in the experiment of 1958, 20 in 1959, and 10 in 1960. Steps involved in computations were as follows:

[1] A straight line (8) was fitted by the method of least-square to a plot of Y against X in the 3-point experiment. Let a_3 and b be the values for A and B in (8) thus obtained. Putting into (14) b , and the values γ_0 and γ_1 in the 3-point experiment, σ' was computed which served as the first approximation of σ .

[2] A straight line passing through the origin (14) was fitted by the method of least-square to a plot of Y against X in the 4-point experiment. Let a_4 be the value of A in (14) thus obtained. Then K' , a tentatively adopted value of K , was given by putting into (13) a_4 , γ_0 and γ_1 in the 4-point experiment, and σ' .

[3] By inserting σ' into (3), the values of ρ_0 and ρ_1 in the 3-point experiment were computed with the purpose of recalculating the value of A in (8) by means of (10) from these

values and K' obtained in the step [2]. This value A was regarded as the slope of the straight line (8) that was attained to by taking into due consideration the result of 4-point experiment. It is to be noted that in fact A does not necessarily coincide with a_3 .

[4] With a change from a_3 to A , if any, in the slope of the straight line (8), b , the experimental value of B therein, was modified as follows:

$$B = b \frac{A}{a_3} \quad (16)$$

which was regarded as the value of the intercept of the straight line (8) that was attained to in view of the result of 4-point experiment.

[5] The finally adopted value of σ was given by putting into (12) γ_0 and γ_1 of the 3-point experiment and B obtained in the step [4].

[6] Through (13), the value of K was determined as the final one by γ_0 and γ_1 in the 3-point experiment and σ obtained in the step [5].

Zajaczkowska (1956) seems to have followed the procedure mentioned above when calculating the values of K and σ , though his description is not entirely clear as to details of the procedure. Frankly speaking, however, the logic inherent in the steps [3] and [4] is beyond the authors' comprehension. As mentioned before,

FIG. 7. Result of the 3- and 4-point experiments. One of the most typical cases with regard to goodness of fit.

$$a_4 = 1.586, \quad a_3 = 1.005, \quad b = .018 \\ A = 1.135, \quad B = .018$$

FIG. 8. Result of the 3- and 4-point experiments. An example of relatively poor fit.

$$a_4 = 2.002, \quad a_3 = .905, \quad b = .013 \\ A = 1.172, \quad B = .017$$

TABLE 1

The values used in estimating the personal constants and the finally adopted values of K and σ

	S	Year	3-point Ex.			4-point Ex.		σ'	K'	A	B	σ	K
			g	a_3	b	g	a_4						
H	TI	58	1.33	1.100	.007	3.34	2.018	7.860	-.778	1.176	.008	8.215	-.771
		59	2.30	.698	.003	3.56	2.077	5.241	-.891	1.186	.005	6.805	-.920
		60	6.19	.778	.006	5.58	2.385	7.352	-.953	1.208	.010	9.167	-.903
	MO	58	1.62	.950	.005	6.80	2.498	7.070	-.983	1.204	.006	7.940	-.963
		58	1.85	1.097	.018	2.26	1.869	12.118	-.373	1.173	.019	12.507	-.330
	EI	59	.72	.730	.012	3.67	2.649	10.232	1.000	1.230	.021	13.259	-.900
		60	2.00	.730	.012	3.41	2.359	10.269	-.891	1.209	.020	13.222	-.710
	HI	59	1.27	1.143	.006	1.24	2.398	7.100	-.947	1.211	.007	7.328	-.976
		60	2.30	1.270	.027	2.58	2.174	14.964	-.337	1.197	.026	14.518	-.321
	TK	59	1.16	.921	.011	2.45	1.536	9.327	-.143	1.213	.015	10.690	-.874
		60	3.33	.952	.003	3.49	1.617	4.850	-.643	1.134	.004	5.310	-.711
	MS	59	1.05	.841	.010	3.22	1.490	9.706	-.115	1.008	.012	10.608	+.034
		60	2.69	.889	.017	3.02	1.738	12.824	-.207	1.167	.022	14.667	-.073
V	TI	58	1.44	1.228	.021	4.15	2.283	13.320	-.723	1.099	.018	12.590	-.770
		59	.95	.762	.002	1.83	2.114	4.537	-.920	1.183	.004	5.630	-.917
		60	2.28	.873	.000	2.76	1.887	1.602	-.946	1.086	.000	1.852	-.758
	MO	58	1.05	1.208	.005	2.42	2.327	6.640	-.941	1.192	.005	7.394	-.930
		58	1.51	.804	.012	2.49	2.374	10.080	-.876	1.208	.019	12.360	-.771
	EI	59	.37	.762	.010	2.47	2.330	9.444	-.888	1.203	.046	11.861	-.691
		60	1.98	.889	.012	2.61	2.310	10.269	-.867	1.206	.017	11.963	-.725
	HI	59	.72	1.159	.005	1.72	2.486	6.164	-.985	1.218	.005	6.300	-1.004
		60	1.60	.905	.013	1.71	2.002	10.482	-.635	1.172	.017	11.918	-.414
	TK	59	.85	1.016	.004	1.97	1.676	5.735	-.646	1.145	.005	6.071	-.691
		60	2.02	1.095	.018	1.66	1.586	11.770	+.060	1.135	.018	11.974	+.221
	MS	59	1.42	.952	.011	2.23	1.578	10.284	-.216	1.134	.013	11.225	-.238
		60	1.93	.679	.003	2.00	1.816	5.549	-.766	1.127	.005	7.137	-.582

TABLE 2

The result of repeated determinations of $\bar{K}$ and σ

S	H				V			
	$\bar{\sigma}$	$\bar{K}$	$\frac{s_{\sigma}}{\bar{\sigma}} \times 100$	$\frac{s_K}{\bar{K}} \times 100$	$\bar{\sigma}$	$\bar{K}$	$\frac{s_{\sigma}}{\bar{\sigma}} \times 100$	$\frac{s_K}{\bar{K}} \times 100$
TI	8.06	-.86	1.21	7.74	6.69	-.82	66.45	5.94
EI	13.24	-.81	.15	12.96	11.91	-.71	.42	2.84
HI	10.93	-.65	32.94	50.77	9.11	-.71	30.85	42.25
TK	8.00	-.79	33.63	10.13	9.02	-.24	32.70	187.50

the scheme of the 3- and 4-point experiments came from Luneburg (1950), but the 3-point experiment is for obtaining σ and the 4-point experiment is for attaining to K in his original plan. That is to say, in Zajackowska's procedure, the 4-point experiment is used to introduce modifications to values of the two parameters in the equation for the 3-point experiment. Zajackowska briefly mentioned that discrepancy between a_3 and A , the slope of (8) directly obtained from the 3-point experiment and the one expected on the basis of the 4-point experiment, disappears if S is trained. It is one of the purposes of the present study to see whether that is the case and the authors computed the values of K and σ according to the above mentioned steps.

RESULTS AND DISCUSSION

Since adequacy of the computational method of K and σ and stability of the values obtained are at issue, not only final values of K and σ computed but also intermediary results will be reported as fully as possible.

As examples of basic data obtained in the experiment, two plots of Y against X are shown in Figs. 7 and 8 with straight lines (8) and (14) fitted. The standard deviations of experimental determinations of Y are marked by bars in the figures. Fig. 7 represents the most typical case the authors obtained as to goodness of fit of straight lines and Fig. 8 the case where the fit was relatively poor. As is clear from the figures, the fit was satisfactory in almost all the cases. Whole the results are given in Table 1 where g is a measure indicating the goodness of fit.

$$g = \frac{1}{n} \sum_{i=1}^n \left\{ |d_i| / \frac{s_i}{\sqrt{N}} \right\} \quad (19)$$

where n represents the number of points, d_i the deviation of Y_i from the fitted straight line, s_i the standard deviation in repetitions involved in the determination of Y_i , and N the number of repetitions. Since $s_i / \sqrt{N}$ yields the standard deviation of Y_i , the fit might be regarded as satisfactory

if g is less than, say, 2. There is no wonder that the fit was poorer in general for Equation (14) because a restriction is being imposed in addition in this case. Although it is clear that the deviations in some cases can not be solely ascribed to the within fluctuations of the repeated determinations, no consistent curvilinearity is apparent even in those cases. Deviations of the points from the straight lines seems random and might be due to errors in experimental conditions. It might have happened that the fixed points; Q_1 , Q_2 etc., are not exactly located as required or that the graduations on the board is slightly out of position when the E read the position of the movable light point.

The slopes in the 3-point experiment are in general smaller than the slopes calculated on the basis of the 4-point experiment, in short, $a_3 < A$. The same result was also obtained by Zajackowska (1956 a). In contradiction to Zajackowska's mentioning, however, there is slightest tendency in the present result that the discrepancy between a_3 and A tends to disappear as S accumulates experience in this kind of experiment. Due to the general tendency: $a_3 < A$, a little larger values are in general obtained for σ than for σ' .

As to K , it is remarkable that with three exceptions estimated values are within the interval from 0 to -1 as expected in (4). Of the exceptional cases, two (MS. in H and TK, 1960, in V) yield positive values for K and the other (HI, 1959, in V) gives K which is just slightly smaller than -1 . Among 30 cases reported in the study of Zajackowska (1956a), we find two cases where $K \leq -1$ and no case where $K > 0$. We can not tell definitely whether K becomes larger or smaller than K' in the above mentioned computational procedure, though the cases where $K > K'$ are a little larger in number in Table 1.

Of K as well as σ , repeatedly estimated values for the same S show satisfactory

stabilities in some cases but considerable fluctuations in others. With S s whose personal constants were repeatedly determined, averages and standard deviations in the repetitions are shown in Table 2. Coefficients of variability amount to more than 50% in three cases.

It might be premature to make any comparison, even tentatively, between the results in the two conditions; H and V. Suffice it to mention the following remarks in this context. Let us denote the largest phenomenal distance one can perceive, max D , by $\mathfrak{D}$. It is remarkable, though veiled by the cover of everyday-staleness, that the sky is sensed as a dome of finite radius and $\mathfrak{D}$ should represent the radius for the individual. That the sky appears to be a flattened dome seems to suggest that $\mathfrak{D}$ differs more or less according to the angle of elevation. A phenomenal distance from the observer to a point P is, in general, given as follows, by putting

$$\xi'_i = \eta'_i = \zeta'_i = 0$$

and omitting the subscript i in (2).

$$\begin{aligned} D &= \frac{2}{\sqrt{-K}} \sin^{-1} \left(\frac{\sqrt{-K}}{2} \frac{\rho}{\sqrt{1 + \frac{K}{4} \rho^2}} \right) \\ &= \frac{2}{\sqrt{-K}} \tan^{-1} \left(\frac{\sqrt{-K}}{2} \rho \right) \end{aligned} \quad (20)$$

By definition, max $\rho=2$, and then

$$\mathfrak{D} = \frac{2}{\sqrt{-K}} \tan^{-1} \sqrt{-K} \quad (21)$$

With one exception each, Table 2 shows that K 's are nearer to -1 in H than in V and σ 's are smaller in V than in H. The same tendency is predominant with the remaining S s as is clear from Table 1. If the results are taken without reserve on qualification, it might be said that the result has something to do with the following facts in everyday observation.

Differentiating (21) with respect to K , we have

$$\begin{aligned} \frac{d\mathfrak{D}}{dK} &= \frac{1}{(-K)(1+K)} \\ &\left(\frac{1+K}{\sqrt{-K}} \tan^{-1} \sqrt{-K} - 1 \right) \end{aligned} \quad (22)$$

hence, for $-1 < K < 0$, under the condition:

$$\tan^{-1} \sqrt{-K} < \frac{\sqrt{-K}}{1+K} \quad (23)$$

which holds unless K is not too near 0, (22) is negative and $\mathfrak{D}$ is a decreasing function of K . The nearer the value of K to -1 , the larger the value of $\mathfrak{D}$. That is to say, $\mathfrak{D}$ is smaller in V than in H. In fact, for example, $D_H=3.23$ for $K_H=-0.8$ and $\mathfrak{D}_V=2.87$ for $K_V=0.7$ or $\mathfrak{D}_H=2.25$ for $K_H=-0.3$ and $\mathfrak{D}_V=2.16$ for $K_V=-0.2$, which, ceteris paribus, may have something to do with the flattened form of the dome in the zenith. On the other hand, in the case of D , the matter is complicated for the reason that D is a function of K as well as σ . From (3) and (20),

$$\frac{\partial D}{\partial \sigma} = \frac{\partial D}{\partial \rho} \frac{\partial \rho}{\partial \sigma} = - \frac{\gamma \rho}{1 + \frac{K}{4} \rho^2} = -S \quad (24)$$

It is evident that S is positive and D is a decreasing function of σ because $\gamma > 0$, $\rho > 0$ and $\rho < 2/\sqrt{-K}$. For a given value of γ , therefore, the smaller the value of σ , the larger the perceived distance D , provided that K remains constant. Whereas, differentiating (20) with respect to K , we obtain

$$\begin{aligned} \frac{\partial D}{\partial K} &= \frac{\rho}{2(-K)(1 + \frac{K}{4} \rho^2)} \\ &\left[\left(1 + \frac{K}{4} \rho^2 \right) \tan^{-1} \left(\frac{\sqrt{-K}}{2} \rho \right) - 1 \right] = T \end{aligned} \quad (25)$$

In so far as, for $-1 < K < 0$,

$$\tan^{-1} \left(\frac{\sqrt{-K}}{2} \rho \right) < \frac{2\rho \sqrt{-K}}{4 + K\rho^2}$$

T is negative and D is a decreasing function of K . The nearer the value of K to -1 , the larger the perceived distance D for a given value of γ provided that σ remains constant. Therefore, for a given value of γ , disregarding the higher terms in Taylor's expansion, we have

$$\begin{aligned}
D_H - D_V &= D\{(K - JK), (\sigma + J\sigma)\} - D\{K, \sigma\} \\
&= \frac{\partial D}{\partial K} (-JK) + \frac{\partial D}{\partial \sigma} J\sigma \\
&= T(-JK) - SJ\sigma \quad (26)
\end{aligned}$$

If $T(-JK) < SJ\sigma$, then $D_V > D_H$ for a given value of γ . Let us consider the case that $K_H = -0.8$, $\sigma_H = 10$ and $K_V = -0.7$ and $\sigma_V = 9$ which approximates the fact in Table 2, then $D_H = 1.77$ and $D_V = 1.79$ for $\gamma = 0.03$ or $D_H = 1.35$ and $D_V = 1.42$ for $\gamma = 0.05$. That D_V appears larger than D_H in these cases may be said, *ceteris paribus*, to have some bearing on the well known fact that vertical lengths are apt to be overestimated within a certain range.

It is to be kept in mind, however, that comparing $\mathfrak{D}_H$ with $\mathfrak{D}_V$ or D_H with D_V is not necessarily legitimate for the reason that the unit of the scale is not necessarily same in both directions, H and V. In each direction, max ρ is arbitrarily defined as 2 and no direct method is available for making a comparison between max ρ 's in both direction.

SUMMARY

As the first of a series of investigations to study experimentally the theory developed by Luneburg for the purpose of describing the binocular space perception in terms of hyperbolic geometry, the results were reported which were obtained in the so called 3- and 4-point experiments. A striking feature of the Luneburg theory seems to consist in that a number of facts in binocular space perception of a S are deduced solely from his two personal constants; K and σ . The former is Gaussian curvature and $-1 < K < 0$ if the theory holds. The latter is a parameter related to the degree of depth perception and $\sigma > 0$. Luneburg suggested two experimental procedures for determining the values of K and σ which are conducted respectively with 3- and 4- small light points in the dark. Following Zajackowska's scheme, the authors tried to obtain by these procedures the values of K and σ with 8 Ss.

The values will be served as predictors for alley experiments in the article to follow. In the present study, however, the description was limited to the following points.

(1) Goodness of fit of the theoretical relationships predicted from the theory to the data of the 3- and 4-point experiments.

(2) Examination in detail of all the steps involved in the computational procedure of K and σ from the data.

(3) Stability of the estimated values of K and σ .

(4) Comparison between the values of K and σ in the horizontal direction and those in the vertical direction.

The goodness of fit was satisfactory and of 26 cases, with three exceptions, estimated values of K were within the range from 0 to -1 as expected from the Luneburg theory, though the repeatedly obtained values for the same Ss showed fluctuations of a considerable degree.

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