(2) With these values of the radii as well as given values of thickness and interval $l$'s, find $s_1'$, $s_2'$ and $s_3'$.

(3) With these $s$'s and $l$'s, calculate $\overline{a}_1$, $\overline{a}_2$ and $\overline{a}_3$ [Use the formulae (I)].

(4) With the values of the radii found in (1) and with the given values of $l$'s, calculate $Q_1$, $Q_2$, $Q_3$, and $Q_4$. [Use the definition formula of $Q$].

(5) Next calculate the formulae (VII).

(6) And then calculate $\overline{a}_1$ by (A*), $\overline{a}_2$ by (B*), $\overline{a}_3$ by (C*), $\overline{a}_4$ by (D*), and differential coefficients by (III), (IV), (V*), (VI*).

(7) With these values, calculate (II) and find $\Delta Q_1$, $\Delta Q_2$, $\Delta Q_3$, and $\Delta Q_4$, putting $\Gamma=0$, $S=0$ and $T=0$.

(8) Find $Q_1+\Delta Q_1$, $Q_2+\Delta Q_2$, $Q_3+\Delta Q_3$, and $Q_4+\Delta Q_4$, by the simple addition of the values found in (4) and (7).

(9) By the aid of definition formulae of optical invariants, find corrected radii from $Q_1+\Delta Q_1$, $Q_2+\Delta Q_2$, $Q_3+\Delta Q_3$ and $Q_4+\Delta Q_4$.

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**A Catadioptric Simple Lens.**

**By**

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In his celebrated work "Theorie der optischen Instrumente nach ABBE," Czapski deduced the focal length\(^{(1)}\) of a system which consists of a refracting surface and a reflecting one. When this system is used instead of a single mirror, it is desirable to remove the spherical aberration at least, in order to obtain a sharp image. Professor Suyehiro\(^{(2)}\) solved this problem in order to determine the form of the concave lens-mirror used in his optical torsionmeter. Although his mathematical ex-

\(^{(1)}\) There is a misprint in the expression of this focal length [P. 84 of Czapski's Theorie der optischen Instrumente nach ABBE (1904)]. It must be written as

\[ \frac{1}{2f} = \frac{1}{2f} = \frac{1}{n-1}p_1 - np_2 + 2d(n-1)p_1p_2 - \frac{d^2(n-1)^2}{n} p_1^2 - \frac{d^2(n-1)^2}{n} p_2^2 \]

pressions are true, there is an ambiguity of sign of the quantities $u$'s, $v$'s, and $r$'s. I want to report here the solution of this problem employing Abbe's optical invariants and detracting the ambiguity of sign.

In the annexed figure, let $ABB'A'$ be a catadioptric simple lens, the surface $BS_1B'$ being silvered. Hence $AS_1A'$ is the refracting surface whose radius is $r_1$ and $BS_2B'$ the reflecting surface whose radius is $r_2$. The positive values of $r_1$ and $r_2$ mean that the surfaces are convex toward the left in the annexed diagram or toward the given object, and vice versa. Next let $L_1$ be the position of the object, $L_1'$ its image after the refraction through the refracting surface $AS_1A'$ for the first time; then $L_1'$ will be the object with respect to the reflecting surface and $L_2'$ the image after reflection; at last $L_2'$ will be the object with reference to the refracting surface for the second time and $L_3'$ the final image. Put $L_1S_1=s_1$, $S_1L_1'=s_1'$, $S_1'\cdot S_2=s_2$, $S_2L_2'=s_2'$, $L_2'S_1=s_3$ and $S_1L_3'=s_3'$. The $s$'s which lie on the left side of the lens are taken negative and those on the right side positive. If we denote the optical invariant of the refracting surface for the first time with $Q_1$, that of the reflecting surface with $Q_2$ and that of the refracting surface for the second time with $Q_1'$, then these are expressed as follows:

\[ Q_1 = \frac{1}{r_1} - \frac{1}{s_1} = n \left( \frac{1}{r_1} - \frac{1}{s_1} \right) \]  
\[ Q_2 = n \left( \frac{1}{r_2} - \frac{1}{s_1} \right) = -n \left( \frac{1}{r_2} - \frac{1}{s_1} \right) \]  
\[ Q_1' = n \left( \frac{1}{r_1} + \frac{1}{s_3} \right) = -\frac{1}{r_1} + \frac{1}{s_3}' \]

where $n$ is the refractive index of the system.

From (1) and (3), we have

\[ Q_1 + Q_1' = \frac{1}{s_3}' - \frac{1}{s_1} = \mathfrak{A}. \]
CATADIOPTRIC SIMPLE LENS.

If we neglect the thickness of the system, \( s_1' = s_2, \ s_2' = s_3 \) and from three relations (1), (2) and (3) we can deduce the relation

\[ 2Q_2 = 2l \quad (5) \]

To this order of approximation the removal of spherical aberration may be expressed by

\[
Q_1^3 - Q_2^3 \left\{ \frac{n+3}{2} \frac{1}{s_1} + \frac{n}{s_2} \right\} + Q_1 \left\{ \frac{4n+5}{4} \frac{2l}{2} + n \frac{3l^2}{s_3} \right\}
- \frac{1}{4(n-1)} \left\{ \frac{(4n^2-3)l^3}{2} + n(2n-1) \frac{3l^2}{s_1} \right\} = 0 \quad (6)
\]

If we put \( s_2' = -ms_1 \), this equation can be verified to be identical with the equation proposed by Professor Suyehiro. (1) Or the equation (6) may be transformed into the form:

\[
(s_1Q_1)^3 - (s_1Q_2)^3 \left\{ \frac{n+3}{2} (s_1l) + n \right\} + (s_1Q_1) \left\{ \frac{4n+5}{4} (s_2l)^2 + n(s_3l) \right\}
- \frac{1}{4(n-1)} \left\{ \frac{4n^2-3}{2} (s_1l)^3 + n(2n-1)(s_3l)^2 \right\} = 0 \quad (7)
\]

From (6) or (7) combined with (1) we can obtain the value of \( r_1 \) of a catadioptric lens free from spherical aberration. To determine \( r_2 \), the equation

\[
\frac{2n}{r_2} = \frac{2(n-1)}{r_1} + \frac{1}{s_1} + \frac{1}{s_3'} \quad (8)
\]

may be of service, which is easily deduced from (1), (2) and (3).

As a numerical example, I will show the proportion of the lens-mirror whose data are inserted in Professor Suyehiro's paper. There \( n = 1.52, \ s_1 = -61 \text{mm} \) and \( s_3' = 61 \text{mm} \times 4 \); the equation (8) becomes

\[
(s_1Q_1)^3 + 1.3050(s_1Q_2)^3 + 2.4281(s_1Q_3) + 0.6011 = 0.
\]

This has only one real root which is equal to \(-0.28083\). Hence

\[
\frac{s_1}{r_1} = 0.71917
\]

and

\[
r_1 = -84.82\text{mm} \quad r_2 = -123.80\text{mm}
\]

As the signs of \( r_1 \) and \( r_2 \) are both negative, these two surfaces are

\[\text{(1) K. Suyehiro, loc. cit. the last equation of p. 498 which is the condition of abolition of spherical aberration.}\]
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converge toward the left or toward the given object. The result well coincides with that of Professor Suyehiro.

It may not be out of place to consider also the case when the object is at infinity, in which case \( s_1 = \infty \). If we denote the focal length with \( f \), then \( s_1' = f \) and \( \alpha = \frac{1}{f} \), \( Q_1 = \frac{1}{r_1} \). The equation (6) is simplified as

\[
(f Q_1)^2 - \frac{n+3}{5} (f Q_1)^2 + \frac{4n+5}{4} (f Q_1) - \frac{8(n-1)}{8(n-1)} = 0 \quad (9)
\]

and (8) becomes

\[
\frac{2n}{r_2} = \frac{2(n-1)}{r_1} + \frac{1}{f} \quad (10)
\]

if is positive when the focus lies on the right side and negative when it lies on the left side. From (9) and (10) we can determine \( r_1 \) and \( r_2 \).

By the principle of the reversibility of the right-path, this result may be applied to the case when the image is to be at infinity, or when it is required to send parallel rays from a point source. In this case, however, the size of the catadioptric lens is frequently so large that it necessary to introduce terms of correction to the radii by taking the thickness into consideration, which can be easily deduced as in other cases.

Let \( t \) be the axial thickness of the lens and put

\[
\frac{h_2}{h_1} = \omega_1, \quad \frac{h_3}{h_1} = \omega_2
\]

Then they can be expressed as

\[
\omega_1 = 1 - \frac{t}{s_1'}, \quad \omega_2 = \omega_1 \left( 1 - \frac{t}{s_2'} \right).
\]

And the image distance is given by

\[
\frac{1}{s_1'} = -\frac{n-1}{n} Q_1' + \frac{2}{n} \frac{Q_1}{\omega_1} + \frac{n-1}{n} \frac{Q_1}{\omega_1 \omega_2}.
\]

From which we have the expression of the reciprocal of the focal length \( f \):

\[
\varphi = \frac{1}{f} = \omega_2 \frac{n-1}{n} Q_1' + \frac{2Q_1}{n} + \frac{n-1}{n} \frac{Q_1}{\omega_1} \quad (A)
\]

and, on the other hand from (1) and (3)

\[
\varphi = \omega_2 (Q_1 + Q_1') \quad (B)
\]

is derived. From these two, we know
and
\[ \frac{\partial Q'_1}{\partial Q'_1} = \frac{n-1}{n} - \frac{n}{\sigma_1 \sigma_2} - n, \quad \frac{\partial Q'_1}{\partial Q'_2} = \frac{2}{\sigma_2} . \]

By the aid of (C), (A) is transformed into
\[ \varphi = 2Q_2 + (n-1)\left(\frac{1}{\sigma_1} - \sigma_2\right)Q_1 \tag{A^*} \]
and
\[ \frac{\partial \varphi}{\partial Q'_1} = (n-1)\left(\frac{1}{\sigma_1} - \sigma_2\right), \quad \frac{\partial \varphi}{\partial Q'_2} = 2. \]

If we express \( Q_1, \sigma_1 \) and \( \sigma_2 \) by \( r_1, r_2 \) and \( t \), the expression \( (A^*) \) of \( \varphi \) becomes nearly equal to that proposed by Czapski\(^{(1)}\), the difference being only in the sign. But when \( \varphi \) is expressed as Harting\(^{(2)}\) did, neglecting the thickness, great discrepancy occurs. This is another proof that the procedure here given is more accurate than that of Harting.

The spherical aberration is expressed by
\[ S = \frac{n-1}{n} \frac{Q_1^3}{2\sigma_1^4} - \frac{Q_1^2}{n} \left(\frac{n-1}{\sigma_1} \frac{Q_1}{n} + \frac{Q_1}{n}\right) \]
\[ \sigma_2 \left[ 2 \left(\frac{n-1}{n}\right) \frac{Q_2}{\sigma_2} + \left(\frac{n-1}{n}\right)^2 \frac{Q_1}{\sigma_1 \sigma_2} + \frac{n-1}{n} \frac{Q'_1}{\sigma_1} \right] \tag{D} \]
By the substitutions
\[ O_1 = \sigma_1^3 \frac{n-1}{n} Q_1, \quad O_2 = \sigma_2^3 \frac{n-1}{n} Q_2 \]
\[ O_1' = \sigma_2^3 \frac{n-1}{n} Q'_1, \quad P = \sigma_2 \left(\frac{n-1}{n}\right)^2 \frac{Q_1}{\sigma_1} \tag{E} \]
this can be expressed as
\[ S = \frac{n-1}{n} \frac{Q_1^3}{2\sigma_1^4} - \frac{Q_2}{n} \left(\frac{O_1 + \sigma_1^3 O_2}{n}\right) \]
\[ - Q_2 \left(\frac{2}{n} O_2 + P + \sigma_2 O_1'\right) \tag{D''} \]
and its partial differential coefficients are

\(^{(1)}\) See the foot-note of p. 495 of this paper.

Let \( \varphi \) and \( S \) be the values of the expressions \((A^*)\) and \((D^*)\) respectively, calculated with the given thickness and the radii estimated neglecting the thickness; \( \varphi \) and \( S \) are their required values, and in general we assume \( S=0 \). As \( \varphi \) and \( S \) are not equal to the required values of \( \varphi \) and \( S \), it must be attained by changing the radii. These relations are expressed by

\[
\frac{\partial S}{\partial Q_1} = 3\left(\frac{n-1}{n}\right)Q_1^2 - 2\left(\frac{n-1}{n}\right)\frac{Q_1}{\omega_1} - 2Q_1\left(\frac{n-1}{\omega_1} + \frac{1}{n}\right) - \frac{2Q_1}{\omega_1}\left(\frac{Q_1}{n} - n\omega_2\right) \quad (F)
\]

\[
\frac{\partial S}{\partial Q_2} = -4\frac{Q_2}{\omega_2} - 6\omega_1\left(\frac{Q_2}{n}\right)^2 - \frac{4Q_1}{\omega_2} - \frac{2Q_1}{n} - 2Q_1\left(\frac{n+1}{n}\right) \quad (G)
\]

\( \bar{Q}_1 \) and \( \bar{Q}_2 \) solved from these equations are to be added to \( Q_1 \) and \( Q_2 \) which are used to calculate the differential coefficients, \( \varphi \) and \( S \). By the aid of the definition-formulae of optical invariants \( Q \)'s, corrected values of the radii are obtained.