§ 5. * Derivation of the Ordinary Formalism.

We shall now show that the ordinary formalism of the quantum electrodynamics can be really derived from ours as the special case in which the surface $C$ reduces into a plane parallel to the $xyz$-plane.

As was stated in I our fundamental equation (III) can be integrated by means of the unitary operator $T[C; C_0]$ defined by

$$T[C; C_0] = \hat{U} \left( 1 - \frac{i}{\hbar} H_{I,I}(X) dV_x \right).$$

(5.1)

Namely, we obtain $\Psi[C]$ by giving its "initial" value $\Psi[C_0]$ in the form:

$$\Psi[C] = T[C; C_0] \Psi[C_0].$$

(5.2)

We shall now transform each field quantity $G$ by means of

$$G(X) \rightarrow T[C; C_0]^{-1} G(X) T[C; C_0]$$

(5.3)

and let $G[X; C]$ denote this transformed field. This field quantity depends, besides on $x, y, z$ and $t$, on the variable surface $C$. The transformation (5.3) induces also the transformation of the generalized $\phi$-vector $\Psi[C]$:

$$\Psi[C] \rightarrow T[C; C_0]^{-1} \Psi[C] = \Psi[C_0].$$

(5.4)

Then the transformed $\Psi$ depends no longer on the surface $C$. In this way we obtain the "Heisenberg picture" of the behavior of our system, in

* §§ 1, 2, 3 and 4 have appeared in the foregoing issues of this journal; Progr. Theor. Phys.; 2 (1947), 101.
which the \( \psi \)-vector depends no longer on \( C \), but the field quantities, on the other hand, depend on \( C \), not only on \( t \).

Also in our generalized Heisenberg picture the dependence of the field quantities on \( x, y, z \) and \( t \) are still given by the field equations (I) of the free fields. The dependences on \( C \), on the other hand, are given by

\[
\frac{\partial G[X; C]}{\partial C_p} = \frac{i}{\hbar} \left[ H_{I,II}(P), G(X; C) \right] \tag{5.5}
\]

as can be easily shown.

Now, the ordinary Heisenberg scheme of our system is obtained if one takes the variable plane \( C_T \) parallel to the \( xyz \)-plane whose intercept with the time axis is \( T \). If the field quantities in this ordinary scheme are denoted by affixing the superscript \( o \), they are given by

\[
\hat{G}(T, xyz) = G[t, x, y, z, C] \bigg|_{C = C_T} \tag{5.6}
\]

The time derivative of \( G \) with respect to the "common time" \( T \) is then given by

\[
\frac{\partial \hat{G}(T)}{\partial T} = \left\{ \frac{\partial G[t, C]}{\partial t} + \int_C \frac{\partial G}{\partial C_p} dF_p \right\} \bigg|_{C = C_T} \bigg|_{t = t_T} = \left\{ \frac{\partial G}{\partial t} + \frac{i}{\hbar} \left[ H_{I,II}, G \right] \right\} \bigg|_{C = C_T} \bigg|_{t = t_T} \tag{5.7}
\]

If we denote by \( \text{Div}' \), \( \text{Curl}' \), etc. the differential operators in which the time differentiation is \( \partial / \partial T \), and by \( \text{Div} \), \( \text{Curl} \), etc. those in which the time differentiation is \( \partial / \partial t \), then we have the relations

\[
\text{Div}' \hat{G} = \left\{ \text{Div} G - \frac{i}{\hbar} \left[ H_{I,II}, G_o \right] \right\} \bigg|_{t = t_T} \bigg|_{C = C_T}
\]

\[
\text{Curl}' \hat{G} = \left\{ \text{Curl} G \bigg|_{t = t_T} \bigg|_{C = C_T} \text{ for the space component}, \right. \\
\left. - \text{grad} \hat{G}_o - \frac{\partial \hat{G}}{\partial t} + \frac{i}{\hbar} \left[ H_{I,II}, \hat{G} \right] \bigg|_{t = t_T} \bigg|_{C = C_T} \text{ for the time component}. \right. \tag{5.8}
\]

Applying these formulas to the potential \( \hat{A} \), noting that \( \hat{A} \) is commutable with \( H_{I,II} \), we obtain
Using the last relation, we have next

\[ \text{Div}' \text{Curl}' \boldsymbol{A} = \text{Div}' \{ \text{Curl} \boldsymbol{A} \}_{t=T, \; \mathcal{C}=\mathcal{C}_T} \]

but applying (5.8), this is expressed as

\[ = \left\{ \text{Div} \text{Curl} \boldsymbol{A} + \frac{i}{\hbar} \left[ \overline{H}_{1, \text{In}} \left( \text{Curl} \boldsymbol{A} \right) \right] \right\}_{t=T, \; \mathcal{C}=\mathcal{C}_T} \]

\[ = \left\{ \text{Div} \text{Curl} \boldsymbol{A} + \frac{i}{\hbar} \left[ \overline{H}_{1, \text{In}} \left( -\frac{\partial \hat{\boldsymbol{A}}}{\partial t} - \text{Grad} \boldsymbol{A} \right) \right] \right\}_{t=T, \; \mathcal{C}=\mathcal{C}_T} \]

\[ = \text{Div} \text{Curl} \boldsymbol{A} + \epsilon \left( \mathbf{j} \right)_{t=T} \]

where the symbol \( \langle a \rangle \) means the four-vector whose time component vanishes and whose space component is \( \hat{a} \). If we notice further the formula

\[ \text{Grad Div} \boldsymbol{A} = \text{Div Curl} \boldsymbol{A} + \Box \boldsymbol{A} , \]

we find

\[ \text{Div}' \text{Curl}' \boldsymbol{A} = \left\{ \text{Grad Div} \boldsymbol{A} - \Box \boldsymbol{A} + \epsilon \langle \mathbf{j} \rangle \right\}_{t=T, \; \mathcal{C}=\mathcal{C}_T} \]

But, since \( \boldsymbol{A} \) satisfies \( \Box \boldsymbol{A} = 0 \), this last relation yields

\[ \text{Div}' \text{Curl}' \boldsymbol{A} = \left\{ \text{Grad Div} \boldsymbol{A} + \epsilon \langle \mathbf{j} \rangle \right\}_{t=T, \; \mathcal{C}=\mathcal{C}_T} \]  \hspace{1cm} (5.10)

Now, the auxiliary condition (IV) requires, on the other hand,

\[ \left\{ \text{Grad Div} \boldsymbol{A}(P) + \epsilon \text{Grad}_P \left[ (\mathbf{j}(P'), \; \mathbf{N}(P')) D(P'-P) dF_P \right] \right\}_{T=0} = 0 . \]  \hspace{1cm} (5.11)

Since the point \( P \) is here considered as lying on \( \mathcal{C} \), the integral can be carried out in the reference system whose space axes are tangent to \( \mathcal{C} \) at \( P \). As in this system \( (\mathbf{j}(P'), \; \mathbf{N}(P')) \) can be replaced simply by \( -\mathbf{j}_0(P') \), and \( \text{Grad}_P D(P'-P) \) has the non-vanishing components \( \partial (\overrightarrow{x}' - \overrightarrow{x}) \) only in the \( \overrightarrow{0} \)-th direction, we obtain
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\[ \text{Grad}_P \oint (\mathbf{J}(P'), \mathbf{N}(P')) D(P' - P) \, dF_{P'} \]

\[ = \begin{cases} -J_o(P) & \text{for the time component} \\ 0 & \text{for the space component.} \end{cases} \]

This gives, returning to the general reference system,

\[ \text{Grad}_P \oint (\mathbf{J}(P'), \mathbf{N}(P')) D(P' - P) \, dF_{P'} = (\mathbf{J}(P), \mathbf{N}(P)) \mathbf{N}(P). \]

The auxiliary condition \( (5\cdot11) \) is thus expressed in

\[ \{ \text{Grad Div} \, \mathbf{A}(P) + \varepsilon (J(P), \mathbf{N}(P)) \mathbf{N}(P) \} \mathbf{T} = 0, \]

which for \( C = C_T \) reduces into

\[ \{ \text{Grad Div} \, \mathbf{A} - \varepsilon J_0 \} \mathbf{T} = 0 \quad (5 \cdot 12) \]

where the symbol \( \langle a_0 \rangle \) means the four-vector whose space component is zero and whose time component is \( a_0 \).

We find in this way that, operated on \( \mathbf{T} \), Grad Div \( \mathbf{A} \) on the right-hand side of \( (5\cdot10) \) can be replaced by \( \varepsilon \langle J_0 \rangle \), and this gives

\[ \{ \text{Div' Curl'} \, \mathbf{A} - \varepsilon J \} \mathbf{T} = 0 \quad (5 \cdot 13) \]

the vectors \( \langle \mathbf{J} \rangle \) and \( \langle J_0 \rangle \) making up the four-vector \( J \).

The last equation \( (5\cdot13) \) corresponds just to the half set of the ordinary Maxwell equation:

\[
\begin{align*}
\text{curl} \overset{\cdot}{\mathbf{H}} - \frac{\partial \overset{\cdot}{\mathbf{E}}}{\partial T} &= \varepsilon \overset{\cdot}{\mathbf{J}} \\
\text{div} \overset{\cdot}{\mathbf{E}} &= \rho.
\end{align*}
\]

The remaining set of equations:

\[
\begin{align*}
\text{curl} \overset{\cdot}{\mathbf{E}} + \frac{\partial \overset{\cdot}{\mathbf{H}}}{\partial T} &= 0 \\
\text{div} \overset{\cdot}{\mathbf{H}} &= 0
\end{align*}
\]
is trivial, since, expressed in terms of the potential, this is

\[ \sum_{\text{cyl}} \text{Grad}' \text{ Curl}' \vec{A} = 0, \]

which is satisfied identically. (\( \sum_{\text{cyl}} \) means cyclic sum)


We now carry out the elimination of the auxiliary condition by the method described in A.

First we expand the operator \( \hat{\mathcal{E}}_x[C] \) into the Fourier integral of the form:

\[
\hat{\mathcal{E}}_x = \frac{1}{(2\pi)^{3/2}} \int \left\{ \hat{\mathcal{E}}(\vec{k}) e^{i(\vec{k}, \vec{x})} + \hat{\mathcal{E}}^+(\vec{k}) e^{-i(\vec{k}, \vec{x})} \right\} \frac{d\vec{k}}{\vec{k}}
\]

\[
\hat{\mathcal{E}}(\vec{k}) = i[\vec{k}, \vec{A}(\vec{k})] - \frac{\epsilon}{2} \frac{i}{(2\pi)^{3/2}} \int (\vec{J}(P), \vec{N}(P)) e^{-i(\vec{k}, \vec{x}_P)} d\vec{F}_P
\]

\[
\hat{\mathcal{E}}^+(\vec{k}) = -i[\vec{k}, \vec{A}^+(\vec{k})] + \frac{\epsilon}{2} \frac{i}{(2\pi)^{3/2}} \int (\vec{J}(P), \vec{N}(P)) e^{i(\vec{k}, \vec{x}_P)} d\vec{F}_P.
\]

(6.1)

As stated in A, we introduce the following four sets of coordinates and momenta(3)

\[
\begin{align*}
R &= (r, \vec{A}) \quad R^+ = (r, \vec{A}^+) \\
S &= (s, \vec{A}) \quad S^+ = (s, \vec{A}^+) \\
K &= (l, \vec{A}^+) \quad K^+ = -(\vec{k}, \vec{A}) \\
L &= (l, \vec{A}) \quad L^+ = (\vec{k}, \vec{A}^+)
\end{align*}
\]

(6.2)

where \( r, s \) and \( l \) are three four-vectors satisfying the relations

\[
\begin{align*}
(r, r) &= (s, s) = 1, \quad (l, l) = 0 \\
(r, s) &= 0, \quad (r, k) = (s, k) = 0 \\
(r, l) &= (s, l) = 0, \quad (l, k) = 1.
\end{align*}
\]

(6.3)

The vector \( \vec{k} \) satisfies, of course, \( (\vec{k}, \vec{k}) = 0 \), because it is the propagation vector of the electromagnetic wave.

It is easily verified that, on account of (6.3), the commutation relations now take the following forms:

\[
\begin{align*}
[R(\hat{k}), R^+(\hat{k}')] &= [S(\hat{k}), S^+(\hat{k}')] \\
[K(\hat{k}), K^+(\hat{k}')] &= [L(\hat{k}), L^+(\hat{k}')] = + \frac{\hbar k}{2} \delta(\hat{k} - \hat{k}'),
\end{align*}
\]

(6.4)

other commutators = 0

which show that our sets (6.2) represent really four canonically conjugate sets.

In terms of these variables the auxiliary condition is expressed by

\[
\begin{align*}
(K^+ + \frac{f}{2})\Psi &= 0 \\
(L^+ - \frac{f^*}{2})\Psi &= 0,
\end{align*}
\]

(6.5)

in which \(f\) stands for

\[
f = \frac{\varepsilon}{(2\pi)^{3/2}} \int e^{-i(k, \chi f')} dF_{f'},
\]

(6.6)

and \(f^*\) is its conjugate complex quantity.

We introduce now the representation in which \(R, S, K\) and \(L\) are diagonal. Then the \(\psi\)-vector \(\Psi\) is represented by a functional of \(R, S, K, L\) and some variables describing the electron field. The auxiliary condition (6.5) in this representation takes the form:

\[
\begin{align*}
\left( -\frac{\hbar k}{2} \frac{\delta}{\delta K} + \frac{f}{2} \right)\Psi &= 0 \\
\left( -\frac{\hbar k}{2} \frac{\delta}{\delta L} - \frac{f^*}{2} \right)\Psi &= 0
\end{align*}
\]

(6.7)

which can be satisfied identically by

\[
\Psi = e^\chi \Omega
\]

with

\[
\chi = -\frac{1}{\hbar} \int (Lf^* - Kf) \frac{d\hat{k}}{\hat{k}}
\]

(6.8)

\(\Omega\) being a functional not containing \(K\) and \(L\).

It is now required to find the generalized Schrödinger equation for \(\Omega\).
We substitute thus (6.8) into (III), and calculate \( \left( H_{1,II} + \frac{\hbar}{i} \frac{\partial}{\partial C_{P}} \right) e^{x\Omega} \).

Noting

\[
\begin{align*}
A(x) &= \frac{1}{(2\pi)^{3/2}} \int \{ A(k) e^{i(k, x)} + A^{+}(k) e^{-i(k, x)} \} \frac{dk}{k} \\
A(k) &= R \mathbf{r} + S \mathbf{s} + L \mathbf{k} - K^{+} \mathbf{l} \\
A^{+}(k) &= R^{+} \mathbf{r} + S^{+} \mathbf{s} + K \mathbf{k} + L^{+} \mathbf{l},
\end{align*}
\]

we decompose \( A(X) \) into three parts:

\[
A(X) = A_{k}(X) + A_{l}(X) + B(X)
\]

with

\[
\begin{align*}
A_{k}(X) &= \frac{1}{(2\pi)^{3/2}} \int \mathbf{k} \left( L e^{i(k, x)} + K e^{-i(k, x)} \right) \frac{dk}{k} \\
A_{l}(X) &= \frac{1}{(2\pi)^{3/2}} \int \mathbf{v} \left( -K^{+} e^{i(k, x)} + L^{+} e^{-i(k, x)} \right) \frac{dk}{k} \\
B(X) &= \frac{1}{(2\pi)^{3/2}} \int \mathbf{s} \left( S e^{i(k, x)} + S^{+} e^{-i(k, x)} \right) \frac{dk}{k} + \frac{1}{(2\pi)^{3/2}} \int \mathbf{s} \left( S e^{i(k, x)} + S^{+} e^{-i(k, x)} \right) \frac{dk}{k}
\end{align*}
\]

As discussed in A, two components \( A_{k} \) and \( A_{l} \) in (6.10) are to be regarded as describing the generalized longitudinal components, while \( B \) the generalized transverse component of the field. The energy \( H_{1,II} = -\epsilon(\mathbf{J}, A) \) is divided also into three parts corresponding to the above decomposition of the field:

\[
H_{1,II} = H_{k} + H_{l} + H_{B}
\]

with

\[
\begin{align*}
H_{k} &= -\epsilon(\mathbf{J}, A_{k}) \\
H_{l} &= -\epsilon(\mathbf{J}, A_{l}) \\
H_{B} &= -\epsilon(\mathbf{J}, B).
\end{align*}
\]

We calculate now successively \( H_{k} e^{x\Omega} \), \( H_{l} e^{x\Omega} \) and \( \frac{\hbar}{i} \frac{\partial}{\partial C_{P}} e^{x\Omega} \) in the following way:
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(i) The calculation of $H_k(P)e^x\Omega$.

According to (6.10) and (6.11), $H_k$ contains neither $K^+$ nor $L^+$. Further, the current density $J(P)$ in $H_k$ commutes with the expression $(J(P'), N(P'))$ in $f$ as can be shown by carrying out the integration in $f$ in the reference system whose space axes are tangent to $C$ at $P$. These two facts result in that $H_k(P)$ commutes with $e^x$, so that we have

$$H_k e^x \Omega = e^x H_k \Omega. \quad (6.12)$$

(ii) The calculation of $H_l(P)e^x\Omega$.

$$H_l e^x \Omega = -\frac{\hbar}{(2\pi)^{3/2}} \int \langle J, U \rangle \left\{ \frac{k^e i(k, x)}{2} \frac{\delta}{\delta K} - \frac{k^e -i(k, x)}{2} \frac{\delta}{\delta L} \right\} \frac{dk}{k} e^x \Omega$$

$$= -\frac{\hbar}{(2\pi)^{3/2}} e^x \int \langle J, U(k) \rangle \left\{ \frac{ke i(k, x)}{2} \frac{\partial e^x}{\partial K} - \frac{ke -i(k, x)}{2} \frac{\partial e^x}{\partial L} \right\} \frac{dk}{k} \Omega$$

$$= -\frac{e^x}{(2\pi)^{3/2}} \int \langle J, U(k) \rangle \frac{dk}{k} \int_c \langle J(P'), N(P') \rangle \left\{ e^{i(k, x_P-x_{P'})} \right\} dF_{P'} \Omega$$

If we put

$$L(X) = \frac{1}{16\pi^3} \int \langle \hat{U}(k) \rangle \left\{ e^{i(k, x)} + e^{-i(k, x)} \right\} \frac{dk}{k}, \quad (6.13)$$

we have, therefore,

$$H_l e^x \Omega = -e^x \int \langle J(P) L(P-P') \rangle (J(P'), N(P')) dF_{P'} \Omega \quad (6.14)$$

(iii) The calculation of $\frac{\hbar}{i} \frac{\delta}{\delta C_p} e^x \Omega$.

We have first

$$\frac{\hbar}{i} \frac{\delta}{\delta C_p} e^x \Omega = e^x \left\{ \frac{\hbar}{i} \frac{\delta L}{\delta C_p} + \frac{\hbar}{i} \frac{\delta}{\delta C_p} \right\} \Omega.$$
But we have

\[ \frac{\hbar}{i} \frac{\delta \mathcal{L}}{\delta C_P} = \frac{i \varepsilon}{(2\pi)^{3/2}} \int L(\mathbf{k}) \frac{dk}{k} \frac{\delta}{\delta C_P} \int c \left( \mathbf{J}(P'), \mathbf{N}(P') \right) e^{i(k, x_P')} dF_P', \]

\[ - \frac{i \varepsilon}{(2\pi)^{3/2}} \int K(\mathbf{k}) \frac{dk}{k} \frac{\delta}{\delta C_P} \int c \left( \mathbf{J}(P'), \mathbf{N}(P') \right) e^{-i(k, x_P')} dF_P'. \]

If one uses here (3.7), we obtain

\[ \frac{\hbar}{i} \frac{\delta \mathcal{L}}{\delta C_P} = - \frac{i \varepsilon}{(2\pi)^{3/2}} \int L(\mathbf{k}) \text{Div} \left( \mathbf{J}(P) e^{i(k, x_P)} \right) \frac{dk}{k} \]

\[ + \frac{i \varepsilon}{(2\pi)^{3/2}} \int K(\mathbf{k}) \text{Div} \left( \mathbf{J}(P) e^{-i(k, x_P)} \right) \frac{dk}{k}, \]

\[ = - \frac{\varepsilon}{(2\pi)^{3/2}} \int \left( \mathbf{J}(P), \mathbf{k} \right) L(\mathbf{k}) e^{i(k, x_P)} \frac{dk}{k} \]

\[ + \frac{\varepsilon}{(2\pi)^{3/2}} \int \left( \mathbf{J}(P), \mathbf{k} \right) K(\mathbf{k}) e^{-i(k, x_P)} \frac{dk}{k}, \]

\[ = \varepsilon \left( \mathbf{J}(P), \mathbf{A}^k(P) \right) = -H_k. \quad (6.15) \]

Now summing up the results (i), (ii), and (iii), we obtain

\[ \left( H_{11}(P) + \frac{\hbar}{i} \frac{\delta}{\delta C_P} \right) \Psi = \left( H_k(P) + H_t(P) + H_B(P) + \frac{\hbar}{i} \frac{\delta}{\delta C_P} \right) e^{\frac{i}{\hbar} \mathcal{Q}} \]

\[ = e^{\frac{i}{\hbar} \mathcal{Q}} \left( H_B(P) + \varepsilon^2 G(P) + \frac{\hbar}{i} \frac{\delta}{\delta C_P} \right) \mathcal{Q} \quad (6.16) \]

where \( G(P) \) is defined by

\[ G(P) = - \int c \left( \mathbf{J}(P), \mathbf{L}(P-P') \right) \left( \mathbf{J}(P'), \mathbf{N}(P') \right) dF_{P'}. \quad (6.17) \]

Multiplying (6.16) by \( e^{-\frac{i}{\hbar} \mathcal{Q}} \) from the left we obtain the required equation for \( \mathcal{Q} \):

\[ \left\{ H_B(P) + \varepsilon^2 G(P) + \frac{\hbar}{i} \frac{\delta}{\delta C_P} \right\} \mathcal{Q} = 0 \quad (6.18) \]
in which the longitudinal parts of the field appear no longer, while it contains, on the other hand, the term $\varepsilon \epsilon G(P)$, which, according to (6.17), represents the effect on the electron field at the point $P$ of the electron field at other points $P'$.


We have thus shown that it is in fact possible to describe the behavior of the electromagnetic field interacting with electrons in a perfectly relativistic form according to our general scheme developed in I. But if we wish to apply similar methods to more general cases, for instance, to the cases of the meson field interacting with the electromagnetic field or the nucleon field, some generalization is necessary. In fact, in the case of the quantum electrodynamics the situation is exceptionally simple owing to the following two facts: (i) the interaction energy density $H_{1,II}(P)$ is a scalar function of the fields and (ii) the integrability condition $[H_{1,II}(P), H_{1,II}(P')] = 0$ is satisfied from the beginning. These simplifying facts do no longer hold in the more general cases mentioned above.

However, if some generalization of the formalism is made, it is possible to develop the similar theory also in these more complicated cases as will be shown in the later paper. The development of such a theory seems to us of interest, not only because we obtain thus formally more satisfactory theory, but also because in this way we can hope that some new aspects of the difficulties underlying the current quantum theory of the fields would reveal itself. Thus, for instance, the question arises: what bearing would have the so-called cut-off hypothesis on our theory?

In fact, Heisenberg has thought that $H_{1,II} dV$ in the transformation functional $\frac{\partial}{\partial \epsilon_0} \left( 1 - \frac{i}{\hbar} H_{1,II} dV \right)$ might be the suspicious element of the current theory. Miyazima has once noticed that, although our theory seems at first sight to allow the introduction of a relativistically invariant cut-off process, taking as $H_{1,II}(P)$ not the energy density just at the world point $P$ but some average over a finite world region surrounding $P$, such a

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(6) T. Miyazima: Riken Ihö (Bull. I.P.C.R.), 23 (1944), 27.
procedure breaks necessarily the condition of integrability of the fundamental equation (III). Also it will be of great interest to examine Dirac's $\lambda$-process\(^{(7)}\) in the light of our theory. This process will be certainly incompatible with the integrability condition of the fundamental equation. In such a situation we must ask: is it possible to get over this difficulty by some modification of the theory? Or, should we attribute physical meanings to the non-integrable $\Psi$? It is also possible that the defect of the current theory is more deeply so that the formal feature of our theory will be maintained still in the future theory.

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