Influences of Non-Newtonian Viscosity and Elasticity on Potential Flow Analogy by Hele-Shaw Cell

Win Shwe MAW, Tsutomu TAKAHASHI, and Masataka SHIRAKASHI

Department of Mechanical Engineering, Nagaoka University of Technology,
Kamitomiokamachi 1603-1, Nagaoka 940-2188, Japan
(Received : December 22, 2003)

The validity of potential flow analogy by the Hele-Shaw flow of non-Newtonian fluids is examined through experiments using two shear-thinning elastic polymer solutions for three different flow configurations, i.e. flows around a circular cylinder and a square cylinder and flows through an abruptly converging-diverging channel (slit). Although the polymer solutions are highly shear-thinning and elastic, their flows well reproduce the corresponding two-dimensional potential flow patterns of the respective flow configurations when the flow rate is very low. The deviation occurs at values of Reynolds number much lower than the critical value for inertia effect, and in the opposite way of the inertia effect. An analysis for inelastic non-Newtonian fluids shows that non-constant viscosity does not affect the potential flow analogy, and the potential flow patterns are observed at flows with considerable values of the first normal stress difference in shear flow. Therefore, the disturbance to the potential flow pattern is not due to the non-Newtonian viscosity or the elasticity in shear flow, but attributed to the elongational stress due to the elasticity of the polymer solutions.

Key Words: Hele-Shaw flow / Potential flow patterns / Viscoelastic fluid / Shear-thinning / Elongational stress

1. INTRODUCTION

The Hele-Shaw cell is a channel between two transparent parallel plates with a small separation. When a plate of an arbitrary configuration is inserted between the two plates so that it completely fills the gap, the resulting pattern of streamlines well reproduces that in the two-dimensional potential flow with the same boundary geometry, provided that the fluid is Newtonian and the inertia force is negligible.1), 2) The validity of this potential flow analogy using Newtonian fluid is fully verified and the Hele-Shaw apparatus has been used exclusively for instruction purpose in fluid mechanics education.3) Recently, flows of non-Newtonian fluids through parallel plates has become of practical importance and research works are carried out on non-Newtonian fluid flows in a small gap between parallel plates, such as flows associated with the polymer film processing and liquid crystal products.4), 5), 6), 7) It is common that such materials have both non-Newtonian viscosity and elasticity which are not consistent with the bases for the potential flow analogy of the Hele-Shaw flow. The present authors observed streak patterns of flows in a Hele-Shaw cell around a circular cylinder and a square cylinder, and flow passing through an abruptly converging-diverging channel (slit), using two viscoelastic polymer solutions, and found that the potential flow analogy holds when flow rate is very low and the flow pattern is disturbed when the flow rate is higher but the Reynolds number is still very low.8) In this study, effects of non-Newtonian viscosity and elasticity on the potential flow analogy is examined by observing flow patterns of two polymer solutions for the three configurations, together with the aid of a theoretical analysis on the Hele-Shaw flow of purely viscous non-Newtonian fluid.

NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td>characteristic length of the body inserted in the gap</td>
</tr>
<tr>
<td>f(x)</td>
<td>similar gap-wise velocity distribution</td>
</tr>
<tr>
<td>h</td>
<td>half gap of Hele-Shaw cell</td>
</tr>
<tr>
<td>K</td>
<td>pseudo-viscosity in power-law model</td>
</tr>
<tr>
<td>N1</td>
<td>first normal stress difference in steady shear flow</td>
</tr>
<tr>
<td>n</td>
<td>power index in power-law model</td>
</tr>
<tr>
<td>O-x1, x2, x3</td>
<td>coordinates of Hele-Shaw cell analysis, Fig.1</td>
</tr>
<tr>
<td>p</td>
<td>isotropic pressure</td>
</tr>
<tr>
<td>Re</td>
<td>Reynolds number, Eq.(2)</td>
</tr>
<tr>
<td>U</td>
<td>characteristic velocity defined by flow rate divided by the cross-sectional area of Hele-Shaw cell</td>
</tr>
</tbody>
</table>
\( \mathbf{v}(v_1, v_2, v_3) \) velocity vector

\( \mathbf{V}(V_1, V_2) \) velocity vector on the central \((x_3=0)\) plane

\( \dot{\gamma} \) shear rate

\( \dot{\gamma}_w \) wall shear rate, \( \dot{\gamma}_w = \frac{2n+1}{n} \cdot \frac{U}{h} \)

\( \mu \) viscosity of fluid

\( \sigma \) shear stress in steady shear flow

\( II_e, III_e \) the second and the third invariants of strain rate tensor

2. EXPERIMENTAL

2.1 Experimental apparatus

The dimensions of the Hele-Shaw cell channel are 140 mm width, 200 mm length and 1 mm gap as shown in Fig.1, where the coordinate system used later in Chap.4 is also presented. A circular cylinder with diameter \( d = 25 \) mm, a square cylinder with side length \( d = 25 \) mm or an abruptly converging-diverging channel (slit) with an opening \( d = 10 \) mm and length \( L = 10 \) mm is inserted in the cell, and test fluid added with 0.3 wt% dye is introduced through an array of thin capillaries upstream the cell to visualize the flow.

2.2 Test fluids

A 0.2 wt% polyacrylamide solution in water (PAA/W-solution) and a 0.2 wt% polyacrylamide solution in water added with 10 wt% rice syrup (PAA/W+RS-solution) are used as viscoelastic fluids. Water is used as a Newtonian fluid for comparison. Rheological properties of the two PAA-solutions in steady shear flow measured by a cone-plate type instrument are shown in Fig.2. These fluids are highly shear-thinning and the shear viscosity is well modeled by the power-law

\[
\mu = K \dot{\gamma}^{n-1}
\]

as shown by the broken lines in the figure.

Table 1 shows the values of power-law constants of the two solutions. In Fig.2, values of the first normal stress difference \( N_1 \) are considerably large at \( \dot{\gamma} > 1 \) \( 1/s \) compared with the shear stress \( \sigma \) in the both PAA-solutions. The dotted lines in Fig.2 are power-law fitting curves used later to estimate \( N_1 \) at \( \dot{\gamma}_w \) obtained from measured flow rate. The Reynolds number \( Re \) for the Hele-Shaw flow is defined as the ratio of the inertia force to the shear stress on the wall, i.e.,

\[
Re = \frac{\rho Ud}{[\mu]_w (\frac{h}{d})^2},
\]

where \([\mu]_w\) is the shear viscosity given by Eq.(1) at the wall shear rate \( \dot{\gamma}_w \) in the plane Poiseuille flow of the power-law fluid.

3. OBSERVED FLOW PATTERNS

3.1 Behaviour of dye streaks with increasing flow rate

In Figs.3,4 and 5, flow patterns for the PAA/W-solution at a very small flow rate are compared with those of water for the three configurations. The directions of flows are downward in all the photographs in this paper. In these photographs of the PAA-solution, it is seen that:

i) dye streaks are not diffused through passing the body,

ii) no separation or no vortex is observed, and

iii) the total flow pattern is symmetric between upstream and downstream of the inserted body.

Table I The values of pseudo-viscosity and power index for the PAA-solutions.

<table>
<thead>
<tr>
<th></th>
<th>PAA/W</th>
<th>PAA/W+RS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K ) (Pa.s(^n))</td>
<td>0.448</td>
<td>0.376</td>
</tr>
<tr>
<td>( n ) (-)</td>
<td>0.46</td>
<td>0.51</td>
</tr>
</tbody>
</table>

Fig.1 Hele-Shaw cell and the coordinate system.

Fig.2 Rheological properties of two PAA-solutions in steady shear flow (shear viscosity \( \mu \), shear stress \( \sigma \), the first normal stress difference \( N_1 \) versus shear rate \( \dot{\gamma} \)).
Consequently, the whole flow pattern of the PAA/W-solution is seen to be essentially identical with its counterpart of water for all the three configurations, showing that the Hele-Shaw flow of the PAA/W-solution well reproduce the potential flow pattern.

When the flow rate is increased, the flow patterns of the PAA/W-solution deviate from the potential flow pattern in a way quite different from the case of water, as shown in Figs. 6, 7 and 8. The streamlines approaching the circular cylinder are shifted slightly further from the upstream stagnation point and they approach closer to the downstream stagnation point, as seen in Fig. 6(a). It is interesting that a similar behaviour of viscoelastic fluid is observed in a two-dimensional flow around a circular cylinder and flow past a sphere. While, the flow of water around the circular cylinder separates near the maximum width position and wake is formed behind it when the flow rate is increased and Re approaches the order of unity as shown in Fig. 6(b).

This behaviour of streak lines of the two solutions is more clearly observed in flows around the square cylinder. The symmetric potential flow pattern is disturbed into asymmetric between upstream and downstream in a similar way as in the case of the circular cylinder when the flow rate is increased. When the flow rate is still increased, a vortex similar to the lip vortex observed in the slit entry flow forms at each front edge. These vortices grow with increasing flow rate and coalesce to form a considerably large convex bubble on the front surface as shown in Fig. 7(a). It should be noted that the streamlines attach to the rear surface approaching to the stagnation point closer than those in Fig. 4(a).

When the flow rate is increased, the behaviour of the flow pattern of PAA/W-solution in the slit flow is also very different from that of water. A pair of lip vortices is formed at both sides of the slit entrance, while the streamlines remain attached to the wall at the downstream section. The lip vortices grow with increasing flow rate and a funnel shaped inflow region is formed as seen in Fig. 7(a). In contrast, the flow of water passing through the slit forms a jet-like pattern due to inertia effect when the Re is higher as seen in Fig. 8(b). Similar behaviour of dye streaks is observed in the slit flow of the PAA/W+RS-solution.

The common feature of PAA-solutions deviating from potential flow pattern for all the configurations is that streamlines are likely to detach from the body in the region of converging streamlines, such as near the front stagnation point of a cylinder or upstream of the converging channel, and they are likely to attach closer to the wall in the region of diverging streamlines such as in the wake of the cylinder or exit of the
slit. This behaviour of the PAA-solutions is quite opposite to that caused by the inertia effect observed in water flows. Another remarkable feature in the PAA-solution flows is that the dye streaks remain definite even when the potential pattern is disturbed, while they are blurred in the case of water flow affected by inertia force. Since the streaks at the lowest flow rates keep definite contour to the downstream end of the cell, the molecular diffusion effect is negligible for all the test fluids. Hence, the blurred or diffused streaks at higher flow rates are caused by difference in the streamlines on planes with different $x_3$-values. Therefore, the dye streaks with definite contour suggest that streamlines of PAA-solutions on planes with different $x_3$-values well coincide each other even when the potential flow pattern is disturbed.

3.2 Critical values for the potential flow pattern
The onset of deviation from the potential streamlines is most clearly seen in the difference between the upstream and the downstream streak lines both for flows of water and PAA-solutions. Hence, asymmetry in dye streaks was examined to determine whether the flow pattern reproduces the potential flow or not. The maximum flow rate for the potential flow pattern and the minimum flow rate for disturbed patterns were thus obtained from many photographs for the three configurations.

The characteristic velocity $U$, the wall shear rate $\dot{\gamma}_w$ and Reynolds number $Re$ corresponding to the maximum potential and minimum disturbed flow patterns are presented in Tables II and III for water and the PAA solutions respectively. The ratio of $N/\sigma$ estimated by the power-law curves in Fig.2 is also added in the latter. The real critical values are somewhere between the maximum potential flow and the minimum disturbed flow values.

4. DISCUSSION

4.1 Effect of inertia force
For the Hele-Shaw flow of Newtonian fluids, it is known that the inertia force effect is negligible when $Re \ll 1$. In Table II, the critical Reynolds number $[Re]$, for water is known to be around 0.2 for the circular and the square cylinder, and considerably lower for the slit flow, say around 0.1, maybe due to the higher velocity through the slit. As seen in Photo (b)'s in Figs.6,7 and 8, the inertia effect causes the streaks separate behind the body and the dye is largely diffused downstream the body.

The values of $[Re]$, for the PAA-solutions estimated from Table III are more than two decades lower than those of water for all the configurations, confirming that the onset of deviation from potential pattern occurs well in creeping flow region, i.e. the deviation is not caused by the inertia effect.

4.2 Effect of non-Newtonian viscosity
The critical values of $\dot{\gamma}_w$ for the potential flow analogy by the PAA-solutions estimated from data in Table III are well in the shear-thinning region in Fig.2 for all the configurations.

Table II Critical values for onset of disturbance of water flow:

<table>
<thead>
<tr>
<th></th>
<th>Circular Cylinder</th>
<th>Square Cylinder</th>
<th>Slit (L/d=1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$ (m/s)</td>
<td>max. potential</td>
<td>min. disturbance</td>
<td>max. potential</td>
</tr>
<tr>
<td>$\dot{\gamma}_w$(1/s)</td>
<td>2.20 x 10^{-2}</td>
<td>2.26 x 10^{-2}</td>
<td>1.55 x 10^{-2}</td>
</tr>
<tr>
<td>$Re$ (-)</td>
<td>2.19 x 10^{-3}</td>
<td>2.25 x 10^{-4}</td>
<td>1.54 x 10^{-1}</td>
</tr>
</tbody>
</table>

Table III Critical values for onset of disturbance of PAA/W and PAA/W+RS flow:

<table>
<thead>
<tr>
<th></th>
<th>PAA/W</th>
<th>PAA/W+RS</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$ (m/s)</td>
<td>max. potential</td>
<td>min. disturbance</td>
<td>max. potential</td>
<td>min. disturbance</td>
</tr>
<tr>
<td>$\dot{\gamma}_w$(1/s)</td>
<td>3.45 x 10^{-3}</td>
<td>4.67 x 10^{-3}</td>
<td>1.53 x 10^{-3}</td>
<td>2.19 x 10^{-3}</td>
</tr>
<tr>
<td>$Re$ (-)</td>
<td>4.73 x 10^{-4}</td>
<td>7.52 x 10^{-4}</td>
<td>1.35 x 10^{-4}</td>
<td>2.36 x 10^{-4}</td>
</tr>
<tr>
<td>$N/\sigma$</td>
<td>8.73</td>
<td>9.6</td>
<td>6.77</td>
<td>7.58</td>
</tr>
</tbody>
</table>
is proved that the Hele-Shaw flow of inelastic fluid with non-Newtonian viscosity, i.e. incompressible purely viscous fluids, gives flow patterns of the two-dimensional potential flow when the inertia terms are negligible, as shown in follows.

In the analysis of the Hele-Shaw flow of Newtonian fluid, it is assumed that:

i) the inertia term can be neglected since the flow is steady and the flow velocity is very low,

ii) the gap $2h$ between the parallel plates in the Hele-Shaw cell is much smaller than the characteristic length $d$ of the body inserted in the gap, i.e. $h/d << 1$,

iii) the fluid is incompressible, and

iv) its viscosity $\mu$ is constant.

For the case of incompressible purely viscous fluids with zero cross-viscosity, the assumption iv) is generalized as

iv)' the viscosity $\mu$ is a function of the second and the third invariants of strain rate tensor, $II_\varepsilon$ and $III_\varepsilon$)

When the coordinate system $O-x_1, x_2, x_3$ shown in Fig.1 is used and the velocity vector is defined as $\mathbf{v}(v_1, v_2, v_3)$, the assumptions ii) and iii) leads everywhere in the gap within a preciseness of $\frac{h}{d}$

From analogy of the Hele-Shaw flow of Newtonian fluids, we assume that the flow of non-Newtonian purely viscous fluids has a similar gap-wise velocity distribution $f_0(x_3)$. That is,

$v_3 = 0$  

(3)

and

$\gamma = \frac{dV_1}{dx_3} = \mu \frac{df_0}{dx_3}$

(6)

since the terms other than $\gamma^2$ in $II_\varepsilon$ and $III_\varepsilon$ are negligible.

Thus, we can put

$\mu = \mu(\gamma)$

(7)

for the Hele-Shaw flow of purely viscous fluids, which includes Newtonian fluids as a special case.

When smaller terms are neglected by applying the assumptions i), ii), iii) and iv)', the equations of motion for the purely viscous fluids reduce to

$\frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_3} \left( \mu \frac{\partial v_1}{\partial x_3} \right)$

(8-a)

$\frac{\partial p}{\partial x_2} = \frac{\partial}{\partial x_3} \left( \mu \frac{\partial v_2}{\partial x_3} \right)$

(8-b)

and

$\frac{\partial p}{\partial x_3} = 0$, which leads to $p = p(x_1, x_2)$.  

(8-c)

From Eqs.(4-a, b) and (8-a, b), we obtain

$V_1(x_1, x_2) = \frac{1}{[\mu]_{\gamma=0}} \left[ \frac{d^2 f_0}{dx_3^2} \right]_{x_3=0}$

(9-a)

$V_2(x_1, x_2) = \frac{1}{[\mu]_{\gamma=0}} \left[ \frac{d^2 f_0}{dx_3^2} \right]_{x_3=0}$

(9-b)

since $\left[ \frac{df_0}{dx_3} \right]_{x_3=0} = 0$.

The coefficient of the right hand side of Eqs.(9-a, b) is independent of $x_1$ and $x_2$ since $[\mu]_{\gamma=0}$ and $\left[ \frac{d^2 f_0}{dx_3^2} \right]_{x_3=0}$ is constant, which shows that the Hele-Shaw flow gives a potential flow streamline pattern on the central plane since the pressure $p$ is a scalar function of $x_1$ and $x_2$. Therefore,
streamlines on arbitrary \( x \) planes also give the potential flow analogy since Eqs.(4-a, b) give that \( \frac{v_2}{v_1} = \frac{V_2}{V_1} \).

In consequence, it is confirmed that the shear-thinning viscosity is not the cause of the disturbance for the potential flow pattern.

4.3 Effect of elasticity

From Table III, it is seen that the values of \( N_1/\sigma \) for the PAA/W-solution are considerably larger than unity at the maximum flow rate for the potential flow pattern. For the slit flow of PAA/W+RS-solution, the maximum potential-pattern \( N_1/\sigma \) value is quite lower than that of the PAA/W-solution, showing that the relative strength of elastic force due to shear is lower in the PAA/W+RS flow than in the PAA/W flow by a factor of 3 or 4. However, the value of \( [Re] \) in the slit flow seems to be a little lower for the PAA/W+RS than for the PAA/W-solution, which indicates that the PAA/W+RS-solution flow is disturbed more easily. While, the deviation from the potential flow pattern of the PAA-solutions is first observed in the region where the streamlines are converging as most clearly seen in the slit flow. These facts infer that elastic effect in shear flow is not the cause of disturbance for the potential flow pattern, and that the origin of the disturbance is attributed to the elongational stress due to fluid elasticity. This inference seems reasonable since \( [Re] \) is much lower in the slit flows than in the other flows and the deviation from the potential flow pattern is most drastic at the slit entry where the streamlines are highly converging.

5. CONCLUSIONS

Flows of two highly shear-thinning PAA-solutions with considerable first normal stress difference around a circular and a square cylinder and a flow passing through a slit reproduced respective potential streamlines when flow rate is very low. This fact and an analysis for purely viscous fluids with zero cross viscosity showed that the shear-thinning viscosity does not affect the potential flow analogy.

The potential flow pattern of the PAA-solutions is disturbed when the flow rate is higher but the Reynolds number is low enough for the inertia effect to be negligibly small. The deviation of the streamlines from the potential pattern is in the opposite way of the inertia effect and most clearly observed where streamlines converge. The streamlines are likely to separate from the wall where they converge, such as near the front stagnation point of the cylinders and upstream the slit entry, while they shift closer to the wall downstream of the bodies remaining attached to the downstream wall. On the other hand, the effect of elasticity in shear flow is not essential since the values of \( N_1/\sigma \) are beyond unity in the potential pattern flows. These behaviours suggest that the deviation from the potential pattern is caused by the elongational stress due to the elasticity of fluid.

REFERENCES