1. INTRODUCTION

The viscosity of many fluids of industrial and/or physiological importance is known to decrease with time even at a constant shear rate. Such fluid systems, which are generally referred to as thixotropic fluids, are quite common in nature and industry. One can mention, for example, magma, drilling muds, foodstuffs, semi-solids, and grease, among others. Physiological fluids such as blood, synovial liquid, and mucus can also exhibit thixotropic behavior under certain conditions. The same is true about Newtonian fluids when modified rheologically by certain polymeric additives. The thixotropic behavior of these fluid systems can be attributed to the fact that they contain a large number of microstructures. These microstructures, which are formed through inter-particle forces, can easily be broken down by the action of the deformation field. On the other hand, the ever-present Brownian forces try to reform the microstructures as soon as they are broken down. The breakdown and rebuild of microstructures give rise to the fluid system exhibiting a time-dependent viscosity. If the time is large enough, a dynamic equilibrium is eventually reached, for any given shear rate, at which the number of broken structures is statistically equal to the number of reformed ones.

Peristaltic pumping of thixotropic fluids is studied numerically in a flexible tube assuming that the fluid of interest is thixotropic and obeys Moore’s rheological model. Assuming the flow to be laminar, axisymmetric and incompressible, the equations of motion are simplified using the long-wavelength approximation. It is shown that at high Reynolds numbers, the axial pressure gradient has the same functional form as the peristaltic wave with an amplitude which decays as the Reynolds number is increased. Using finite difference method (FDM) to solve the equations of motion, it is concluded that the peristaltic pumping of thixotropic fluids is governed by the ratio of the breakdown to the buildup parameters only, not their separate absolute values. An increase in the viscosity ratio (i.e., the ratio of zero-shear to infinite-shear viscosities) is found to decrease the axial velocity with no significant effect on the wall shear stress. But, an increase in the speed of the peristaltic wave is predicted to increase both the axial velocity and also the maximum wall shear stress.

Key Words: Thixotropic fluid / Moore’s model / structural parameter / peristaltic flow
appears to be no published work addressing peristaltic flow of thixotropic fluids.

In the present work, we are going to investigate peristaltic flow of thixotropic fluids, to the best of our knowledge, for the first time. We are going to rely on the Moore’s structural model\(^{24}\) to represent thixotropic fluids—thanks to its simplicity and also the fact that it encompasses all the basic features of such fluid systems. In recent years, this rheological model has successfully been used for simulating certain complex flows of thixotropic fluids.\(^9,10\) The basic concept in this fluid model is to relate the time-dependent drop in the fluid’s viscosity to the breakdown of a structural parameter. The structural parameter itself satisfies a kinetic equation which also incorporates a parameter denoting the structure buildup. It is the main objective of the present work to see how the thixotropic material properties appearing in the Moore’s model can affect the performance of peristaltic pumps.

To reach the above objective, the work has been organized as follows: We are going to start with presenting the mathematical formulation of our fluid mechanics problem. This is followed by introducing the simplifying assumptions used in the course of this work in order to make the equations of motion tractable, without losing the physical insight. The method of solution is then described in some details. Numerical results are presented next together with discussing their physical significance. The work is concluded with highlighting its main findings.

2. MATHEMATICAL FORMULATION

We consider a flexible tube with uniform wall thickness in which a wave propagates along the wall at a constant speed, \(c\) (see Fig. 1). The radius of the wall is assumed to be of the following form,

\[
R(z, t) = a_0 + a_1 \cos \left( \frac{2\pi}{\lambda} (z - ct) \right) \tag{1}
\]

where \(a_0\) is the radius of the undisturbed tube, \(a_1\) is the amplitude of the peristaltic wave, \(\lambda\) is the wavelength, \(t\) is the time, and \(z\) is the axial coordinate.

The flow is assumed to be laminar and incompressible. It is also assumed to be symmetric (i.e., \(\partial/\partial \theta = 0\)). Denoting the velocity components in the radial and axial directions by \(u(r, z, t)\) and \(w(r, z, t)\), respectively, the Cauchy equations of motion together with the continuity equation can be written as:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} - \frac{1}{\rho} \frac{\partial}{\partial r} \left[ \tau_{rr} \right] + \frac{\partial}{\partial z} \left[ \tau_{rz} \right] \tag{2}
\]

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - \frac{1}{\rho} \frac{\partial}{\partial z} \left[ \tau_{rr} \right] + \frac{\partial}{\partial r} \left[ \tau_{rz} \right] \tag{3}
\]

\[
\frac{\partial u}{\partial r} + \frac{w}{r} + \frac{\partial w}{\partial z} = 0 \tag{4}
\]

where \(\rho\) is the density, \(p\) is the isotropic pressure, and \(\tau_{rr}, \tau_{rz}\) and \(\tau_{zz}\) are the viscous stress components. The three stress components appearing in the above equations can be related to the velocity field provided that the constitutive equation of the fluid is given. In the present work, we assume that the fluid of interest is thixotropic and obeys Moore’s rheological model.\(^{24}\) In this fluid model the stress tensor is related to the rate-of-deformation tensor \((2d_{ij})\) and also the fluid’s viscosity \((\mu)\) by the Newtonian constitutive equation, i.e., \(\tau_{ij} = 2\mu d_{ij}\). Unlike Newtonian fluids, however, the viscosity of a Moore’s fluid is assumed to be a function of a structural parameter, \(S(t)\), such that we have:\(^{24}\):

\[
\mu(t) = \mu_0 + mS(t) \tag{5}
\]

where \(\mu_0\) is the infinite-shear viscosity corresponding to complete structure breakdown (i.e., \(S = 0\)). In this equation the coefficient “\(m\)” is equal to \(\mu_0 - \mu_\infty\) where \(\mu_0\) is the zero-shear viscosity of the fluid corresponding to complete structure buildup (i.e., \(S = 1\)). In the Moore’s model, the structural parameter satisfies the following kinetic equation:\(^{24}\):

\[
\frac{DS}{Dt} = a(1 - S) - b |\dot{\gamma}| S \tag{6}
\]

where \(D/DT\) represents material derivative. In this equation, \(\dot{\gamma} = \Pi_{2D}\) is an effective shear (or deformation) rate with
II\textsubscript{2D} being the second invariant of the deformation-rate tensor. In Eq. 6, “a” and “b” are the two (positive) material properties representing structure buildup and structure breakdown, respectively. This equation suggests that unlike the rebuild process, the structure breakdown process is shear-dependent—which is often found to be the case.\textsuperscript{1-3}) It should also be noted that, denoting the dimension of time by T, “a” has the dimension of T\textsuperscript{-1} whereas “b” is dimensionless. Therefore, the ratio b/a can conveniently be taken as a characteristic time of our thixotropic fluid. (It can be interpreted as the time needed by the fluid for its viscosity change to occur at any given shear rate.) One should note that thixotropic effects are expected to have a significant influence on the flow if the characteristic time of the fluid is sufficiently larger than the characteristic time of the flow (say, the inverse of \(\dot{\gamma}\)).

From Eq. 6 one can conclude that under steady-state conditions, the structural parameter takes an equilibrium value equal to \(S_e = 1/(1 + b\dot{\gamma})\). Also, by rewriting Eq. 6 in the form of \(dS/dt = a - (a + b\dot{\gamma})S\) it can immediately be seen that the rate equation is a linear differential equation for \(S\). Therefore, \(S\)-constant curves are straight lines passing through the origin revealing the fact that for a given structural parameter, the behavior is indeed Newtonian. It needs to be emphasized that, Moore’s model is capable of predicting the basic facets of thixotropic fluid systems. For example, when a thixotropic fluid is sheared at a constant rate, it is known that its viscosity decreases by the progress of time until it reaches to a steady-state value corresponding to an equilibrium state for the structural parameter \(S_e\). To show this explicitly, one can integrate the rate equation, \(dS/dt = a - (a + b\dot{\gamma})S\), to obtain \(25)\):

\[
S = \frac{S_0 - S_e}{(a + b\dot{\gamma})} \exp\left(-\frac{1}{(a + b\dot{\gamma})}\right),
\]

where \(S_0\) is the initial value of the structural parameter. From this equation and Eq. 5, one can conclude that in steady shear, at any given \(\dot{\gamma}\), the viscosity of a Moore’s fluid decreases exponentially with time, say from its initial value \(\mu_0\) to an equilibrium value \(\mu_e\). From Eq. 7, this drop in viscosity is seen to be controlled by the time constant \((a + b\dot{\gamma})^{-1}\). This time constant can therefore be interpreted as a sort of relaxation time—a term commonly used to represent memory effects in viscoelastic fluids (One should note that Moore’s model represents inelastic fluids only, so that this term should be used with extreme caution). For fluids with small relaxation time, \(S\) remains close to \(S_e\). Therefore, for thixotropic fluids a decrease in the relaxation time means that the fluid forgets its initial configuration more significantly so that it rapidly reaches to its equilibrium state.

The fluid mechanics problem as posed by the above system of equations is too formidable to render itself to an analytical or even numerical solution, at least in its current general form. Due to the qualitative nature of the present work, we look for a less-demanding set of equations in which we are going to employ certain approximations to simplify these equations. Chief among them is the assumption that the wavelength ratio (which is defined as \(\delta = 2\pi a_0/\lambda\)) is sufficiently small such that certain terms can be dropped off from the governing equations, Eqs. 2-6. To decide on the term(s) which can be neglected, it is a good idea to make the above equations dimensionless. Here we substitute,\(w' = \frac{w}{c}; u' = \frac{u}{c0}; z' = \frac{2\pi z}{\lambda}; r' = \frac{r}{a_0};\)

\[
R' = \frac{R}{a_0}; \quad \tau' = \frac{2\pi \tau}{\lambda}; \quad \tau_{\sigma} = \frac{a_0}{\mu_0 c0}; \quad \tau_{\nu} = \frac{a_0}{\mu_e c}; \quad \frac{a_0}{\mu_0 c0}; \quad \frac{a_0}{\mu_e c}; \quad Q' = \frac{Q}{2\pi ca^2}.
\]

where \(Q\) is the volumetric flow rate. Having dropped the “*” sign above dimensionless parameters for convenience, the dimensionless form of the governing equations become,

\[
Re\delta \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z}\right) = \frac{\partial P}{\partial x} - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\partial (r\tau_{\nu})}{\partial r} + \frac{\partial (\tau_{\sigma})}{\partial z}\right),
\]

\[
Re\delta \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z}\right) = \frac{\partial P}{\partial z} - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\partial (r\tau_{\nu})}{\partial r} + \frac{\partial (\tau_{\sigma})}{\partial z}\right),
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = 0,
\]

where the Reynolds number is defined as,

\[
Re = \frac{\rho c a_0}{\mu_e}
\]

In dimensionless form, the viscosity function, and also the transport equation for the structural parameter become:

\[
\mu_e = 1 + (\mu_e - 1)\times S
\]

\[
\frac{DS}{Dt} = A(1 - S) - BS \sqrt{\beta^2 \frac{\partial u}{\partial z} + \frac{\partial w}{\partial t}}
\]
where \( A \) and \( B \) denote the modified build-up and break-down parameters defined by,

\[
A = a \left( \frac{2\pi L}{c} \right) ; \quad B = b \left( \frac{2\pi L}{c} \right) \sqrt{\frac{c}{a_0}}.
\]

(15)

In dimensionless form, the stress components become,

\[
\tau_\mu = -2(1 + (\mu_r - 1) \times S) \frac{\partial w}{\partial z},
\]

(16)

\[
\tau_\mu = -(1 + (\mu_r - 1) \times S) \left( \frac{\partial w}{\partial \tau} + \delta^2 \frac{\partial u}{\partial z} \right),
\]

(17)

\[
\tau_\mu = -2(1 + (\mu_r - 1) \times S) \frac{\partial u}{\partial \tau},
\]

(18)

where \( \mu_a \) is the apparent viscosity, and \( \mu_r = \mu_b/\mu_c \) is the viscosity ratio. It needs to be mentioned that when the viscosity ratio is equal to one, Moore’s model reduces to the so-called Newtonian fluid model. (The viscosity of the corresponding Newtonian fluid will be equal to \( \mu_0 \) provided that \( S = 1 \)). Also, in dimensionless form, the profile of the tube’s wall becomes

\[
R(z, t) = 1 + \alpha \cos(z - t),
\]

(19)

where \( \alpha = a_1/a_0 \) is called the amplitude ratio. Assuming that \( \delta \) is sufficiently small, we can safely drop all \( \delta^2 \) and \( \delta^3 \) terms from Eqs. 9-18. From Eq. 9 one can then conclude that \( \partial p/\partial \tau \approx 0 \). The axial pressure gradient, \( \partial p/\partial z \), which results from the wall’s peristaltic motion, can be related to the viscoelastic properties of the wall material. To that end, we model the flexible wall as a viscoelastic membrane similar to that used in Refs. 26-28. We also assume that the wall is inextensible so that its movement is restricted to the radial direction only. As is well established in the literature, the mechanical pressure generated in the fluid as a result of the wall’s peristaltic motion is given by

\[
p_0 - p = \frac{T}{Re^2} \frac{\partial^2 R}{\partial z^2} - M \frac{\partial^2 R}{\partial \tau^2} - \frac{\eta}{Re^3} \frac{\partial^4 R}{\partial \tau^4} - D \frac{\partial^2 R}{\partial \tau^2} - K R
\]

(20)

where \( p_0 \) is the pressure exerted by the muscle (or, roller) at the outer wall, \( p \) is the pressure generated in the liquid, and \( T, M, \eta, D \), and \( K \) are the longitudinal tension per unit length, mass of the wall per unit area, the damping coefficient, the flexural rigidity, and the spring stiffness of the wall, respectively. This equation can be simplified if it can be assumed the wall is so thin that its mass can be neglected, i.e., \( M \approx 0 \). It can further be simplified by assuming that the flow is occurring at high Reynolds numbers. (This assumption also justifies ignoring the contribution of the viscous normal stress from the fluid side in the mechanical pressure.) To see which terms can be neglected at high Reynolds numbers, Eq. 20 is written in dimensionless form as

\[
p_0 - p = \frac{\eta}{Re^3} \frac{\partial R}{\partial \tau} - \frac{\eta}{Re^3} \frac{\partial^3 R}{\partial \tau^3} - \frac{\eta}{Re^3} \frac{\partial^4 R}{\partial \tau^4} - \frac{K}{Re} R
\]

(21)

where “*” sign above dimensionless quantities has been dropped for convenience. As can be seen in this equation, at sufficiently high Reynolds numbers, we can write,

\[
p_0 - p \approx -\frac{\eta}{Re^3} \frac{\partial R}{\partial \tau}.
\]

(22)

(This approximation becomes plausible even at moderate Reynolds numbers provided that the amplitude ratio, \( a_1/a_0 \), is sufficiently small.) Now, assuming that \( p_0 \) is constant, the axial pressure gradient can be obtained as,

\[
\frac{\partial p}{\partial \tau} = -\frac{\eta}{Re^3} \frac{\partial R}{\partial \tau} = \frac{\eta \alpha}{Re^3} \cos(z - t) = G \cos(z - t)
\]

(23)

where \( G \) depends on the viscous properties of the fluid and the wall, and also the parameters pertaining to the peristaltic wave.

In the next section we will describe the numerical method used to solve the above set of (simplified) governing equations, i.e., Eqs. 10-19. As to the boundary conditions required to close the problem, we are going to employ on the no-slip and no-penetration conditions at the inner wall of the tube. The radial gradient of the structural parameter can also be set equal to zero at the wall based on the so-called no-flux condition \(^4\); i.e.,

\[
\frac{\partial w}{\partial \tau} = 0, \quad \frac{\partial S}{\partial \tau} = 0
\]

(24)

At the centerline, \( r = 0 \), the appropriate boundary conditions are \(^4\):

\[
u = 0, \quad \frac{\partial R}{\partial \tau} = 0
\]

(25)

At the inlet ant outlet sections of the channel, we are going to employ periodic boundary conditions. As to the initial conditions, it is assumed that the system is initially at rest with the structure being complete everywhere; that is, at \( t = 0 \) we have,

\[
u = 0, \quad w = 0, \quad S = 1
\]

(26)
3. METHOD OF SOLUTION

As the first step in our search for a numerical solution, the x-coordinate is transformed in such a way that square mesh can be used when discretizing the governing equations. To achieve this goal, we substitute,

$$x = \frac{r}{R(z,t)}. \quad (27)$$

With this transformation, and after applying the long-wavelength assumption, the z-momentum equation becomes,

$$\frac{\partial w}{\partial t} = \frac{1}{R} \frac{\partial}{\partial t} \left( \frac{R}{\partial R} \frac{\partial w}{\partial z} \right) + w \frac{x}{R} \frac{\partial R}{\partial z} \frac{\partial w}{\partial x} - \frac{w}{R} \frac{\partial w}{\partial z} \left( \frac{\partial P}{\partial z} + \frac{1}{\partial x} \frac{xR}{R} \frac{\partial R}{\partial x} \frac{\partial w}{\partial x} \right). \quad (28)$$

Similarly, the continuity equation becomes,

$$\frac{1}{R} \frac{\partial u}{\partial t} + u \frac{x}{R} \frac{\partial w}{\partial x} - w \frac{x}{R} \frac{\partial R}{\partial z} \frac{\partial w}{\partial x} = 0. \quad (29)$$

The transformed forms of the stress components, and also the structural parameters become

$$\tau_{xz} = -2(1 + (\mu_{i} - 1) \times S) \left( \frac{\partial w}{\partial z} + w \frac{x}{R} \frac{\partial R}{\partial x} \frac{\partial w}{\partial x} \right). \quad (30)$$

$$\tau_{zz} = -(1 + (\mu_{i} - 1) \times S) \left( \frac{1}{R} \frac{\partial w}{\partial x} \right). \quad (31)$$

As to the transformed form of the kinetic equation we have,

$$\frac{\partial S}{\partial t} = \left( \frac{1}{R} \frac{\partial}{\partial t} \left( \frac{R}{\partial R} \frac{\partial S}{\partial z} \right) \right) \frac{\partial S}{\partial x} - \frac{2}{R} \frac{\partial S}{\partial x} \left( 1 - S \right) - 2BS \frac{1}{R} \frac{\partial w}{\partial x}. \quad (32)$$

A closed form solution can be obtained for the radial velocity component by multiplying the continuity equation, Eq. 29, by and then integrating this equation as,

$$x \frac{\partial u}{\partial x} + u + xR \frac{\partial w}{\partial z} - x^{2} \frac{x}{R} \frac{\partial R}{\partial x} \frac{\partial w}{\partial x} = 0. \quad (33)$$

The result is,

$$u = -\frac{1}{\int_{0}^{1} xR \frac{\partial w}{\partial z} \frac{dx}{x} + \int_{0}^{1} x^{2} \frac{x}{R} \frac{\partial R}{\partial x} \frac{\partial w}{\partial x} dx}. \quad (34)$$

Using the partial integrating, this equation takes the following form,

$$u = -\frac{R}{x} \left[ x \frac{\partial w}{\partial z} \frac{dx}{x} + x \frac{\partial R}{\partial z} \frac{w}{x} - \frac{2}{x} \frac{x}{R} \frac{\partial R}{\partial x} \int_{0}^{1} xw dx \right]. \quad (35)$$

where, having imposed the no-slip boundary condition, we can write,

$$-\int_{0}^{1} x \frac{\partial w}{\partial z} \frac{dx}{x} = 1 \frac{\partial R}{\partial t} \frac{\partial w}{\partial z} \int_{0}^{1} xw dx. \quad (36)$$

In the next step, we introduce an auxiliary function which satisfies the condition \( \int_{0}^{1} xf(x)dx = 1 \). The following function meets this purpose \(^{29}\),

$$f(x) = 4(1 - x^{2}) \quad (37)$$

Using this auxiliary function, Eq. 36 can be written as,

$$-\int_{0}^{1} x \frac{\partial w}{\partial z} \frac{dx}{x} = 1 \frac{\partial R}{\partial t} \frac{\partial w}{\partial z} \int_{0}^{1} xf(x)dx + 2 \frac{\partial R}{\partial z} \int_{0}^{1} xw dx \quad (38)$$

From this equation, one can conclude that:

$$\frac{\partial w}{\partial z} = -f(x) \frac{1}{R} \frac{\partial R}{\partial t} \frac{\partial w}{\partial z}. \quad (39)$$

By substituting this equation into Eq. 35, and having replaced \( f(x) \) by its explicit from Eq. 37, the radial velocity is finally obtained as,

$$u(x,z,t) = x \left( \frac{\partial R}{\partial z} + \frac{2}{R} \frac{\partial R}{\partial x} \frac{x}{R} \frac{\partial w}{\partial x} \right). \quad (40)$$

Having determined the radial velocity in the above form, we are now ready to solve the governing equations using the finite difference method. To approximate the derivatives we are going to employ the central differences.\(^{30}\) For example, we substitute,

$$\frac{\partial w}{\partial x} = \frac{(w)_{i+1,j} - (w)_{i-1,j}}{\Delta x}, \quad \frac{\partial w}{\partial z} = \frac{(w)_{i,j+1} - (w)_{i,j-1}}{\Delta z}. \quad (41)$$

where the subscripts \( i \) and \( j \) refer to the spatial discretization in the x- and z-directions. Also, the superscript “k” refers to the time step. The time derivative of the axial velocity is approximated by,

$$\frac{\partial w}{\partial t} = \frac{(w)_{i,j}^{k+1} - (w)_{i,j}^{k}}{\Delta t}. \quad (42)$$

For other variables, we use similar expressions.\(^{30}\) The axial and radial coordinates, and also the time step, are discretized as:

$$z_{i} = (i - 1)\Delta z, \quad i = 1, 2, ..., M + 1 \quad \text{that} \quad \Delta z = 2\pi$$

$$x_{j} = (j - 1)\Delta x, \quad j = 1, 2, ..., N + 1 \quad \text{that} \quad \Delta x = 1$$

$$t_{k} = (k - 1)\Delta t, \quad k = 1, 2, ...$$
The discretized forms of Eqs. 28-32, and also Eq. 41 become:

\[
(w)_{j+1}^k = (w)_{j}^k + \Delta t \left[ \left( \frac{\partial R}{\partial t} \right)_{ij} + \left( \frac{w_{j+1}^k}{R_{ij}} \right) \frac{x_j}{R_{ij}^3} \left( \frac{\partial R}{\partial x} \right)_{ij} - \left( \frac{\partial w_{j+1}^k}{\partial z} \right)_{ij} + \left( \frac{\partial w_{j+1}^k}{\partial x} \right)_{ij} \right]
\]

(44)

The discretized form of the boundary and initial conditions become,

\[
\left( \frac{\tau_{ij}}{\mu} \right)_{ij} = -2 \left( \frac{1}{\mu} \right)_{ij} \left( \frac{\partial w_{j+1}^k}{\partial x} \right)_{ij} \]

(45)

\[
\left( \frac{\tau_{ij}}{\mu} \right)_{ij} = - \left( \frac{1}{\mu} \right)_{ij} \left( \frac{\partial w_{j+1}^k}{\partial x} \right)_{ij}
\]

(46)

\[
\left( u_{ij} \right)_{ij} = x_j \left( \frac{\partial R}{\partial x} \right)_{ij} \left( w_{j+1}^k \right)_{ij} + \left( \frac{\partial R}{\partial x} \right)_{ij} (2 - x_j)
\]

(47)

\[
\left( S_{ij} \right)_{ij} = \Delta t \left[ \frac{1}{R_{ij}^3} \left( \frac{\partial S_{ij}}{\partial x} \right)_{ij} + \frac{1}{R_{ij}^3} \left( \frac{\partial S_{ij}}{\partial z} \right)_{ij} \right]
\]

(48)

The discretized form of the boundary and initial conditions become,

\[
\left( u_{ij} \right)_{N+1} = \left( \frac{\partial R}{\partial t} \right)_{ij}, \quad (w)_{N+1} = 0
\]

(49)

\[
\left( u_{ij} \right)_{ij} = 0, \quad \left( \tau_{ij} \right)_{ij} = 0, \quad (w)_{ij} = (w)_{ij}, \quad \left( S_{ij} \right)_{ij} = \left( S_{ij} \right)_{ij}
\]

(50)

\[
\left( u_{ij} \right)_{ij} = 0, \quad (w)_{ij} = 0, \quad \left( S_{ij} \right)_{ij} = 1
\]

(51)

At the start of the simulations, the stress components are calculated from Eqs. 45 and 46. The axial velocity is then calculated from Eq. 44. In the next step, the radial velocity and the structural parameter are computed from Eqs. 47 and 48 respectively. At each time step, the dimensionless volumetric flow rate, and also the wall shear stress are obtained as,

\[
Q^k_{\text{v}} = \left( R_{ij} \right)^3 \int x_j w^k_{ij} dx_j |_{r=M/2}
\]

(52)

\[
\tau_{\text{wall}}^k = \left( \frac{\tau_{ij}}{\mu} \right)_{ij} \left|_{r=M/2} \cos \left( \tan^{-1} \left( \frac{\partial R}{\partial z} \right)_{ij} \right) |_{r=M/2}
\]

(53)

As to the time step needed to ensure convergence of the numerical results, we have relied on a trial-and-error approach [starting from values suggested by the CFL criterion as given in Ref. 32] to find out that a time step as small as \( \Delta t = 0.00001 \) is good enough for virtually all computations carried out in this work. To be on the safe side, all numerical results reported in this work have been obtained for a time step of \( \Delta t = 0.00001 \) although the results are virtually the same as those obtained with \( \Delta t = 0.00001 \) (see Figure 2). For the spatial discretization, we have found it adequate to employ on \( N = 30 \) and \( M = 160 \) in all of our simulations.

4. RESULTS AND DISCUSSIONS

The code developed in this work had to be verified first before being used for our thixotropic fluid of interest. This was done by comparing the code’s output in computing the axial velocity profile in start-up pipe flow of Newtonian fluids. As is well established in the literature [see, for example, Ref. 31] for a Newtonian fluid of viscosity \( \mu_0 \) which is driven by a step pressure gradient in a pipe having a radius of \( a_0 \), the axial velocity can be obtained at any instant of time as,

\[
w(r,t) = -\frac{a_0^2}{4\mu_0} \frac{\partial p}{\partial x} \left[ 1 - \left( \frac{r}{a_0} \right)^2 \right] - 8 \sum_{k=1}^{\infty} \frac{J_0(a_0 \alpha_k r)}{a_0 \alpha_k J_1(a_0)} e^{-\alpha_k^2 \alpha_0^2 t} \]

(54)

where \( J_0 \) and \( J_1 \) are the zeroth- and first-order Bessel functions of the first kind, respectively, with being the roots of the zeroth-order Bessel function of the first kind. Figure 3 shows a comparison between the numerical velocity profiles obtained in this work with the analytical solution reported in Ref. 31. A good agreement is observed to exist between our numerical results with those obtained using Eq. 54, thereby the code is verified.

Having validated the code, we are now at a stage to present our brand-new numerical results in which we are going to show the effect of the thixotropic parameters appearing in the Moore’s model on peristaltic pumping. For illustrative
purposes, we are going to present data for a typical channel with $\alpha = 0.01$ and $\delta = 0.02$ only. In all simulations to be presented shortly we set $Re = 10$. The axial location where the velocity profiles will be plotted is set at the middle of the channel, i.e., at $z = \pi$. Our main objective is to investigate the effects of the thixotropic parameters, $A$ and $B$, in the Moore’s model on the flow characteristics. We are also interested in studying the effect of $\mu_r$ and $G$ on the velocity field and volumetric flow rate in the above-mentioned channel. Our preliminary numerical results have shown that, it is in practice the ratio $B/A$ which controls the fluid’s response to peristaltic movement of the wall not the separate values of $A$ and $B$ [see also, Refs. 9 and 10]. For this reason, we are going to present results in terms of this ratio only.

Figure 4 shows the time evolution of the velocity field computed at $z = \pi$ for a typical $B/A = 1$, $G = 10$, and $\mu_r = 4$. As can be seen in this figure, the magnitude of the radial velocity is smaller, say by two orders of magnitudes, than the magnitude of the axial velocity. It is also interesting to note that, the axial and radial velocities can be either positive or negative, depending on the time step.

Figure 5 shows the time evolution of the structural-parameter profile obtained at $z = \pi$ for two different $B/A$ ratios of 0.1 and 10 for the same $P_1 = 10$, and $\mu_r = 4$, as used in Fig. 4. This figure shows that at $t = 0$ the structural parameter is equal to one for both $B/A$ ratios. But, it decreases by the progress of time, eventually reaching to its equilibrium profile. From Fig. 5 it can be concluded that for $B/A = 0.1$, the structural parameter needs roughly $t = \pi/8$ to relax to its equilibrium shape. For $B/A = 1$, on the other hand, the structures need roughly $t = \pi/4$ to reach the equilibrium state. This is not surprising realizing the fact that by an increase in $B/A$, the relaxation time of the fluid is increased, so that the fluid takes a longer time to reach the equilibrium state.

Figure 6 shows the effect of the $B/A$ ratio on the axial and radial velocity profiles obtained at a typical $z = \pi$ location. To ensure that the equilibrium condition has indeed been reached, the velocity profiles are plotted at $t = \pi/2$. From this figure, it can be concluded that the magnitude of both velocity components ($u$ and $w$) will increase by an increase in $B/A$. As previously mentioned, this ratio can be interpreted as a characteristic time for thixotropic fluids obeying Moore’s model. Therefore, one would expect to see a more significant thixotropic effect when this ratio is large. One should also note that, for a given $A$, an increase in $B/A$ means smaller structures at any given radius, and smaller structures mean

![Fig. 3. A comparison between our numerical results with published analytical results in start-up pipe flow of Newtonian fluids.](image)

![Fig. 4. Time evolution of the axial and radial velocity profiles obtained at $z = \pi$.](image)

![Fig. 5. Time evolution of the structural parameter at $z = \pi$ for two different $B/A$ ratios.](image)

![Fig. 6. Effect of the thixotropic ratio $B/A$ on the axial and radial velocity profiles at $z = \pi$ and $t = \pi$.](image)
lower viscosity. Thus, the velocity of the fluid elements is predicted to increase by an increase in B/A, as shown in Fig. 6.

Figure 7 shows the effect of the thixotropic ratio B/A on the induced flow rate and wall shear stress distribution calculated at \( z = \pi \) location. As can be seen in this figure, the average volumetric flow rate increases by an increase in B/A with the price being that the amplitude of its fluctuation also increases to some extent. Interestingly, however, the wall shear stress is predicted to remain almost unchanged when the thixotropic properties of the fluid are altered.

Figure 8 shows the effect of the viscosity ratio on the velocity profiles calculated at \( z = \pi \) location and \( t = \pi/2 \) for a typical B/A = 1. Both velocity components are predicted to decrease by an increase in the viscosity ratio. For a given \( \mu_\infty \), an increase in the viscosity ratio is equivalent to an increase in \( \mu_0 \). Thus, the drop in the velocity components is not surprising. The same can be said about the drop in the volumetric flow rate when the viscosity ratio is increased (see Fig. 9). But, again, the wall shear stress is predicted not to be affected that much even by the viscosity ratio, as can be seen in Fig. 9.

Figure 10 shows the effect of G (i.e., the amplitude of the axial pressure gradient in Eq. 23) on the velocity field at \( z = \pi \) location and \( t = \pi/2 \) for \( \mu_\text{s} = 4 \). As can be seen in this figure, by an increase in G, the absolute value of both velocity components are increased. Figure 11 shows that the volumetric flow rate remains unchanged although its range of fluctuations is increased by an increase in G. Interestingly, the maximum wall shear stress is also increased by an increase in G (see Figure 11). To explain these effects one should note that for a given tube and a given fluid, an increase in G can be caused by an increase in the wave speed, c (see Eq. 23). These results show that the wave speed, c, is very important in peristaltic pumping.
5. CONCLUDING REMARKS

In the present work, we have tried to numerically investigate peristaltic pumping of thixotropic fluids. Our scope has been quite limited in that we are primarily interested in qualitatively illustrating the effects of different parameters on the flow characteristics only. Based on the results obtained in this work, one can conclude that the thixotropic behavior of a fluid can significantly affect its behavior in peristaltic flow. It was found that, the response of thixotropic fluids to the peristaltic flow depends on the ratio between the breakdown parameter to the buildup parameters only, not their separate absolute values. The viscosity ratio, defined as the ratio of zero-shear viscosity to infinite-shear viscosity, was shown to influence the flow kinematics but not the wall shear stress. That is, the axial and radial velocity profiles are predicted to decrease by an increase in the viscosity ratio. More interestingly, the speed of the peristaltic wave was shown to dramatically affect the velocity and stress fields.

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