CONSTITUTIVE MODEL CONSIDERING SAND CRUSHING

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ABSTRACT

The behavior of sand crushing will appear when the confining pressure is up to a certain value, which results in disappearing of the positive dilatancy of sand. Adopting a new hardening parameter with the crushing stress, an elastoplastic constitutive model considering sand crushing is proposed. Comparing the conventional triaxial compression test results with the model prediction, it shows that the proposed model can reasonably describe the dilatancy of sand from positive to negative.

Key words: crushing, dilatancy, hardening, model, sand (IGC: D6)

INTRODUCTION

Many triaxial tests indicate that sand presents crushing at a high confining pressure (Daouadji et al., 2001; Fukumoto, 1992; Liu et al., 2005) and the particle crushing results in negative dilatancy. The peak strength of sand also decreases with the confining pressure increasing. Therefore, it is necessary to develop an elastoplastic constitutive model to describe the mechanical behavior of sand crushing.

The Cam-clay model, proposed by Roscoe and Burland (1968), is suitable for the normally consolidated clay. As it is known, the stress-strain relationship under isotropic stress is the foundation of the Cam-clay model and the normal compression line is straight in the semi-logarithmic compression plane. It suits for the normally consolidated clay (Wroth and Houlsby, 1985), but not for sand in a large-scale stress range. According to the previous work (Nakai, 1989); the normal compression line and the unloading-reloading lines are straight in a semi-exponential plane.

So far, lots of hardening parameters being able to describe dilatancy of soils have been assumed and many plasticity models have been proposed. Yao et al. (1999, 2007, 2008) has developed a new hardening parameter, a revised plastic volumetric strain. The validity of the hardening parameter is confirmed by triaxial compression and extension tests along various stress paths. It is suitable for both clay and sand.

In order to consider the behavior of sand crushing, in this paper, the hardening parameter H is revised with a crushing stress p_c , to enable the hardening parameter H to describe positive dilatancy at a low stress and negative dilatancy at a high stress and the peak strength decreasing

when the confining pressure increases. Seven soil parameters in the model will be determined via isotropic consolidation tests and conventional triaxial compression tests. The results predicted by the proposed model are compared with the test results to verify the proposed model. In this paper, the stress is to be interpreted as effective stress.

THE CONSTITUTIVE MODEL FOR SAND CRUSHING

In this section, based on a new yield surface proposed by authors and the hardening parameter developed by Yao et al. (1999, 2007, 2008), we propose a constitutive model for sand, which can describe the crushing behavior of sand particles.

The Stress-strain Relationship in Isotropic Consolidation

Referencing the work (Nakai, 1989; Sun et al., 2001), the relationship between the plastic (or elastic) volumetric strain ε_v^p (or ε_v^e) and the mean stress p under isotropic consolidation condition could be assumed as Eqs.(1) and (2):

$$\varepsilon_v^e = C_e \left[\left(\frac{p}{p_a} \right)^m - \left(\frac{p_0}{p_a} \right)^m \right] \quad (1)$$

$$\varepsilon_v^p = (C_t - C_e) \left[\left(\frac{p}{p_a} \right)^m - \left(\frac{p_0}{p_a} \right)^m \right] \quad (2)$$

where p_0 is the initial mean stress, p_a is the atmospheric pressure, C_t is the compression index, C_e is the swelling index and m is a coefficient for sand. The value of atmospheric pressure p_a is given as 0.1 MPa.

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The manuscript for this paper was received for review on August 20, 2007; approved on May 28, 2008.

Written discussions on this paper should be submitted before March 1, 2009 to the Japanese Geotechnical Society, 4-38-2, Sengoku, Bunkyo-ku, Tokyo 112-0011, Japan. Upon request the closing date may be extended one month.

The Hardening Parameter Considering Sand Crushing

For sand, both the plastic volumetric strain ε_v^p and the plastic deviator strain ε_d^p depend on stress-paths, so that neither of them is suitable to be the hardening parameter of sand (Yao et al., 1999, 2007, 2008). The primary factor which influences the deformation of soil is not the deviator stress q but the stress ratio η ($\eta = q/p$). Many triaxial tests indicate that the positive dilatancy of sand will not appear when the initial confining pressure is at a high degree. It is because of the particles breakage of sand (Fukumoto, 1992; Liu et al., 2005), which results in the volume of sand sample decreasing continuously.

Revising the hardening parameter developed by Yao et al. (1999) as

$$H = \int dH = \int \Theta d\varepsilon_v^p = \int \frac{M_c^4 M_f^4 - \eta^4}{M_f^4 M_c^4 - \eta^4} d\varepsilon_v^p \quad (3)$$

Simplifying the differential form of Eq. (3), we obtain

$$d\varepsilon_v^p = \frac{1}{\Theta} dH = \frac{M_f^4 M_c^4 - \eta^4}{M_c^4 M_f^4 - \eta^4} dH \quad (4)$$

Where

$$\Theta = \frac{M_c^4 M_f^4 - \eta^4}{M_f^4 M_c^4 - \eta^4} \quad (5)$$

$$M_f = M \left(\frac{p}{p_c} \right)^{-n} \quad (6)$$

$$M_c = M \left(\frac{p}{p_c} \right)^n \quad (7)$$

in which M_c is the stress ratio at characteristic state point, M_f the stress ratio at shear failure (Yao et al., 2007, 2008), M the stress ratio at critical state, p_c the reference crushing stress, and n the material parameters of sand. The parameters of the proposed model are M , p_c and n .

Substituting $M_f = q_f/p$ into Eq. (6) gets the function of q_f as Eq. (8) on the p - q plane, which is an exponential function as shown in Fig. 1.

$$q_f = Mp_c^n p^{1-n} \quad (8)$$

Substituting $M_c = q_c/p$ into Eq. (7) gets the function of q_c as Eq. (9) on the p - q plane, which is also an exponential function as shown in Fig. 1.

$$q_c = Mp_c^{-n} p^{1+n} \quad (9)$$

In Fig. 1, the abscissa denotes the mean stress p , the ordinate denotes the deviator stress q , the solid curve denotes the variable M_f on p - q plane and the dash one denotes the variable M_c . The straight line between the curves of M_c and M_f is the critical line (M line).

According to different initial mean stress p_{OA} , p_{OC} and p_{OE} , there are three kinds of drained triaxial compression stress paths AB, CD and EF as shown in Fig. 1. The stress-strain curves according to three stress paths are shown as Figs. 2(a), (b) and (c) respectively, where Fig. 2(a) corresponds to the stress path AB, Fig. 2(b) the stress path CD and Fig. 2(c) the stress path EF.

Considering the differential of the hardening parameter dH is always larger than or equal to zero, there are four states existed for Eq. (4) according to the value of the stress ratio η as follows. ① At the initial state of stress path: when $\eta = 0$, $d\varepsilon_v^p = dH > 0$ (isotropic consolidation). ② Along stress path AB: when $0 < \eta < M_c < M_f$, $d\varepsilon_v^p > 0$ (negative dilatancy); when $0 < \eta = M_c < M_f$, $d\varepsilon_v^p = 0$ (characteristic state); when $0 < M_c < \eta < M_f$, $d\varepsilon_v^p < 0$ (positive dilatancy). That is, point A is under isotropic compression, the stress path from A to K is of negative dilatancy, point K is at characteristic state, the path from K to B is of positive dilatancy and point B is at failure state. ③ Along stress path CD: when $0 < \eta < M_c = M_f$, $d\varepsilon_v^p > 0$ (negative dilatancy); when $0 < \eta = M_c = M_f$

Fig. 1. The M_f (and M_c) curves and the stress paths of triaxial compression on p - q plane

Fig. 2. The stress-strain curves along different stress paths in triaxial compression tests

M , $d\varepsilon_v^p=0$ (critical state). That is, point C is under isotropic compression, the path from C to D is of negative dilatancy, point D is at critical state and also at failure state. ④ Along stress path EF: when $0 < \eta \leq M_f$, $d\varepsilon_v^p > 0$ (negative dilatancy). That is, point E is under isotropic compression, the path from E to F is of negative dilatancy, point F is at failure state.

When the stress state is at point D, the stress ratio of characteristic point M_c is equal to the peak stress ratio M_f , so that the coefficient M can be determined at point D in Fig. 1. After M and p_c are obtained, the coefficient n can be obtained by the linear relationship between $\ln M_f$ and $\ln p$.

The Elastoplastic Constitutive Model

In the proposed model, the stress-dilatancy equation is expressed as

$$\frac{d\varepsilon_v^p}{d\varepsilon_d^p} = \frac{M_c^2 - \eta^2}{2\eta} \quad (10)$$

The orthogonality condition is

$$dp \cdot d\varepsilon_v^p + dq \cdot d\varepsilon_d^p = 0 \quad (11)$$

The expression of the plastic potential function, which is obtained via the solution of the differential equation, composed of Eqs. (10) and (11), is written as

$$f = g = (2n+1) \frac{p_c^{2n}}{M^2} \cdot \frac{q^2}{p} + p^{2n+1} - p_x^{2n+1} = 0 \quad (12)$$

where p_x is the mean stress at isotropic stress.

The associated flow rule is adopted in the proposed model. The yield surface and the potential surface on p - q plane are shown as in Fig. 3. In Fig. 3, the solid curves are a series of yield surface, and the dashed one is the M_c curve, which is the track of the peak point of series of yield surface.

When $n=0$, Eq. (12) will be simplified as

$$p_x = p + \frac{q^2}{M^2 p} \quad (13)$$

which is the potential function of the Cam-clay model.

Associating Eqs. (2) and (12) can be written as

Fig. 3. The yield surface and the potential surface on p - q plane

$$\frac{C_t - C_e}{p_a^m} \left\{ \left[\frac{(2n+1)p_c^{2n}}{M^2} \cdot \frac{q^2}{p} + p^{2n+1} \right]^{\frac{m}{2n+1}} - p_0^m \right\} - \varepsilon_v^p = 0 \quad (14)$$

Adopt the revised hardening parameter H with crushing stress p_c to replace the plastic volumetric strain ε_v^p in Eq. (11). The yield function (potential function) is given as

$$f = g = \frac{C_t - C_e}{p_a^m} \left\{ \left[\frac{(2n+1)p_c^{2n}}{M^2} \cdot \frac{q^2}{p} + p^{2n+1} \right]^{\frac{m}{2n+1}} - p_0^m \right\} - H = 0 \quad (15)$$

The elastic modulus E in the proposed model can be deduced as

$$E = \frac{3(1-2\nu)p_a^m}{mC_e p^{m-1}} \quad (16)$$

The plastic strain increment is written as

$$d\varepsilon_{ij}^p = \Lambda \frac{\partial g}{\partial \sigma_{ij}} \quad (17)$$

where Λ is the proportionality constant, which can be deduced from Eq. (15) as

$$\Lambda = \frac{C_t - C_e}{p_a^m} \frac{mp^{m-2n-1}}{(2n+1)} \frac{M_f^4 M_c^2 + \eta^2}{M_c^4 M_f^4 - \eta^4} \times \left[\frac{(2n+1)\eta^2}{M_c^2} + 1 \right]^{\frac{m-2n-1}{2n+1}} \left(\frac{M_c^2 - \eta^2}{3} \delta_{ij} + \frac{3}{p} s_{ij} \right) d\sigma_{ij} \quad (18)$$

where δ_{ij} is Kronecker's delta, s_{ij} is the deviatoric stress tensor expressed as

$$s_{ij} = \sigma_{ij} - p \cdot \delta_{ij} \quad (19)$$

The stress gradient $\partial g / \partial \sigma_{ij}$ is written as

$$\frac{\partial g}{\partial \sigma_{ij}} = (2n+1) \frac{p_c^{2n}}{M^2} \left(\frac{M_c^2 - \eta^2}{3} \delta_{ij} + \frac{3}{p} s_{ij} \right) \quad (20)$$

When $n=0$, from Eqs. (6) and (7), it can be obtained that $M_c = M_f = M$, then Eq. (18) could be simplified as

$$\Lambda = \frac{C_t - C_e}{p_a^m} \frac{mp^{m-1}}{M^2 - \eta^2} \left[\frac{\eta^2}{M^2} + 1 \right]^{m-1} \left(\frac{M^2 - \eta^2}{3} \delta_{ij} + \frac{3}{p} s_{ij} \right) d\sigma_{ij} \quad (21)$$

and Eq. (20) could be simplified as

$$\frac{\partial g}{\partial \sigma_{ij}} = \frac{1}{M^2} \left(\frac{M^2 - \eta^2}{3} \delta_{ij} + \frac{3}{p} s_{ij} \right) \quad (22)$$

It can be indicated that the expression form of Eq. (21) in the proposed model is similar to the Cam-clay model, and the expression form of Eq. (22) is the same as the Cam-clay model.

PREDICTION VERSUS EXPERIMENTS

The following seven soil parameters used in the proposed model: C_t , C_e , m , M , p_c , n and the Poisson ratio ν , all of which can be determined via conventional triaxial tests except for ν . The value of ν is assumed to be 0.3. The parameters C_e , C_t and m are from Sun et al. (2007). The parameters M , p_c and n are determined by drained triaxial

Fig. 4. The drained triaxial compression test results for Toyoura sand under different initial confining pressures

compression tests at different initial stresses. The value of p_c (reference crushing stress) is the mean stress of point D as shown in Fig. 1, the point D indicates that M_f and M_c are equal to M when p is up to p_c .

Figure 4 shows the drained triaxial compression tests results (Sun et al., 2007) for Toyoura sand when the confining pressure is at 0.2 MPa, 0.5 MPa, 1 MPa, 2 MPa, 4 MPa and 8 MPa respectively. A lot of triaxial tests on Toyoura sand are performed using a medium-level pressure triaxial test apparatus. The average diameter of the Toyoura sand is 0.2 mm. The uniformity coefficient is 1.3 and specific gravity is 2.65. The maximum and minimum void ratios are 0.95 and 0.58, respectively. Specimens are prepared by pouring the saturated sand into a mold in several layers and by compacting each layer using a rod with a diameter of 6 mm. The initial void ratio is about 0.68. The triaxial test apparatus is capable of applying the confining-pressure up to 8 MPa. The size of the specimens is 10 cm in height and 5 cm in diameter. For each specimen, the drained triaxial compression tests are performed up to the failure.

The dilatancy is positive when confining pressure is at 0.2 MPa, 0.5 MPa, 1 MPa and 2 MPa respectively, and only negative when it is at 4 MPa and 8 MPa. Besides, the internal friction angle, as the strength index of sand, decreases with the mean stress increasing.

Analyzing the test results as shown in Fig. 4, it is found that both the stress ratio q/p and the strain increment ratio $d\varepsilon_v/d\varepsilon_a$ approach to be constant values when the Toyoura sand is at failure state under different confining pressures. Besides, drawing the points according to failure states in $(d\varepsilon_v/d\varepsilon_a)_f$ - q_f/p plane can get a straight line as shown in Fig. 5(a).

Ignoring the effect of elastic deformation, we assume that the value of the peak stress ratio q_f/p will be equal to M when $(d\varepsilon_v/d\varepsilon_a)_f=0$. Utilizing the equation of the straight line in Fig. 5(a), we can get the value of M (as $M=1.35$).

The values of n and p_c are deduced as follow. Eq. (6) can be rewritten as

$$\ln M_f = -n \ln p + n \ln p_c + \ln M \tag{23}$$

According as the values of M_f and p in test results, draw

Fig. 5. (a) The relationship between q_f/p and $-(d\varepsilon_v/d\varepsilon_a)_f$ under different confining pressures and (b) the relationship between $\ln M_f$ and $\ln p$

Table 1. The seven parameters of the proposed model for Toyoura sand

Isotropic consolidation	Triaxial compression	Poisson ratio
$C_c=0.0016$ $C_i=0.0044$ $m=0.5$	$M=1.35$ $p_c=5.85 \text{ MPa}$ $n=0.085$	$\nu=0.3$

the points in $\ln p$ - $\ln M_f$ plane shown in Fig. 5(b). With the linear fitting result, we get the values of n and p_c as $n=0.085$, $p_c=5850 \text{ kPa}$ respectively.

All of the model parameters for Toyoura sand are listed in Table 1. The test results (Sun et al., 2007) and the predicted relationships between axial strain ε_a , radial strain ε_r , volumetric strain ε_v and principal stress ratio σ_a/σ_r are shown in Fig. 6.

Analyzing the prediction results as shown in Fig. 6, it can be seen that: ① The predicted curves by the proposed model agree well with the test results for Toyoura sand under conventional triaxial compression conditions except the ε_a - ε_v curves at confining pressure of 8 MPa. ② When the confining pressure of sand rises, the peak principal stress ratio decreases. ③ When the confining pressure of sand rises, the volumetric dilatant degree of sand decreases.

As above, the proposed model can generally describe the stress-strain behavior of sand and the dilatancy, which changes from positive to negative according to the variability of principal mean stress p .

Fig. 6. The test results and the predicted stress-strain curves by proposed model for Toyoura sand

Fig. 7. The predicted stress-strain curves by proposed model for Toyoura sand under undrained triaxial compression conditions

The proposed model is considered as a simple constitutive model. In the tests, the range of confining pressure is from 0.2 MPa to 8 MPa, and the predicted curve of volumetric strain does not fit well with the test data when the confining pressure is at 8 MPa. So we suggest that the

model is applied when the value of confining pressure is less than 8 MPa in test conditions or engineering practice.

The stress-strain behavior of Toyoura sand under undrained conventional triaxial compression conditions is also predicted by the proposed model. Figure 7(a) shows the stress paths in p - q plane under different constant confining pressures and Fig. 7(b) shows the relationship between deviator stress q and axial strain ϵ_a .

CONCLUSIONS

Based on the analysis of the test results for Toyoura sand and the prediction using the constitutive model proposed in this paper, we can make the following conclusions:

- 1) The shear failure stress ratio M_f and the characteristic point stress ratio M_c are both the exponential functions of the mean stress p . The shear failure M_f decreases with the mean stress p increasing and the negative stress-dilatancy increases with the particle crushing for sand.
- 2) Taking the revised hardening parameter into the elastoplastic model, it can be reasonably described that the stress-dilatancy characteristics under different confining pressures.
- 3) The prediction using the proposed model can fit well to the test results for Toyoura sand in drained triaxial compression and it is also predicted the stress-strain relationships under undrained triaxial compression conditions.
- 4) The plastic yield function proposed in this paper can be used to describe the behavior of the sand crushing, and it also can be considered as a generalized function

of the Cam-clay model.

ACKNOWLEDGEMENTS

This paper is supported by the National Natural Science Foundation of China, NSFC (No. 50639050 and No. 10672010), and the National Basic Research Program of China (Grant No. 2007CB714203).

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