APPROXIMATE FORMS OF WAVE EQUATIONS FOR WATER-SATURATED POROUS MATERIALS AND RELATED DYNAMIC MODULUS

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SYNOPSIS

On the basis of the information concerning the compressibilities of water, solid particle and solid skeleton, the material constants appearing in the theory of water-filled porous elastic materials developed by M. A. Biot were computed and their order of magnitude was examined. Such check-up permitted to simplify considerably the theory for the case of dynamical problems and led to wave equations which are identical in form with those derived for the elastic theory. This shows that the elastic theory can serve as a substitute for the poro-elastic theory, especially when the latter is associated with the undrained condition like dynamical problems. Young's modulus and Poisson's ratio in the equivalent elastic theory were related with the constants of the materials constituting a two-phase porous medium. Furthermore, the expression for the pore pressure coefficient B proposed by A. W. Skempton was derived and extended to the case of one-dimensional compression. Finally, a relationship between Poisson's ratio and the pore pressure coefficients was disclosed.

1. INTRODUCTION

In many engineering problems dealing with ground vibrations such as earthquakes, machine foundations, compaction machines, or artificial detonations, engineers are required to assess, directly or indirectly, the deformation characteristics of earth materials composing the ground under consideration. When the problems involve determination of the material properties at a stress level sufficiently high to initiate failure, application of the elastic theory is apparently inadequate. However, when concern is of behaviors at a low stress level, the material characteristics are often prescribed, although far from complete, in terms of Young's modulus and Poisson's ratio on the basis of the hypothesis that the materials are linear in their stress-strain relationships. Earth materials like rocks and soils can be regarded as a kind of composite material consisting of two ingredients in which fluid (liquid or gas) fills, partially or fully, the pores interspersed uniformly in the skeleton made up of the parent materials of the earth. In case both of these ingredient materials exhibit linear relationships between stress and strain, it is also possible to establish a theory

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in the way similar to that which has been accomplished in development of the linear theory of elasticity. Theoretical development along this line has extensively been advanced by M. A. Biot^{1,2)} and some other investigators,^{3,4)} and the validity of the theory was confirmed experimentally to some extent by those working in the field of rock mechanics. However, it seems that the use of the poro-elastic theory has been handicapped by the absence of well-established means to determine the constants involved. Primarily because of this, the use of the theory has been delayed in the field of soil mechanics. On the other hand, Nagumo⁵⁾ recently tried to define three basic compressibilities of the materials constituting a porous medium by considering some specific test conditions. Nevertheless, no attempt was made to relate them with the phenomenological constants appearing in Biot's treatment. In an attempt to fill the gap, the author⁶⁾ has proposed the relationships between the three compressibilities and the Biot's constants. As a result, several concepts which Biot has developed qualitatively for linear poro-elasticity have been examined in a more realistic manner. One of the most important conclusions deduced from the previous study was that the wave travelling through a water-filled porous material with frequencies of about 1.0 to 50 cycles per second propagates without any change in the pore volume, i.e., in undrained condition. The wave has its own velocity of propagation which is related with the deformation modulus obtained in undrained condition. Another point which was suggested in the previous study was that the stress-strain relationship of a porous medium in undrained condition closely resembles that of an elastic material. Therefore, it was expected that the problem of dynamic poro-elasticity can be reduced to that of the conventional elasticity by simply replacing Young's modulus and Poisson's ratio appearing in the theory of elasticity by the corresponding modulus expressed in terms of poro-elastic constants. One of the objectives of the present study is to examine thoroughly several of the porous elastic constants on the numerical basis and to bring out the similarity between the wave equations of the classical elasticity and those of poro-elasticity under undrained condition.

Whether a porous water-saturated material is stressed statically or dynamically, the building-up of pore pressure is an important aspect of the problem. Generally, the development of pore pressure is attributed to two primary causes; the one due to compressive stress and the other due to shear stress. The pore pressure due to the latter cause is specified by what is called "pore pressure coefficient A ," while that due to the former is determined by the coefficient called B . In the realm of the linear theory, the coefficient A is equal to $1/3$. The detailed investigation by A. W. Skempton⁷⁾ has shown that the coefficient B is a function of porosity and compressibilities of water and the skeleton, and that B is quite close to unity in most cases. This indicates that almost all of the external all-round stress is transferred to the water of the pores and the change of the apparent volume is caused primarily by the volume change of the water. Since the volume change on the part of the water is

small enough, it was suggested that the saturated soil can be considered to undergo no volume change whatsoever while it is subjected to deformation. In terms of linear elasticity, this is equivalent to putting Poisson's ratio equal to 0.5. In statical problems the hypothesis that the saturated soil is incompressible is relevant, but for those problems in which the material is stressed dynamically, this assumption is no longer adequate. To consider the saturated soil incompressible is equivalent to claiming that the longitudinal wave must propagate with an infinite velocity, which is apparently in contradiction to what is actually experienced. In order to determine mathematically the velocity of the longitudinal wave, even small compressibility of the constituent materials can not be neglected. By taking into account the compressibility of each material, the formula for the velocity of longitudinal wave will be derived and its order of magnitude will be examined. The formula in turn furnishes a simple approximate expression for determining Poisson's ratio. Upon passage of the longitudinal wave, there will be a rise in pore pressure within the pore water. The extent of the pore pressure development is specified by the coefficient B' defined in one-dimensional stress condition, instead of the Skempton's coefficient B for all-round stress condition. These coefficients will be derived together from the poro-elastic theory and the relationship between them and Poisson's ratio will be discussed.

2. STRESS-STRAIN RELATIONSHIPS OF POROUS MATERIALS

The stress-strain relations for an elastic porous material are given by⁶⁾

$$\sigma_{ij} = 2\mu e_{ij} + [(\alpha - \frac{2}{3}\mu)e + \alpha_c \varepsilon] \delta_{ij} \quad (1)$$

$$-p = \alpha_c e + k_c \varepsilon \quad (2)$$

σ_{ij} is the tensor of the stress acting on the solid part in the unit area of the soil mass and $-p$ denotes the pressure which acts on the fluid part in the unit surface area. μ is the shear modulus and α , α_c and k stand for the elastic constants associated with compressibilities of the constituents of which the soil consists. e_{ij} is the strain tensor of the solid part of the soil, and related with the displacement u_i by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (3)$$

where the subscript following a comma denotes partial differentiation with respect to coordinate variables. The volume changes of the solid and fluid parts are denoted by e and ε , respectively, which are given by

$$e = e_{11} + e_{22} + e_{33}, \quad \varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \quad (4)$$

The coefficients α , α_c , k_c and μ appearing in Eqs. (1) and (2) were introduced simply on the basis of the existence of elastic potential energy and it is convenient to use them when the equations are handled in terms of displacements rather than stresses. Since the physical implication of these constants are not clear, they must be related with some intrinsic properties of the constituent materials, say, soil and water. By

considering some particular stress conditions, the above coefficients can be expressed as functions of compressibilities of the constituent materials⁶⁾:

$$\alpha = \frac{(n-1)^2 C_p + (n-1)C_s + C_l}{C_b(C_l - C_s) + C_p C_s} \quad (5)$$

$$\alpha_c = \frac{-n[(n-1)C_p + C_s]}{C_b(C_l - C_s) + C_p C_s} \quad (6)$$

$$k_c = \frac{nC_b}{C_b(C_l - C_s) + C_p C_s} \quad (7)$$

where C_b denotes bulk compressibility of the soil skeleton, C_s the compressibility of the solid particle, C_p the pore compressibility, C_l the fluid compressibility and n the porosity. The four compressibilities appearing in Eqs. (5) to (7) are not all independent quantities. They are related by⁶⁾

$$C_b = nC_p + C_s \quad (8)$$

Thus, it becomes possible to transform the values of these compressibilities into the alternative values α , α_c and k_c . The constants α , α_c and k_c are the same as those introduced by M. A. Biot,¹⁾ and the list of correspondence between these two pairs is provided in Table 1.

3. EVALUATION OF MATERIAL CONSTANTS

The use of the material constants as expressed by α , α_c and k_c are preferable to that of the compressibilities themselves, mainly because of the easiness in analytical manipulation. Therefore, it is necessary to see the order of magnitude of these constants for the possible range of compressibilities of soils or rocks. Unfortunately, sufficient information is not presently available concerning these values. The only data that are arranged by A. W. Skempton⁸⁾ seems to be useful for the present calculation. He presented a list of compressibilities, as shown in Table 2, for various kinds of soil and rock materials at the pressure of 1 kg/cm². As indicated, the compressibility decreases as the pressure to which the material is subjected is increased. Nevertheless, the compressibility shown in the table will be reasonable and typical for most earth materials. It is known from this table that the compressibility C_s of the solid particle lies in the narrow range being close to 2.0×10^{-6} per kg/cm², while the bulk compressibility can change over a wide range. On the basis of the above observation, the ratio of the compressibility of the solid particle to that of the water will be taken as

$$\frac{C_s}{C_l} = \frac{2.0 \times 10^{-6}}{48 \times 10^{-6}} = 0.0417$$

throughout the subsequent computation. The ratio C_s/C_b changes over a wide range from the order of 10^{-5} to about 0.5. Equations (5), (6) and (7) are calculated for

such a range of C_s/C_b and for the porosity n ranging between 0.2 and 0.7. The results of calculation are shown in Figs. 1 to 3 in non-dimensional form by plotting the values αC_l , $\alpha_c C_l$, and $k_c C_l$ versus C_s/C_b . Also plotted in Fig. 4 is the value of

Fig. 1. Values of αC_l

Fig. 2. Values of $\alpha_c C_l$

Fig. 3. Values of $k_c C_l$ Fig. 4. Values of $(\alpha + 2\alpha_c + k_c) C_l$

$(\alpha + 2\alpha_c + k_c) C_l$ which will become necessary later in this paper. It is seen from these diagrams that the value of αC_l lies somewhere between 0.2 and 7.0, depending upon the value of C_s/C_b and the porosity n . The value of $\alpha_c C_l$ does not change greatly, remaining in the range from 0.2 to 0.7 for all combinations of n and C_s/C_b .

postulated above. There is a negligible amount of change for the value of $k_e C_i$ in all the ranges of C_s/C_b and n being considered.

4. EQUATIONS OF MOTION

The equations of motion written each for solid and fluid parts are

$$\rho_s \frac{\partial^2 u_i}{\partial t^2} = \sigma_{ij,j} + b \frac{\partial}{\partial t} (u_i - v_i) \quad (9)$$

$$\rho_w \frac{\partial^2 v_i}{\partial t^2} = -p_{,i} - b \frac{\partial}{\partial t} (u_i - v_i) \quad (10)$$

where ρ_s and ρ_w designate the bulk mass density of the solid and fluid, respectively. b denotes a constant related with the permeability k of the soil and the acceleration of the gravity g in the following way:

$$b = \frac{n \rho_w g}{k} \quad (11)$$

In Eqs. (9) and (10), v_i represents the displacement component of the fluid phase and is related with the volume change ε by

$$\varepsilon = v_{i,i} \quad (12)$$

If the relationships (1) and (2) are substituted into Eqs. (9) and (10), the following equations are obtained:

$$\rho_s \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu \nabla^2 \mathbf{u} + \left(\alpha + \frac{\mu}{3} \right) \text{grad. div } \mathbf{u} + \alpha_c \text{grad. div } \mathbf{v} + \frac{\partial}{\partial t} (\mathbf{u} - \mathbf{v}) \quad (13)$$

$$\rho_w \frac{\partial^2 \mathbf{v}}{\partial t^2} = \alpha_c \text{grad. div } \mathbf{u} + k_c \text{grad. div } \mathbf{v} - \frac{\partial}{\partial t} (\mathbf{u} - \mathbf{v}) \quad (14)$$

where $\mathbf{u}$ and $\mathbf{v}$ represent the displacement vectors of the solid and fluid parts, respectively, and the symbol ∇^2 designates the following operator:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Performing the divergence operation on both sides of Eqs. (13) and (14), the equations for compressional wave can be obtained:

$$\left. \begin{aligned} \rho_s \frac{\partial^2 e}{\partial t^2} &= \left(\alpha + \frac{4}{3} \mu \right) \nabla^2 e + \alpha_c \nabla^2 \varepsilon + b \frac{\partial}{\partial t} (e - \varepsilon) \\ \rho_w \frac{\partial^2 \varepsilon}{\partial t^2} &= \alpha_c \nabla^2 e + k_c \nabla^2 \varepsilon - b \frac{\partial}{\partial t} (e - \varepsilon) \end{aligned} \right\} \quad (15)$$

When the curl operator is applied to Eqs. (13) and (14) the equation for shear wave can be obtained as follows:

$$\left. \begin{aligned} \rho_s \frac{\partial^2 w}{\partial t^2} &= \mu \nabla^2 u + b \frac{\partial}{\partial t} (u - W) \\ \rho_w \frac{\partial^2 W}{\partial t^2} &= -b \frac{\partial}{\partial t} (w - W) \end{aligned} \right\} \quad (16)$$

where $w = \text{rot } u$, $W = \text{rot } v$.

5. APPROXIMATE FORMS OF WAVE EQUATIONS AND FREQUENCY CHARACTERISTICS

The basic equations of motion as expressed by Eqs. (15) and (16) can be reduced to some simplified sets of equations by which fundamental characteristics of velocity of propagation and damping loss can be investigated. In order to simplify consideration without loss of generality, the wave propagation in x -direction alone will be considered. In such a case, the compressional wave produces a deformation pattern

Fig. 5. Modes of deformation when waves pass through the ground

- (a) Mode of displacement for compressional wave
(b) Mode of displacement for shear wave

as shown in Fig. 5(a), whereas the pattern such as shown in Fig. 5(b) is produced when shear wave propagates. In this simplified condition, the solutions of Eqs. (15) and (16) will be expressed in the form

$$\left. \begin{aligned} e_x &= A_e \exp[i(lx + \omega t)] \\ \epsilon_x &= A_\epsilon \exp[i(lx + \omega t)] \end{aligned} \right\} \quad (17)$$

$$\left. \begin{aligned} w_x &= A_w \exp[i(lx + \omega t)] \\ W_x &= A_W \exp[i(lx + \omega t)] \end{aligned} \right\} \quad (18)$$

where A_e , A_ϵ , A_w and A_W are constants, e_x and ϵ_x denote x -components of e and ϵ , w_x and W_x designate x -components of w and W , respectively, and i is a complex number. When Relations (17) and (18) are substituted into Eqs. (15) and (16), two

simultaneous equations for the constants A_s , A_e and A_w , A_w are obtained as follows:

$$\left\{ \begin{aligned} \left[-\rho_s \left(\frac{\omega}{l} \right)^2 - \frac{bi}{l} \left(\frac{\omega}{l} \right) + \left(\alpha + \frac{4}{3} \mu \right) \right] A_s + \left[\frac{bi}{l} \left(\frac{\omega}{l} \right) + \alpha_e \right] A_e &= 0 \\ \left[\frac{bi}{l} \left(\frac{\omega}{l} \right) + \alpha_e \right] A_s + \left[-\rho_w \left(\frac{\omega}{l} \right)^2 - \frac{bi}{l} \left(\frac{\omega}{l} \right) + k_c \right] A_e &= 0 \end{aligned} \right\} \quad (19)$$

$$\left\{ \begin{aligned} \left[-\rho_s \left(\frac{\omega}{l} \right)^2 - \frac{bi}{l} \left(\frac{\omega}{l} \right) + \mu \right] A_w + \left[\frac{bi}{l} \left(\frac{\omega}{l} \right) \right] A_w &= 0 \\ \left[\frac{bi}{l} \left(\frac{\omega}{l} \right) \right] A_w + \left[-\rho_w \left(\frac{\omega}{l} \right)^2 - \frac{bi}{l} \left(\frac{\omega}{l} \right) \right] A_w &= 0 \end{aligned} \right\} \quad (20)$$

If A_s and A_e are eliminated from Eq. (19), the following equation for ω/l usually called "frequency equation" is obtained:

$$-i \left(\frac{\omega}{\omega_c} \right) (1 - X)^2 X^2 + \left(\frac{\omega}{\omega_c} \right)^2 \left(1 - \frac{\theta_1}{\theta_2} X^2 \right) X^2 + \theta_2 \left(\frac{\omega}{\omega_c} \right)^2 = 0 \quad (21)$$

where

$$\frac{1}{\omega_c} = \frac{\rho_s k_c + \rho_w (\alpha + \frac{4}{3} \mu)}{\alpha + 2\alpha_e + k_c + \frac{4}{3} \mu} \cdot \frac{1}{b} \quad (22)$$

$$\theta_1 = \frac{V_d^2}{V_n^2}, \quad \theta_2 = \frac{V_d^2}{V_l^2} \quad (23)$$

$$X = \frac{1}{V_l} \left(\frac{\omega}{l} \right) \quad (24)$$

$$\left. \begin{aligned} V_d^2 &= \frac{\alpha k_c - \alpha_e^2 + \frac{4}{3} \mu k_c}{\rho_s k_c + \rho_w (\alpha + \frac{4}{3} \mu)} \\ V_l^2 &= \frac{\alpha + 2\alpha_e + k_c + \frac{4}{3} \mu}{\rho_s + \rho_w} \\ V_n^2 &= \frac{k_c}{\rho_w} + \frac{\alpha + \frac{4}{3} \mu}{\rho_s} \end{aligned} \right\} \quad (25)$$

By solving the fourth order algebraic Equation (21) with respect to the variable X , the value of ω/l can be obtained from Eq. (24), permitting further to evaluate the wave velocity and damping characteristics of porous materials. Before proceeding formally along this line of approach, the relative order of magnitude of each term in Eq. (21) will be examined to see if any of the terms have only a negligible effect on the solutions. First of all, the value R defined by

$$R = \frac{\alpha + 2\alpha_e + k_c + \frac{4}{3} \mu}{\alpha + \frac{4}{3} \mu + k_c} \frac{1 - n}{n} G \quad (26)$$

Fig. 6. Values of $R = \frac{\alpha + 2\alpha_c + k_c + \frac{4}{3}\mu}{\alpha + \frac{4}{3}\mu + k_c + \frac{1-n}{n}G}$

Table 1. A list of correspondence between notations used by Biot and in this paper

Biot's notation	The notation adopted in this paper
A	$\alpha - \frac{2}{3}\mu$
Q	α_e
R	k_e
N	μ

Table 2. Compressibilities of different earth materials at the pressure of 1 kg/cm² (after A. W. Skempton)

Material	Compressibility per kg/cm ² × 10 ⁻⁶		C_s/C_b
	C_b	C_s	
Quartzitic sandstone	5.8	2.7	0.46
Quincy granite (100 ft. deep)	7.5	1.9	0.25
Vermont marble	17.5	1.4	0.08
Concrete (approx. values)	20	2.5	0.12
Dense sand	1,800	2.7	0.0015
Loose sand	9,000	2.7	0.0003
London clay (over-cons.)	7,500	2.0	0.00025
Gosport clay (norm.-cons.)	60,000	2.0	0.00003

will be computed because it is frequently used in the subsequent discussions. In Eq. (26) G denotes the specific gravity of the solid particle. The additional information necessary to get numerical values of Eq. (26) is that concerning the value of G and shear modulus μ of porous material which is being considered. Since the modulus is considered to vary over the wide range from very soft clays to hard rocks, μ will be assumed to change from 50 kg/cm² up to 100,000 kg/cm². The specific gravity G of the solid particle will be assumed to be 2.5 throughout this paper. The computed values of R are shown in Figs. 6 (a) and 6 (b), with R plotted against μC_1 .

Fig. 7. Values of $\theta_2 = \frac{(ak_c - a_c^2 + \frac{4}{3}\mu k_c)(1 + \frac{1-n}{n}G)}{(a + 2a_c + k_c + \frac{4}{3}\mu)(a + \frac{4}{3}\mu + k_c \frac{1-n}{n}G)}$

These figures demonstrate that the values of R change between about 0.9 and 1.6 for all the ranges of the variables assumed for the computation. The assignment of all the possible ranges of the variables as mentioned above permits to compute the values of θ_2 as shown in Fig. 7 (a) and Fig. 7 (b). These figures indicate that the values of θ_2 is less than about 0.34 for all the possible values of the variables assumed for calculation. The constant $\rho_s/(\rho_s + \rho_w)$ representing the proportion of the solid mass contained within the gross unit mass of the porous material was computed for various values of n by using the relationship:

$$\frac{\rho_s}{\rho_s + \rho_w} = \frac{1}{1 + \frac{n}{1-n} \cdot \frac{1}{G}} \quad (27)$$

The result is plotted in Fig. 8. Then, the value θ_1/θ_2 is given by

$$\frac{\theta_1}{\theta_2} = \left(\frac{V_l}{V_n} \right)^2 = R \frac{\rho_s}{\rho_s + \rho_w} \quad (28)$$

Observation of the range of variation of R from Fig. 6 and that of $\rho_s/(\rho_s + \rho_w)$ from Fig. 8 indicates that θ_1/θ_2 changes between about 0.8 and 0.9. Since Eq. (21) is of the fourth order in the variable X , there must be, in principle, four solutions which may be complex numbers. Among them, the solution whose real part is close to 1.0 seems most significant. For this particular range of X , the last term in Eq. (21) is obviously smaller in magnitude than the term including θ_1/θ_2 , because the value of θ_2 is smaller than θ_1/θ_2 as examined above in detail. Consequently when a solution is sought around $X = 1.0$, the last term may be put out of consideration. Then, Eq. (21) becomes

$$-i(1 - X^2) + \frac{\omega}{\omega_c}(1 - \theta_c X^2) = 0 \quad \text{where} \quad \theta_c = \frac{\theta_1}{\theta_2} = \frac{V_l^2}{V_n^2} \quad (29)$$

Fig. 8. Values of $\frac{\rho_s}{\rho_s + \rho_w} = \frac{1}{1 + \frac{n}{1-n} \cdot \frac{1}{G}}$

In the same fashion as above, elimination of A_w and A_w between two of Eq. (16) yields the frequency equation for the shear wave mode:

$$-i(1 - Y^2) + \frac{\omega}{\omega_s}(1 - \theta_s Y^2) = 0 \quad (30)$$

where

$$\left. \begin{aligned} \theta_s &= \frac{V_s^2}{V_p^2}, & Y &= \frac{1}{V_s} \left(\frac{\omega}{l} \right) \\ V_p^2 &= \frac{\mu}{\rho_s}, & V_s^2 &= \frac{\mu}{\rho_s + \rho_w}, & \omega_s &= \frac{b}{\rho_w} \end{aligned} \right\} \quad (31)$$

It is interesting to note that there is a strong similarity between Eqs. (29) and (30): the fundamental equations specifying the wave velocities and the damping characteristics are the same in form for the compressional and shear modes. If the approximate Eq. (29) is transformed back to the form of partial differential equation for e or ε by simply replacing ω and l by the operators $(1/i)(\partial/\partial t)$ and $(1/i)(\partial/\partial x)$, respectively, we obtain

$$\left[b(\rho_s + \rho_w) \frac{\partial}{\partial t} \left(-\frac{\partial^2}{\partial t^2} + V_l^2 \nabla^2 \right) - \rho_s \rho_w \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2}{\partial t^2} - V_n^2 \nabla^2 \right) \right] e = 0 \quad (32)$$

Equation (32) is valid also for the variable ε . It is to be noted that Eq. (32) is nothing but the one which can be obtained by eliminating either e or ε between two equations of Eq. (15), whereby disregarding the term associated with a long-term deformation. In the similar way to the above, Eq. (30) can be transformed back to the form of a partial differential equation for w_x or W_x as follows:

$$\left[b(\rho_w + \rho_s) \frac{\partial}{\partial t} \left(-\frac{\partial^2}{\partial t^2} + V_s^2 \nabla^2 \right) - \rho_s \rho_w \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2}{\partial t^2} - V_r^2 \nabla^2 \right) \right] w_x = 0 \quad (33)$$

Fig. 9. Equivalent viscoelastic model of a porous medium

The wave Eqs. (32) and (33) have been derived for those materials whose properties are specified by a model of the water-filled porous body. Exactly the same form of equation as Eqs. (32) and (33) can also be derived for the wave which travels through the material whose properties are represented by a simple viscoelastic model. The equation for the wave travelling through a bar whose properties are represented by the viscoelastic model as shown in Fig. 9 was derived by Kolsky⁹⁾ as follows:

$$\rho \eta_v \frac{\partial^3 u}{\partial t^3} + \rho(E'_a + E_v) \frac{\partial^2 u}{\partial t^2} - \eta_v E'_a \frac{\partial^3 u}{\partial x^2 \partial t} - E_v E'_a \frac{\partial^2 u}{\partial x^2} = 0 \quad (34)$$

where, u denotes the displacement of the bar in the x -direction, ρ is the unit mass of the material, E'_a , E_v and η_v represent viscoelastic constants of the material as shown in Fig. 9. By comparing the coefficients appearing in Eqs. (32) and (33) with those

in Eq. (34) the equivalent viscoelastic constants can be determined for the model of the porous media. The correspondence between the constants in Eq. (34) and those in Eqs. (32) and (33) is indicated on the list of Table 3. In Table 3, τ represents a retardation time defined by η_v/E_v . It should be noticed that the corresponding constants listed in Table 3 have the same unit of dimension as the viscoelastic constants; the constants $\mu\sqrt{\rho_v/\rho_s}$ has the same unit of dimension as E'_s , and so on. Of particular importance is the fact that the reciprocals of the characteristic frequencies ω_c and ω_s , defined by Eqs. (22) and (31) correspond to the retardation time in the viscoelastic model of Fig. 9. In order to get better understanding of the wave equations (32) and (33), it may be wise to refer to the behavior of the model in Fig. 9, by replacing the constants E'_s , E_v , etc., by the corresponding ones listed in Table 1. The behavior of the model in Fig. 9, when vibrating with an angular frequency ω , can best be illustrated by showing how the velocity and the damping are influenced by the frequency. For the compressional mode the relationships between these quantities are given by⁶⁾

$$\frac{V_c}{V_i} = \frac{1 + (\omega/\omega_c)^2}{1 + \theta_c(\omega/\omega_c)^2}, \quad \frac{\delta_c}{\pi} = \frac{1 - \theta_c}{1 + \theta_c(\omega/\omega_c)^2} \left(\frac{\omega}{\omega_c} \right) \quad (35)$$

and for the shear mode they are

$$\frac{V_h}{V_s} = \frac{1 + (\omega/\omega_s)^2}{1 + \theta_s(\omega/\omega_s)^2}, \quad \frac{\delta_h}{\pi} = \frac{1 - \theta_s}{1 + \theta_s(\omega/\omega_s)^2} \left(\frac{\omega}{\omega_s} \right) \quad (36)$$

Fig. 10. Variation of velocities of propagation and damping loss with frequency

where V_c and V_h denote the velocity of the compressional and shear waves, respectively. The logarithmic decrements of these two modes are denoted, respectively, by δ_c and δ_h . As is obvious from Eqs. (35) and (36) the wave velocities and the dampings are functions of ω/ω_c or ω/ω_s with the constants θ_c or θ_s appearing in these equations in the identical manner. The constant θ_s defined in Eq. (31) is written in another way as

$$\theta_s = \frac{\rho_s}{\rho_s + \rho_w} \quad (37)$$

From Fig. 8 it is observed that the value θ_s changes between about 0.5 and 0.9. Furthermore, the previous examination has shown that θ_c varies between about 0.8 and 0.9. In view of such range of variation of θ_s and θ_c , numerical computation was carried out for the velocities of propagation and damping loss as expressed in Eqs. (35) and (36). The result is shown in Fig. 10 with the velocity and damping loss plotted against angular frequency of propagation.

6. FURTHER SIMPLIFICATION OF WAVE EQUATION FOR LOW FREQUENCY RANGE

Whether the frequency of a wave is low or high must be judged on the basis of the relative magnitude of actual frequency to the characteristic one which is inherent to the material properties under consideration. The characteristic frequency ω_c of the compressional wave propagating through the porous material is given by Eq. (22) as follows:

$$\omega_c = \frac{\alpha + 2\alpha_c + k_c + \frac{4}{3}\mu}{k_c\rho_s + (\alpha + \frac{4}{3}\mu)\rho_w} \cdot b \quad (38)$$

With reference to Eqs. (11) and (26), the above equation becomes

$$\omega_c = \frac{ng}{k} \cdot R \quad (39)$$

From Eqs. (11) and (31), the characteristic frequency of the shear wave is given by

$$\omega_s = \frac{ng}{k} \quad (40)$$

Since the ratio ω_c/ω_s is equal to the value R which varies, as shown in Fig. 6, within relatively narrow limits of about 0.9 and 1.6, ω_c has the same order of magnitude as the value ω_s . Hence, it will be sufficient to examine the order of magnitude of either ω_c or ω_s alone. For the sake of convenience, the value of ω_s will be examined more thoroughly. Equation (40) indicates that ω_s depends on the permeability coefficient of the material which varies within wide limits from about 10^{-1} cm/sec for extremely permeable cohesionless soils to about 10^{-7} cm/sec for extremely fine-grained cohesive soils. Assuming the porosity, for instance, to be 0.5, the frequency

f_s defined by

$$f_s = \frac{\omega_s}{2\pi} \quad (41)$$

can change as,

$$\left. \begin{aligned} f_s &= \frac{0.5 \times 980}{6.28} \times (10 \sim 10^7) \text{ cycle/sec} \\ &= 780 \sim 7.8 \times 10^8 \text{ cycle/ec} \end{aligned} \right\} \quad (42)$$

In view of the fact that the primary mode of earthquake or artificial vibrations consists of the wave propagating with the frequency ranging from about 1.0 to 50 cycles per second, the characteristic frequency as demonstrated above is by far greater than the actual frequency encountered in practice. Even for the material with a permeability coefficient as large as 10^{-1} cm/sec, the ratio ω/ω_s is of the order of 10^{-2} to 10^{-3} . For materials with larger permeability, the ratio ω/ω_s becomes even smaller. Hence, it can generally be concluded that the ratio between the actual frequency and the characteristic one is so small that it is permitted to consider the ratio ω/ω_s close to zero, or approximately equal to zero. With this fact in mind, observation of Fig. 10 shows that the velocities of compressional and shear waves tend respectively to become V_l and V_s , and that the damping loss, irrespective of the values of θ_c and θ_s , has the tendency to vanish as ω/ω_c or ω/ω_s decreases. This fact is equivalent to being able to neglect the second terms with ω/ω_c and ω/ω_s in Eqs. (29) and (30). By disregarding these terms, the solutions of Eqs. (29) and (30) can readily be obtained as follows:

$$\left. \begin{aligned} X^2 &= \frac{1}{V_l^2} \left(\frac{\omega}{l} \right)^2 = 1.0 \\ Y^2 &= \frac{1}{V_s^2} \left(\frac{\omega}{l} \right)^2 = 1.0 \end{aligned} \right\} \quad (43)$$

Hence, the positive solutions for ω/l become

$$\left. \begin{aligned} (\omega/l)_c &= V_l \\ (\omega/l)_s &= V_s \end{aligned} \right\} \quad (44)$$

for compressional and shear modes, respectively. The solutions of the frequency equation have been determined so that two equations of Eq. (19) or (20) are simultaneously satisfied with non-zero amplitudes of the wave motion. Consequently, if the solutions given by Eq. (44) are substituted in Eqs. (19) and (20), the ratio between the amplitudes of solid and fluid phases can readily be determined as follows:

$$\left. \begin{aligned} A_e &= A_\epsilon \\ A_w &= A_w \end{aligned} \right\} \quad (45)$$

This result, substituted further back into Eqs. (17) and (18), indicates that

$$\left. \begin{aligned} e_x &= \varepsilon_x \\ w_x &= W_x \end{aligned} \right\} \quad (46)$$

With reference to the compatibility relations given by Eqs. (3) and (12), Eq. (46) expressed in terms of strain is transformed to relations between displacement components:

$$\left. \begin{aligned} u_x &= v_x \\ u_y &= v_y \end{aligned} \right\} \quad (47)$$

It is to be noted that the displacement component u_x or u_y could be different from v_x or v_y by a constant amount, corresponding to relative rigid-body movements between solid and fluid phases. Nevertheless, Eq. (47) can be valid whenever the displacements are measured from the starting value common to u_x and v_x . When the dynamic behaviors are of concern, the current static state of displacements can be set as an initial state and the subsequent change in displacements can be taken to be u_x and v_x . Such stipulation permits to derive Eq. (47) directly from Eq. (46). Equation (47) indicates that the displacement of the solid phase is identical with that of the fluid phase whenever the porous material is stressed in dynamic conditions like those encountered in earthquakes or artificial vibrations. In other words, there is no relative displacement between the solid and fluid; they move tightly together with each other without being separated. This fact is equivalent to saying that the fluid-saturated material is stressed in undrained condition when vibrated with frequencies normally met in what is called dynamic problems. When a wave propagates with a frequency far smaller than the characteristic one, the wave length becomes so large that the gradient between the points of the highest and lowest pressures is small and, consequently, no water can migrate through the pores of the material. For this reason, the damping loss resulting from the relative motion of the solid and fluid phases disappears, as apparently visualize in the diagram shown in Fig. 10.

The wave equations corresponding to the situations as described above can be derived directly from the solutions of the frequency equation given by Eq. (43). By replacing ω and l by the operators $(1/i)(\partial/\partial t)$ and $(1/i)(\partial/\partial x)$, respectively, Eq. (43) is converted back to the form of partial differential equations as follows:

$$\left. \begin{aligned} \frac{\partial^2 e_x}{\partial t^2} &= V_l^2 \frac{\partial^2 e_x}{\partial x^2} \\ \frac{\partial^2 \omega_x}{\partial t^2} &= V_s^2 \frac{\partial^2 \omega_x}{\partial x^2} \end{aligned} \right\} \quad (48)$$

The above equations can alternatively be derived by adding two separate equations in Eqs. (15) and (16) with reference to Eq. (46). Eq. (48) is the basic wave equations for water-saturated porous rocks or soils having linear characteristics between stress

and strain. Worthy of particular notice is the fact that Eq. (48) derived for a porous material under dynamic condition is completely identical in form with the conventional wave equations for a purely elastic solid. Hence, the knowledge we have concerning the classical wave equation for an elastic theory can all be used, without modifications, for the problems in which porous elastic materials are subjected to dynamic loading.

7. RELATIONSHIPS BETWEEN ELASTIC CONSTANTS AND PORO-ELASTIC CONSTANTS IN UNDRAINED CONDITION

From the discussion demonstrated above it is realized that the water-logged porous material, as it is stressed in undrained condition, exhibits the characteristics much like elastic materials. Therefore, there must be some correspondence relationships between the elastic constants and the poro-elastic constants defined above, because the condition "undrained" imposes severe restriction on the behavior of a porous water-filled body. When the relationship extended from Eq. (46) to more general three-dimensional case is used in Eqs. (1) and (2), the following stress-strain relationships are obtained:

$$\left. \begin{aligned} \sigma_{ij} &= 2\mu e_{ij} + (\alpha + \alpha_c - \frac{2}{3}\mu)e\delta_{ij} \\ -p &= (\alpha_c + k_c)e \end{aligned} \right\} \quad (49)$$

It is possible and convenient to regard the two-phase porous system as one single gross system in which the total stress given by $\sigma_{ij} - p$ is related with the gross volume change represented by e . In view of this concept, Eq. (49) is contracted to

$$\sigma_{ij}^t = \sigma_{ij} - p\delta_{ij} = 2\mu e_{ij} + (\alpha + 2\alpha_c + k_c - \frac{2}{3}\mu)e\delta_{ij} \quad (50)$$

where σ_{ij}^t represents the stress tensor of total stress. On the other hand, the stress-strain relationships for an elastic body are expressed in terms of the so-called Lamé constants λ and μ as follows:¹⁰⁾

$$\sigma_{ij}^t = 2\mu e_{ij} + \lambda e\delta_{ij} \quad (51)$$

By comparing Eqs. (50) and (51), the equivalent Lamé constants are defined as

$$\left. \begin{aligned} \lambda &= \alpha + 2\alpha_c + k_c - \frac{2}{3}\mu \\ \mu &= \mu \end{aligned} \right\} \quad (52)$$

The alternate expressions, in terms of Lamé constants, of elastic constants such as Young's modulus E , Poisson's ratio ν and volume compressibility K are given by¹⁰⁾

$$\left. \begin{aligned} E &= \frac{3\lambda + 2\mu}{\lambda + \mu} \cdot \mu, & \nu &= \frac{\lambda}{2(\lambda + \mu)} \\ K &= \frac{3\lambda + 2\mu}{3} \end{aligned} \right\} \quad (53)$$

Replacing the Lamé constants λ and μ appearing in Eq. (53) by the new coefficients

defined by Eq. (52) for undrained state of water-filled porous materials, E , ν and K are expressed in terms of α , α_e , k_e and μ , as follows:

$$\left. \begin{aligned} E &= 3\mu \left(1 - \frac{\mu}{3(\alpha + 2\alpha_e + k_e) + \mu} \right) \\ K &= \alpha + 2\alpha_e + k_e \\ \nu &= \frac{1}{2} \left(1 - \frac{\mu}{\alpha + 2\alpha_e + k_e + \frac{1}{3}\mu} \right) \end{aligned} \right\} \quad (54)$$

Eq. (54) defines a set of elastic constants for water-filled porous materials, expressed as function of $\alpha + 2\alpha_e + k_e$ and μ . As indicated by Eqs. (5), (6) and (7) and shown in Fig. 4, the constant $\alpha + 2\alpha_e + k_e$ can be evaluated by three fundamental constants of materials constituting the porous system. Plotted against the quantity μC_i in Fig. 11 are the values of Young's modulus computed by using the first relation of Eq. (54). The values of Poisson's ratio calculated by using the last relation of Eq. (54) are plotted in Fig. 12 against the quantity μC_i . As seen from these diagrams, Young's modulus and Poisson's ratio depend highly on the shear modulus μ rather than the quantity C_s/C_b , especially when the value μC_i is greater than 10^{-1} . This value corresponds to the shear modulus of about 2000 kg/cm². Consequently, when the water-filled porous material has the larger value of shear modulus, such as rock or concrete, exceeding about 2000 kg/cm², Poisson's ratio could be somewhat smaller

Fig. 11. Ratio between Young's modulus and shear modulus, E/μ

Fig. 12. Poisson's ratio of undrained saturated porous materials

Fig. 12. (continued)

than 0.5. Most soils have the shear modulus far smaller than 2000 kg/cm², and as a consequence, Poisson's ratio is quite close to 0.5. In order to facilitate computation of Poisson's ratio with sufficiently good accuracy for the soils with smaller shear modulus, approximate formula as follows is derived by expanding the expression in Eq. (54) in power series of μ/K :

$$\nu \doteq \frac{1}{2} \left(1 - \frac{\mu}{\alpha + 2\alpha_e + k_e} \right) \quad (55)$$

If Eqs. (5), (6) and (7) are further used disregarding the terms including C_s/C_b , and C_l/C_b , Eq. (55) is expressed in terms of compressibility of water, porosity and shear modulus as follows:

$$\nu = \frac{1}{2}(1 - \mu n C_l) \quad (56)$$

Comparison of the values computed by Eq. (56) with those obtained accurately shows that Eq. (56) gives the value of Poisson's ratio within the error of 1% when the value μC_l is less than 0.05. Equation (56) indicates that Poisson's ratio can be determined if shear modulus, porosity and water compressibility are known for a soil or rock under investigation. In view of the fact that difficulty generally arises in determining Poisson's ratio either in the laboratory or in the field, the use of the formula given by Eq. (56) will greatly facilitate the investigation of soil or rock properties under dynamic loading condition.

8. PORE PRESSURE CHARACTERISTICS

When the compressional wave is travelling through a porous saturated material, the stress is transmitted from one point to another within the material where a part of stress is carried by the solid skeleton while the remaining part is transmitted through the pore water. Inasmuch as the change in stress on the part of the solid skeleton affects the deformation behavior, it is of importance to assess the amount of change in effective stress when the porous material is stressed in undrained condition. The change in effective stress can readily be obtained if the change in pore pressure is made known. In what follows, what is known as pore pressure coefficient B defined as the ratio between the pore pressure and the total stress will be investigated. Consider an element of a porous material stressed undrained by uniform all-round pressure. For this particular state of stress, Eq. (49) is reduced to

$$\sigma = (\alpha + \alpha_c)e, \quad -p = (\alpha_c + k_c)e \quad (57)$$

where

$$\sigma = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

Then, the coefficient B proposed by Skempton is

$$B = \frac{-p/n}{\sigma - p} = \frac{1}{n} \frac{\alpha_c + k_c}{\alpha + 2\alpha_c + k_c} \quad (58)$$

When the compressional wave is travelling in the ground, the material will be subjected to the stress in such a manner as shown in Fig. 5 (a) with the lateral movement being confined. Such an oedometer stress condition can be obtained from Eq. (49) as follows:

$$\left. \begin{aligned} \sigma_x &= (\alpha + \alpha_c + \frac{4}{3}\mu)e_x \\ -p &= (\alpha_c + k_c)e_x \end{aligned} \right\} \quad (59)$$

For this state of stress, the pore pressure coefficient B' may be defined by

$$B' = \frac{-p/n}{\sigma_x - p} = \frac{1}{n} \frac{\alpha_c + k_c}{\alpha + 2\alpha_c + k_c + \frac{4}{3}\mu} \quad (60)$$

From the definition of B and B' , it is apparent that the coefficient B specifies the pore pressure development in all-round stress condition, while B' is associated with the building-up of pore pressure in the oedometer condition. It is possible to express the coefficients in terms of compressibilities of the ingredients constituting the porous material as follows:

$$B = \frac{1}{1 + \frac{C_l - C_s}{C_r}} \quad (61)$$

$$B' = \frac{1}{1 + \frac{C_l - C_s}{C_p} + \frac{4}{3}\mu \frac{C_b}{C_p} \left[(C_l - C_s) + \frac{C_s}{C_b} \cdot C_p \right]} \quad (62)$$

The coefficient B defined by Eq. (61) is an extended form of the pore pressure coefficient proposed by A. W. Skempton. If the term having the ratio C_s/C_b is dropped as being negligibly small, Eq. (61) is reduced to the same form as originally defined by Skempton:

$$B = \frac{1}{1 + \frac{nC_i}{C_b}} \quad (63)$$

Likewise, Eq. (62) becomes

$$B' = \frac{1}{1 + \frac{nC_i}{C_b} + \frac{4}{3}\mu[(1-n)C_s + nC_i]} \quad (64)$$

The result of numerical computation for B' as given by Eq. (62) is shown in Fig. 13 with B' plotted against μC_i . Since B' tends to approach B as the value of μ becomes small, the asymptotic values on the left part of the diagram are approximately equal to the value of B . The diagrams indicate that the pore pressure coefficient depends on the value of μC_i as well as the compressibility ratio C_s/C_b . For a much smaller value of C_s/C_b as is the case with soft clays, B' is very close to 1.0, provided μC_i is less than 10^{-2} , whereas the decrease in the value B' occurs significantly for μC_i larger than 10^{-1} . For hard materials like rocks having the larger value of C_s/C_b , the coefficient B' is considerably smaller as seen in the diagram of Fig. 13 (b).

Fig. 13. Pore pressure coefficient B' in oedometer state of stress

Fig. 13. (continued)

It will be of interest to see what relationship exists between the pore pressure coefficients and Young's modulus. By eliminating the constant $\alpha + 2\alpha_e + k_e$ between the first of Eq. (54) and Eqs. (58) and (60), Young's modulus can be expressed as

$$\frac{E}{3\mu} = \frac{4B'/B}{1 + 3B'/B} \quad (65)$$

Likewise, the last of Eq. (54) yields the relationship between Poisson's ratio and the

Fig. 14. Values of B'/B

Table 3. A list of correspondence of the material constants between viscoelastic and poro-elastic models

Constants in the visco-elastic model Eq. (34)	Corresponding constants in the porous model	
	Compressional mode Eq. (33)	Shear mode Eq. (33)
E_a'	$\left(\alpha + 2\alpha_c + k_c + \frac{4}{3}\mu\right) \sqrt{\frac{\theta_2}{\theta_1} - 1}$	$\mu \sqrt{\frac{\rho_w}{\rho_s}}$
E_v	$\left(\alpha + 2\alpha_c + k_c + \frac{4}{3}\mu\right) / \sqrt{\frac{\theta_2}{\theta_1} - 1}$	$\mu \sqrt{\frac{\rho_s}{\rho_w}}$
η_v	$\frac{\rho_w \rho_s V \pi^2}{b} / \sqrt{\frac{\theta_2}{\theta_1} - 1}$	$\frac{\mu}{b} \sqrt{\rho_s \rho_w}$
τ	$\frac{\rho_x}{b} \cdot \frac{\theta_2}{\theta_1} / \left(1 + \frac{\rho_w}{\rho_s}\right) = \frac{1}{\omega_c}$	$\frac{\rho_w}{b} = \frac{1}{w_s}$
ρ	$(\rho_w + \rho_s) \frac{\theta_1}{\theta_2} \sqrt{\frac{\theta_2}{\theta_1} - 1}$	$\sqrt{\rho_w \rho_s}$

pore pressure coefficients as follows:

$$\nu = \frac{3B'/B - 1}{3B'/B + 1} \quad (66)$$

In order to facilitate transformation between the values of $E/3\mu$ or ν and the ratio B'/B , Fig. 14 was provided in which B'/B is plotted versus the value of μC_1 . Apparently, the value B'/B decreases sharply when the value μC_1 is increased above about 0.1. The relations as given by Eqs. (65) and (66) are shown on the diagram in Fig. 15.

Fig. 15. Relations between Young's modulus $E/3\mu$, Poisson's ratio, and pore pressure coefficients

9. CONCLUSIONS

As a result of thorough examination of wave equations for porous water-saturated materials by using the numerical values of constants, it was found that the equation can be simplified greatly for the usual range of frequencies and reduced to the one which is identical in form with the wave equations for elastic materials. Therefore, use of the elastic theory is suggested as a substitute for the poro-elasticity, especially when the medium is stressed in undrained condition as encountered in dynamical problems. To be used for such cases, the expressions for Young's modulus and Poisson's ratio in terms of poro-elastic constants were given. These expressions indicate that Poisson's ratio depends roughly on the water compressibility, shear modulus and the porosity of the skeleton of the porous medium. Insofar as Poisson's ratio is estimated on the basis of the information concerning the material properties of the porous system, there is no need to perform a test solely to determine Poisson's ratio, which is fairly difficult in most cases. It may be sufficient to find only the shear modulus of the porous system. When the wave is propagating in undrained condition, there is a rise in pore water pressure. Expressions were obtained for the pore pressure coefficients under all-round compression as well as under one-dimensional compression, and they were related to each other. Also derived were the relationships between Young's modulus, Poisson's ratio and the pore pressure coefficients for undrained stress condition of a porous water-filled material.

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12. NOTATIONS

- σ_{ij} : tensor of the stress acting on the solid part
 $-p$: stress acting on the fluid part
 e_{ij} : strain tensor of the solid part
 u_i : displacement component of the solid part
 v_i : displacement component of the fluid part
 e : volume change of the solid part
 ε : volume change of the fluid part
 $\mathbf{u}$: displacement vector of the solid part
 $\mathbf{v}$: displacement vector of the fluid part
 $\mathbf{w}$: rotation of the displacement vector of the solid part
 $\mathbf{W}$: rotation of the displacement vector of the fluid part
 δ_{ij} : Kronecker's delta
 μ : shear modulus
 α, α_c, k_c : moduli of poro-elasticity related with volume compressibility
 C_b : bulk compressibility of the solid skeleton
 C_s : compressibility of the solid particle
 C_p : pore compressibility
 C_l : compressibility of water
 n : porosity
 ρ_s : bulk mass density of the solid part
 ρ_w : bulk mass density of the fluid part
 b : constants associated with the friction between two phases
 K : permeability coefficient
 g : acceleration of gravity
 A_e, A_c : amplitudes of vibration in compressional mode
 A_w, A_s : amplitudes of vibration in shear mode

- l : variable related with the wave length
- ω : angular frequency of vibration
- ω_c : characteristic frequency in compressional mode
- ω_s : characteristic frequency in shear mode
- V_l, V_d, V_n : velocities of propagation of compressional waves
- V_p, V_s : velocities of propagation of shear waves
- V_c : compressional wave velocity
- V_h : shear wave velocity
- δ_c : logarithmic decrement associated with compressional wave
- δ_h : logarithmic decrement associated with shear wave
- σ_{ij}^t : tensor of total stress
- E : Young's modulus
- ν : Poisson's ratio
- K : volume compressibility
- B : pore pressure coefficient under all-round pressure
- B' : pore pressure coefficient under one-dimensional stress
- θ_1, θ_2 : constants defined by Eq. (23)
- θ_c, θ_s : constants defined by Eqs. (29) and (31)
- R : constants defined by Eq. (26)

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