FINITE ELEMENT METHOD APPLIED TO BIOT'S CONSOLIDATION THEORY

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SYNOPSIS

A variational principle equivalent to the governing equations in Biot's consolidation theory is derived. On the basis of the variational principle, a numerical method effective to nonhomogeneous, anisotropic soil in one-, two-, or three-dimension is developed. Two examples are given to illustrate the validity and practicality of the method.

INTRODUCTION

The closed-form solutions of the three-dimensional consolidation theory developed by Biot (1941, 1955, 1956) were obtained only in simple cases due to the complexity of the governing equations (see, for example, Paria's review, 1963).

In this paper, the finite element method used in stress analysis of solids (e.g., Pian and Tong, 1969) is applied to Biot's consolidation theory and a numerical method effective to nonhomogeneous, anisotropically elastic soil of arbitrary geometry in one-, two-, or three-dimension is developed. In the first section, a variational principle characterizing Biot's consolidation theory is derived. In the second section, a direct method is applied to the variational principle in order to obtain governing simultaneous equations of approximate solutions. In the third section, two examples are given to illustrate the validity and practicality of the numerical method.

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VARIATIONAL PRINCIPLE

The standard indicial system with respect to a rectangular Cartesian reference frame is employed: Repeated subscripts imply summation, Kronecker's delta is denoted by δ_{ij} , differentiation with respect to space is indicated by subscripts preceded by comma and differentiation with respect to time by superposed dots, no distinction is made between covariant and contravariant tensors. The spatial coordinate vector is denoted by x , and the time by t . The quantities such as water head, displacement of soil, etc., are the

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increments from the initial stationary state.

Three assumptions are made: The soil is nonhomogeneous, anisotropically elastic, and saturated by incompressible water; the deformation of the soil does not depend on pore water pressure, but on effective stress; the water flows through soil according to Darcy's law. The second assumption is made in accordance with the effective stress concept in soil mechanics, though it deviates from Biot's assumption (Biot, 1955).

The volume of the soil is denoted by V and its boundary surface by S . Surface traction $\bar{T}_i$ and displacement of soil $\bar{u}_i$ are prescribed on S_T and S_u , respectively; normal component of stream velocity of water relative to soil $\bar{v}$ and water head $\bar{h}$ are prescribed on S_v and S_h , respectively, where

$$S = S_T + S_u = S_v + S_h \quad (1)$$

The region is divided into N elements. The volume and the boundary surface of the n -th element are denoted by V^n and S^n , respectively. The terms S_T^n , S_u^n , S_v^n and S_h^n are the portions of S^n belonging, respectively, to S_T , S_u , S_v and S_h . The interelement boundary surface of the n -th element is denoted by S_I^n (cf. Fig. 1).

Fig. 1. Division of region into finite elements

$$S^n = S_T^n + S_u^n + S_I^n = S_v^n + S_h^n + S_I^n \quad (2)$$

$$V = \sum_n V^n \quad (3)$$

$$S_T = \sum_n S_T^n \quad (4)$$

$$S_u = \sum_n S_u^n \quad (5)$$

$$S_v = \sum_n S_v^n \quad (6)$$

$$S_h = \sum_n S_h^n \quad (7)$$

where $\sum_n$ indicates the sum of N terms.

As for soil, displacement vector, body force vector, strain tensor, effective stress tensor, permeability coefficient tensor, and elasticity tensor with respect to effective stress are denoted by u_i , F_i , ε_{ij} , σ_{ij} , k_{ij} and E_{ijkl} , respectively, where tensile stress and tensile strain are taken to be positive. Water head and stream velocity vector of water relative to soil are denoted by h and v_i , respectively. Constant unit weight of water is γ_w . The quantities u_i , F_i , ε_{ij} , σ_{ij} , h , v_i , $\bar{u}_i$, $\bar{T}_i$, $\bar{v}$, and $\bar{h}$ are functions of x and t ; the quantities k_{ij} and E_{ijkl} are functions of x . The tensors ε_{ij} , σ_{ij} , and k_{ij} are symmetric and the tensor E_{ijkl} has the usual symmetric properties of the elasticity tensor of a non-porous solid (Biot, 1955). The outward unit normal vector on the boundary surface of an element is denoted by n_i .

Relaxing the continuity requirement of σ_{ij} and v_i on the interelement boundary surface S_I^n , Biot's three-dimensional consolidation theory for anisotropically elastic soil (Biot, 1955) can be rewritten as follows. The field equations in V^n are

$$(\sigma_{ij} - \gamma_w h \delta_{ij})_{,j} + F_i = 0 \quad (8)$$

$$\sigma_{ij} = E_{ijkl} \varepsilon_{kl} \quad (9)$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (10)$$

$$v_i = -k_{ij} h_{,j} \quad (11)$$

$$\dot{\varepsilon}_{ii} = -v_{i,j} \quad (12)$$

Equations (8), (9) and (10) show the equilibrium condition, the effective stress-strain relation, and the strain-displacement relation, respectively. Equation (11) is Darcy's law for an anisotropic porous solid (Biot, 1955). Equation (12) is the relation between volumetric strain rate of soil and stream velocity of water relative to soil. Elimination of v_i in Eqs. (11) and (12) yields

$$\dot{\varepsilon}_{ii} = (k_{ij} h_{,i})_{,j} \quad \text{in } V^n \quad (13)$$

As the boundary conditions,

$$\bar{h} = h \quad \text{on } S_h^n \quad (14)$$

$$\bar{v} = -n_i k_{ij} h_{,j} \quad \text{on } S_v^n \quad (15)$$

$$\bar{T}_i = n_j (\sigma_{ij} - \gamma_w h \delta_{ij}) \quad \text{on } S_T^n \quad (16)$$

$$\bar{u}_i = u_i \quad \text{on } S_u^n \quad (17)$$

On the interelement boundary surface S_I^n ,

$$\{(n_i v_i)\}^n + \{(n_i v_i)\}^{n+1} = 0 \quad (18)$$

$$\{n_i (\sigma_{ij} - \gamma_w h \delta_{ij})\}^n + \{n_i (\sigma_{ij} - \gamma_w h \delta_{ij})\}^{n+1} = 0 \quad (19)$$

Equation (18) expresses the continuity of normal components of stream velocity of water

relative to soil on S_I^n and Eq. (19) expresses the equilibrium of surface tractions on S_I^n . Substitution of Eq. (11) in Eq. (18) yields

$$(n_i k_{ij} h_{,j})^n + (n_i k_{ij} h_{,j})^{n+1} = 0 \quad \text{on } S_I^n \quad (20)$$

The initial condition is

$$\varepsilon_{ii} = 0 \quad \text{in } V^n \quad (21)$$

Equation (13) with the initial condition (21) is equivalent to the following equation (Gurtin, 1964a).

$$\varepsilon_{ii} = (k_{ij} * h_{,j})_{,i} \quad \text{in } V^n \quad (22)$$

For the functions A and B in space and time, $A * B$ implies convolution and is defined by the following equation (e.g., Churchill, 1958).

$$A(x, t) * B(x, t) = \int_0^t A(x, t - \tau) B(x, \tau) d\tau \quad (23)$$

Hence the governing equations are Eqs. (8), (9), (10), (14), (15), (16), (17), (19), (20), and (22).

In the variational principle stated below, an integral $\int_A \dots dx$ is a volume or a surface integral, according as A is a volume or a surface. The continuity and differentiability requirement with respect to space and time are assumed as follows. With respect to space, the functions u_i and h are continuous in V ; u_i , h , and their derivatives up to the second order with respect to space are continuous in V_n ; E_{ijkl} , k_{ij} , their first partial derivatives, and F_i are continuous in V^n . With respect to time, u_i , its first partial derivative with respect to time, and h are continuous. With respect to space, $\bar{T}_i$ is continuous on S_T^n , $\bar{v}$ on S_T^n , $\bar{u}_i$ on $\sum_n S_{u_i}^n$, and $\bar{h}$ on $\sum_n S_h^n$. With respect to time, $\bar{T}_i$, $\bar{v}$, $\bar{u}_i$, and $\bar{h}$ are continuous.

The variational principle equivalent to the above-mentioned governing equations is as follows: For the functional Π defined by

$$\begin{aligned} \Pi = \sum_n \left[\int_{S_v^n} \gamma_w * \bar{v} * h dx + \int_{S_T^n} \bar{T}_i * u_i dx + \right. \\ \left. + \int_{V^n} \left(\frac{1}{2} \gamma_w k_{ij} * h_{,i} * h_{,j} + \gamma_w h * \varepsilon_{ii} - \frac{1}{2} \sigma_{ij} * \varepsilon_{ij} + F_i * u_i \right) dx \right] \quad (24) \end{aligned}$$

with side conditions (9), (10), (14) and (17),

$$\delta \Pi = 0 \quad (25)$$

The first variation of Π becomes, by the divergence theorem,

$$\begin{aligned} \delta \Pi = \sum_n \left[\int_{S_v^n} \gamma_w * (\bar{v} + n_i k_{ij} h_{,j}) * \delta h dx + \right. \\ \left. + \int_{S_I^n} \{ \gamma_w * n_i k_{ij} h_{,j} * \delta h - n_j (\sigma_{ij} - \gamma_w h \delta_{ij}) * \delta u_i \} dx + \right. \end{aligned}$$

$$\begin{aligned}
 & + \int_{S_T^n} \{\bar{T}_i - n_j(\sigma_{ij} - \gamma_w h \delta_{ij})\}^* \delta u_i dx + \\
 & + \int_{V^n} [\gamma_w \{\varepsilon_{ii} - (k_{ij}^* h_{,j})_{,i}\}^* \delta h + \{(\sigma_{ij} - \gamma_w h \delta_{ij})_{,j} + F_i\}^* \delta u_i] dx = 0 \quad (26)
 \end{aligned}$$

By the fundamental lemma of the calculus of variations which includes convolution (Gurtin, 1964b), satisfaction of Eq. (26) for each δh and δu_i which satisfy

$$\delta h = 0 \quad \text{on} \quad S_h^n \quad (27)$$

$$\delta u_i = 0 \quad \text{on} \quad S_u^n \quad (28)$$

is equivalent to Eqs. (8), (15), (16), (19), (20), and (22).

FINITE ELEMENT FORMULATION

To the variational principle developed in the preceding section, a direct method (e.g., Kantorovich and Krylov, 1958) is applied in order to derive the governing equations of approximate solutions. Utilizing matrix representation, the functional Π can be rewritten as follows.

$$\begin{aligned}
 \Pi = \sum_n \left[\int_{S_u^n} \gamma_w^* \bar{v}^* h dx + \int_{S_T^n} \{\bar{T}_i\}^T \{u_i\} dx + \right. \\
 \left. + \int_{V^n} \left(\frac{1}{2} \gamma_w \{h_{,i}\}^T [k_{ij}]^* \{h_{,i}\} + \gamma_w h^* \varepsilon_{ii} - \frac{1}{2} \{\sigma_i\}^T \{\varepsilon_i\} + \{F_i\}^T \{u_i\} \right) dx \right] \quad (29)
 \end{aligned}$$

The superscript T denotes the transpose of a matrix. The column vectors $\{u_i\}$, $\{\sigma_i\}$, $\{\varepsilon_i\}$ and $\{F_i\}$ are, as for soil, the displacement vector, the reduced effective stress tensor, the reduced strain tensor, and the body force vector, respectively. The column vectors $\{\bar{T}_i\}$ and $\{h_{,i}\}$ are the prescribed surface traction vector and the water head gradient vector. The symmetric matrix $[k_{ij}]$ is the permeability coefficient tensor. The effective stress-strain relations (9) can be expressed as

$$\{\sigma_i\} = [E_{ij}] \{\varepsilon_i\} \quad (30)$$

where the symmetric matrix $[E_{ij}]$ is the reduced elasticity tensor with respect to effective stress.

The column vector $\{\phi^n(t)\}$ is the set of the displacement vectors at all nodal points belonging to the n -th element. The column vector $\{\psi^n(t)\}$ is the set of water heads at all nodal points belonging to the n -th element. The column vector $\{\alpha(t)\}$ is the set of the displacement vectors and water heads at all nodal points belonging to the *entire* region. The column vectors $\{\phi^n(t)\}$ and $\{\psi^n(t)\}$ can be expressed by $\{\alpha(t)\}$ through the transformation matrices $[\Phi^n]$ and $[\Psi^n]$ as follows.

$$\{\phi^n(t)\} = [\Phi^n] \{\alpha(t)\} \quad (31)$$

$$\{\psi^n(t)\} = [\Psi^n] \{\alpha(t)\} \quad (32)$$

Defining the shape functions of the n -th element $[a^n(\mathbf{x})]$ and $[b^n(\mathbf{x})]$, the displacement vector

and the water head in V^n can be written as

$$\{u_i\} = [a^n(\mathbf{x})]\{\phi^n(t)\} \quad (33)$$

$$h = \{b^n(\mathbf{x})\}^T \{\phi^n(t)\} \quad (34)$$

The shape functions must be such that the displacement vector and the water head and their derivatives up to the second order with respect to space are continuous in the n -th element and that when the nodal displacement vectors and the nodal water heads on the interelement boundary surface of two neighboring elements are compatible, the displacement vector and the water head across the interelement boundary surface are compatible. The shape function $[a^n(\mathbf{x})]$ is usually called "the displacement function" in the finite element method for stress analysis of solids (e.g., Zienkiewicz and Cheung, 1967). Using Eqs. (31) and (32),

$$\{u_i\} = [A^n(\mathbf{x})]\{\alpha(t)\} \quad (35)$$

$$[A^n(\mathbf{x})] = [a^n(\mathbf{x})][\Phi^n] \quad (36)$$

$$h = \{B^n(\mathbf{x})\}^T \{\alpha(t)\} \quad (37)$$

$$\{B^n(\mathbf{x})\}^T = \{b^n(\mathbf{x})\}^T [\Psi^n] \quad (38)$$

The vectors $\{\varepsilon_i\}$, $\{h_{,i}\}$ and the scalar ε_{ii} are obtained by appropriate space differentiation of Eqs. (35) and (37).

$$\{\varepsilon_i\} = [C^n(\mathbf{x})]\{\alpha(t)\} \quad (39)$$

$$\varepsilon_{ii} = \{D^n(\mathbf{x})\}^T \{\alpha(t)\} \quad (40)$$

$$\{h_{,i}\} = [G^n(\mathbf{x})]\{\alpha(t)\} \quad (41)$$

From Eq. (30)

$$\{\sigma_i\} = [E_{ij}][C^n(\mathbf{x})]\{\alpha(t)\} \quad (42)$$

Utilizing these matrix representations,

$$\begin{aligned} & \sum_n \left[\int_{S_v^n} \gamma_w \bar{v}^* h d\mathbf{x} + \int_{S_T^n} \{\bar{T}_i\}^T \{u_i\} d\mathbf{x} + \int_{V^n} \left(\frac{1}{2} \gamma_w \{h_{,i}\}^T [k_{ij}] \{h_{,i}\} + \gamma_w h^* \varepsilon_{ii} \right. \right. \\ & \quad \left. \left. - \frac{1}{2} \{\sigma_i\}^T \{\varepsilon_i\} + \{F_i\} \{u_i\} \right) d\mathbf{x} \right] \\ &= \{\alpha(t)\}^T \sum_n \gamma_w \int_{S_v^n} \{B^n(\mathbf{x})\}^* \bar{v}(\mathbf{x}, t) d\mathbf{x} + \{\alpha(t)\}^T \sum_n \int_{S_T^n} [A^n(\mathbf{x})]^T \{\bar{T}_i(\mathbf{x}, t)\} d\mathbf{x} \\ & \quad + \frac{1}{2} \{\alpha(t)\}^T \sum_n \gamma_w \int_{V^n} [G^n(\mathbf{x})]^T [k_{ij}(\mathbf{x})] [G^n(\mathbf{x})] d\mathbf{x} \{\alpha(t)\} \\ & \quad + \{\alpha(t)\}^T \sum_n \gamma_w \int_{V^n} \{B^n(\mathbf{x})\} \{D^n(\mathbf{x})\}^T d\mathbf{x} \{\alpha(t)\} \\ & \quad - \frac{1}{2} \{\alpha(t)\}^T \sum_n \int_{V^n} [C^n(\mathbf{x})]^T [E_{ij}(\mathbf{x})] [C^n(\mathbf{x})] d\mathbf{x} \{\alpha(t)\} \end{aligned}$$

$$\begin{aligned}
 & + \{\alpha(t)\}^T * \sum_n \int_{V^n} [A^n(x)]^T \{F_i(x, t)\} dx \\
 = & -\frac{1}{2} \{\alpha(t)\}^T * [L] \{\alpha(t)\} + \{\alpha(t)\}^T * [M] \{\alpha(t)\} \\
 & + \frac{1}{2} \{\alpha(t)\}^T * [N] * \{\alpha(t)\} + \{\alpha(t)\}^T * \{P(t)\}
 \end{aligned} \tag{43}$$

where

$$[L] = [L]^T = \sum_n \int_{V^n} [C^n(x)]^T [E_{ij}(x)] [C^n(x)] dx \tag{44}$$

$$[M] = \sum_n \gamma_w \int_{V^n} \{B^n(x)\} \{D^n(x)\}^T dx \tag{35}$$

$$[N]^T = [N]^T = \sum_n \gamma_w \int_{V^n} [G^n(x)]^T [k_{ij}(x)] [G^n(x)] dx \tag{46}$$

$$\begin{aligned}
 \{P(t)\} = & \sum_n \left(\gamma_w \int_{S_v^n} \{B^n(x)\} * \bar{v}(x, t) dx + \int_{S_T^n} [A^n(x)]^T \{\bar{T}_i(x, t)\} dx \right. \\
 & \left. + \int_{V^n} [A^n(x)]^T \{F_i(x, t)\} dx \right)
 \end{aligned} \tag{47}$$

Since $(1/2)\sigma_{ij}\varepsilon_{ij}$ and $-v_i h_{,i}$ are positive definite, matrices $[L]$ and $-[N]$ are positive definite. Substituting the known $\bar{u}_i$ and $\bar{h}$ on S_u and S_h into $\{\alpha(t)\}$ of Eq. (43), the functional Π can be obtained. The column vector $\{\alpha(t)\}$ can be divided into the unknown column vector $\{\alpha_1(t)\}$ and the known column vector $\{\alpha_2(t)\}$.

$$\{\alpha(t)\} = \begin{Bmatrix} \alpha_1(t) \\ \dots \\ \alpha_2(t) \end{Bmatrix} \tag{48}$$

In accordance with the rearrangement of $\{\alpha(t)\}$, the matrices $[L]$, $[M]$, $[N]$ and the vector $\{P(t)\}$ are rearranged and Eq. (43) is rewritten in the following form.

$$\begin{aligned}
 \Pi = & -\frac{1}{2} \begin{Bmatrix} \alpha_1(t) \\ \dots \\ \alpha_2(t) \end{Bmatrix}^T * \begin{bmatrix} L_{11} & L_{12} \\ \dots & \dots \\ L_{21} & L_{22} \end{bmatrix} \begin{Bmatrix} \alpha_1(t) \\ \dots \\ \alpha_2(t) \end{Bmatrix} + \begin{Bmatrix} \alpha_1(t) \\ \dots \\ \alpha_2(t) \end{Bmatrix}^T * \begin{bmatrix} M_{11} & M_{12} \\ \dots & \dots \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} \alpha_1(t) \\ \dots \\ \alpha_2(t) \end{Bmatrix} \\
 & + \frac{1}{2} \begin{Bmatrix} \alpha_1(t) \\ \dots \\ \alpha_2(t) \end{Bmatrix}^T * \begin{bmatrix} N_{11} & N_{12} \\ \dots & \dots \\ N_{21} & N_{22} \end{bmatrix} \begin{Bmatrix} \alpha_1(t) \\ \dots \\ \alpha_2(t) \end{Bmatrix} + \begin{Bmatrix} \alpha_1(t) \\ \dots \\ \alpha_2(t) \end{Bmatrix}^T * \begin{Bmatrix} P_1(t) \\ \dots \\ P_2(t) \end{Bmatrix}
 \end{aligned} \tag{49}$$

The first variation of the functional Π is

$$\begin{aligned}
 \delta \Pi = & \{\delta \alpha_1(t)\}^T * \left\{ \left(-[L_{11} \dots L_{12}] + [M_{11} \dots M_{12}] + \begin{bmatrix} M_{11} \\ \dots \\ M_{21} \end{bmatrix}^T \right) \begin{Bmatrix} \alpha_1(t) \\ \dots \\ \alpha_2(t) \end{Bmatrix} \right. \\
 & \left. + [N_{11} \dots N_{12}] * \begin{Bmatrix} \alpha_1(t) \\ \dots \\ \alpha_2(t) \end{Bmatrix} + \{P_1(t)\} \right\} \\
 = & \{\delta \alpha_1(t)\}^T * ([Q] \{\alpha_1(t)\} + [N_{11}] * \{\alpha_1(t)\} + \{R(t)\}) = 0
 \end{aligned} \tag{50}$$

where

$$[Q] = [Q]^T = -[L_{11}] + [M_{11}] + [M_{11}]^T \quad (51)$$

$$\{R(t)\} = (-[L_{12}] + [M_{12}] + [M_{21}]^T)\{\alpha_2(t)\} + [N_{12}]\{\alpha_2(t)\} + \{P_1(t)\} \quad (52)$$

Hence

$$[Q]\{\alpha_1(t)\} + [N_{11}]\{\alpha_1(t)\} + \{R(t)\} = 0 \quad (53)$$

Equation (53) is the governing equation of approximate solutions. One of the solution techniques is as follows.

The step-by-step solution technique with respect to time developed by Zienkiewicz and Parekh (1970) is applied to Eq. (53). Differentiating Eq. (53) with respect to time,

$$[Q]\{\dot{\alpha}_1(t)\} + [N_{11}]\{\alpha_1(t)\} + \{\dot{R}(t)\} = 0 \quad (54)$$

In the time interval $(t, t + \Delta t)$

$$[Q]\left\{\dot{\alpha}_1\left(t + \frac{1}{2}\Delta t\right)\right\} + [N_{11}]\left\{\alpha_1\left(t + \frac{1}{2}\Delta t\right)\right\} + \left\{\dot{R}\left(t + \frac{1}{2}\Delta t\right)\right\} = 0 \quad (55)$$

$$\left\{\dot{\alpha}_1\left(t + \frac{1}{2}\Delta t\right)\right\} = \frac{1}{\Delta t}(\{\alpha_1(t + \Delta t)\} - \{\alpha_1(t)\}) \quad (56)$$

$$\left\{\alpha_1\left(t + \frac{1}{2}\Delta t\right)\right\} = \frac{1}{2}(\{\alpha_1(t + \Delta t)\} + \{\alpha_1(t)\}) \quad (57)$$

Substitution of Eqs. (56), (57) into Eq. (55) yields

$$\left(\frac{\Delta t}{2}[N_{11}] + [Q]\right)\{\alpha_1(t + \Delta t)\} = -\left(\frac{\Delta t}{2}[N_{11}] - [Q]\right)\{\alpha_1(t)\} - \Delta t\left\{\dot{R}\left(t + \frac{1}{2}\Delta t\right)\right\} \quad (58)$$

If $\{\alpha_1(t)\}$ is known, $\{\alpha_1(t + \Delta t)\}$ can be obtained by Eq. (58).

NUMERICAL EXAMPLES

One-dimensional consolidation

In Fig. 2(a), the load of 5 ton/m² is applied instantaneously on the upper surface of the clay layer. The clay layer of thickness $H = 15$ m is underlain by an impervious, rigid rock. Drainage is allowed only on the upper surface. The coefficient of volume compressibility m_v is equal to 5.0×10^{-3} m²/ton, which is the only elasticity coefficient necessary for calculation in a one-dimensional case. The permeability coefficient k is equal to 1.0×10^{-9} m/sec. The unit weight of water γ_w is equal to 1.0 ton/m³. The downward distance from the upper surface is denoted by z .

The clay layer is divided into eight line elements. The vertical displacement and the water head at the i -th nodal point are denoted by $w_i(t)$ and $h_i(t)$, respectively. In Fig. 2(b), the n -th element of length l has the i -th and the j -th nodal points. The column vectors $\{\phi^n(t)\}$ and $\{\psi^n(t)\}$ of the element are

$$\{\phi^n(t)\} = \begin{Bmatrix} w_i(t) \\ w_j(t) \end{Bmatrix} \quad (59)$$

Fig. 2. One-dimensional consolidation

$$\{\phi^n(t)\} = \begin{Bmatrix} h_i(t) \\ h_j(t) \end{Bmatrix} \quad (60)$$

The shape functions employed are

$$[a(x)]^T = \{b^n(x)\} = \begin{Bmatrix} 1 - \frac{s}{l} \\ \frac{s}{l} \end{Bmatrix} \quad (61)$$

where s is the downward distance from the upper nodal point (the i -th nodal point).

The numerical solution by Eq. (58) is shown by dots in Figs. 3 and 4. The average degree of consolidation U is shown in Fig. 3 and the solid line is computed by the following equations which closely approximate the exact value (e.g., Scott, 1963).

$$U = \sqrt{\frac{4T}{\pi}} \quad (T \leq 0.2) \quad (62)$$

$$U = 1 - \frac{8}{\pi^2} \exp\left(-\frac{1}{4}\pi^2 T\right) \quad (T \geq 0.2) \quad (63)$$

where T is the time factor. The degree of consolidation U_z is shown in Fig. 4. The solid line is the partial sum of fifty terms in the exact solution (e.g., Scott, 1963);

$$U_z = 1 - \sum_{m=0}^{\infty} \frac{2}{M} \left(\sin \frac{Mz}{H} \right) \exp(-M^2 T) \quad (64)$$

$$M = \frac{1}{2}\pi(2m + 1)$$

The numerical solution by Eq. (58) agrees well with the exact solution.

Axi-symmetric consolidation

The circular uniform load p which depends on time t acts on the upper surface of the clay layer underlain by impervious rigid rock.

Fig. 3. Average consolidation ratio in one-dimensional consolidation

Fig. 4. Consolidation ratio in one-dimensional consolidation

A ring whose cross section is triangular is chosen as an element as shown in Fig. 5. A cylindrical coordinate system is employed, where r and z denote respectively the radial and axial coordinates of a point: The corresponding components of displacement are u_r , u_z and those of stream velocity of water relative to soil v_r , v_z ; the strain tensor ϵ_{rr} , ϵ_{zz} , $\epsilon_{\theta\theta}$, ϵ_{zr} , and the effective stress tensor σ_{rr} , σ_{zz} , $\sigma_{\theta\theta}$, σ_{zr} ; the permeability coefficient tensor k_{rr} , k_{zz} , k_{zr} .

The shape functions to a ring in Fig. 5 are

Fig. 5. Typical triangular axis-symmetric element

$$[a^n(x)] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_k & 0 \\ 0 & N_i & 0 & N_j & 0 & N_k \end{bmatrix} \quad (65)$$

$$\{b^n(x)\} = \begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix} \quad (66)$$

where

$$N_i = (a_i + b_i r + c_i z)/2\Delta \quad (67)$$

$$2\Delta = \det \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_k & z_k \end{bmatrix} \quad (68)$$

$$a_i = r_j z_k - r_k z_j$$

$$b_i = z_j - z_k \quad (69)$$

$$c_i = r_k - r_j$$

In Fig. 6, the clay layer is 20 m thick. The origin of the coordinate system is point A. The circular uniform load p of radius 10 m acts on the ground surface, where

$$p = \begin{cases} 1.0 \times 10^{-6} t \text{ (ton/m}^2\text{)} & (t \leq 0.5 \times 10^7 \text{ sec}) \\ 5.0 & (\text{ton/m}^2) \quad (t > 0.5 \times 10^7 \text{ sec}) \end{cases} \quad (70)$$

The vertical boundary of the clay layer is assumed to be sixty meters away from the axis of symmetry. The boundary conditions are as follows.

Fig. 6. Axi-symmetric finite element idealization

On the axis of symmetry

$$u_r = 0, \quad v_r = 0 \quad (r = 0 \text{ m}) \quad (71)$$

On the surface of the rock

$$u_r = u_z = 0, \quad v_z = 0 \quad (z = 20 \text{ m}) \quad (72)$$

On the vertical boundary sixty meters away from the axis of symmetry

$$u_r = 0, \quad \sigma_{rz} = 0, \quad h = 0 \quad (r = 60 \text{ m}) \quad (73)$$

On the ground surface

$$\begin{aligned} h &= 0 & (z = 0 \text{ m}) \\ \sigma_{zz} &= \begin{cases} p & (z = 0 \text{ m}, \quad r \leq 10 \text{ m}) \\ 0 & (z = 0 \text{ m}, \quad r > 10 \text{ m}) \end{cases} \end{aligned} \quad (74)$$

The clay layer is divided into ninetythree elements with fifty-nine nodal points.

The clay layer is assumed to be a transversely isotropic material with five elasticity coefficients. Using the notation of Lekhnitskii (1963), the stress-strain relation in the axi-symmetric case is

$$\begin{aligned} \varepsilon_z &= \sigma_z/E_2 - \nu_2\sigma_r/E_2 - \nu_2\sigma_\theta/E_2 \\ \varepsilon_r &= -\nu_2\sigma_z/E_2 + \sigma_r/E_1 - \nu_1\sigma_\theta/E_1 \\ \varepsilon_\theta &= -\nu_2\sigma_z/E_2 - \nu_1\sigma_r/E_1 + \sigma_\theta/E_1 \\ 2\varepsilon_{zr} &= \tau_{zr}/G_2 \end{aligned} \quad (75)$$

The elasticity coefficients and the permeability coefficients employed in the calculation are

$$E_2 = \begin{cases} 2.0 \times 10^2 & (\text{ton/m}^2) & (z \leq 10 \text{ m}) \\ 2.0 \times 10^2 (0.5 + z/20) & (\text{ton/m}^2) & (z \geq 10 \text{ m}) \end{cases}$$

$$\begin{aligned}
 E_1 &= 2.5E_2 \\
 G_2 &= 0.5E_1 \\
 \nu_1 &= 0.3 \\
 \nu_2 &= 0.2 \\
 k_z &= \begin{cases} 1.0 \times 10^{-8} & (\text{m/sec}) \quad (z \leq 10 \text{ m}) \\ 1.0 \times 10^{-8} (1.2-0.02z) & (\text{m/sec}) \quad (z \geq 10 \text{ m}) \end{cases} \\
 k_r &= 5.0k_z \\
 k_{rz} &= 0
 \end{aligned} \tag{77}$$

The unit weight γ_w is equal to 1.0 ton/m³.

The volume integral over elements is performed numerically by Gaussian quadrature formula (Stroud and Secrest, 1966).

In Fig. 7, the settlements of the points A and B of Fig. 6 are shown. The settlement of the ground surface in one-dimensional consolidation under the same uniform load p as that of Eq. (70) acting all over the ground surface is also shown. The consolidation is more rapid in the axi-symmetric case than in the one-dimensional case. One of the reasons may be that the horizontal permeability coefficient is higher.

Fig. 7. Variation of settlement of ground surface

In Fig. 8, the variations of water heads at points C, D, E, and F with respect to time are shown. The water heads at points C and D reach their maxima as soon as the load p reaches its maximum, and the water heads at points E and F which are farther from the axis of symmetry reach their maxima at later times.

Fig. 8. Variation of water head in axi-symmetric consolidation

In Figs. 9 and 10, the distributions of displacement and water head are shown. The water head at $t = 0.75 \times 10^8$ sec is so small that the diagram of water head distribution is not shown.

Fig. 9. Displacement of soil

Fig. 9. Displacement of soil

Fig. 9. Displacement of soil

Fig. 10. Distribution of water head

Fig. 10. Distribution of water head

CONCLUSIONS

The numerical method developed in this paper is of practical value to complex problems in Biot's consolidation theory, i.e., the problems dealing with nonhomogeneous, anisotropic soil of arbitrary geometry in one-, two-, or three-dimension.

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NOTATION

- E_{ij} = reduced elasticity tensor with respect to effective stress
- E_{ijkl} = elasticity tensor with respect to effective stress
- F_i = body force vector of soil saturated by water
- S = boundary surface of region occupied by soil
- S_h = subsurface on which water head is prescribed
- S_T = subsurface on which surface traction is prescribed
- S_u = subsurface on which displacement is prescribed
- S_v = subsurface on which normal component of stream velocity of water relative to soil is prescribed
- S^n = boundary surface of the n -th element
- S_h^n = portion of S^n belonging to S_h
- S_T^n = portion of S^n belonging to S_T
- S_u^n = portion of S^n belonging to S_u
- S_v^n = portion of S^n belonging to S_v
- S_I^n = interelement surface of the n -th element
- $\bar{T}_i$ = prescribed surface traction
- V = volume of region occupied by soil
- V^n = volume of the n -th element
- $[a^n(x)]$ = shape function for displacement of the n -th element
- $\{b_u(x)\}$ = shape function for water head of the n -th element
- h = water head
- $\bar{h}$ = prescribed water head
- k_{ij} = permeability coefficient tensor
- n_i = outward unit normal vector on boundary surface of an element
- t = time
- u_i = displacement vector
- $\bar{u}_i$ = prescribed displacement vector
- v_i = stream velocity of water relative to soil
- $\bar{v}$ = prescribed normal component of stream velocity of water relative to soil

- $\mathbf{x}$ = spatial coordinate vector
 $[\Phi^n]$ = transformation matrix for displacement of the n -th element
 $[\Psi^n]$ = transformation matrix for water head of the n -th element
 $\{\alpha(t)\}$ = set of displacement vectors and water heads at all nodal points belonging to the entire region
 $\{\alpha_1(t)\}$ = unknown part of $\{\alpha(t)\}$
 $\{\alpha_2(t)\}$ = known part of $\{\alpha(t)\}$
 γ_w = unit weight of water
 ε_i = reduced strain tensor of soil
 ε_{ij} = strain tensor of soil
 σ_i = reduced effective stress tensor
 σ_{ij} = effective stress tensor
 $\{\phi^n(t)\}$ = set of displacement vector at all nodal points belonging to the n -th element
 $\{\phi^n(t)\}$ = set of water heads at all nodal points belonging to the n -th element

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APPENDIX

After contribution of this paper, the authors found the paper by Sandhu and Wilson (1969) which dealt with Biot's consolidation theory by almost the same procedure.

The minor imperfections, though not essential, of the paper by Sandhu and Wilson, which are corrected in the present paper, are as follows. Notations defined in the present paper are employed.

1. The continuity and differentiability requirement of admissible functions u_i and h in functional II was not indicated distinctly.
2. It was not indicated distinctly that u_i and h in II must satisfy boundary conditions.
3. The paper dealt with only plane-strain finite element method.
4. It was not indicated distinctly that the shape functions for displacement and water head must be such that displacement and water head are continuous across the interelement boundary.
5. In the finite element formulation, the nodal point displacement vector and the nodal point water head (which are written as $\{\alpha(t)\}$ in the present paper collectively) must be divided into unknown and known parts (which are $\{\alpha_1(t)\}$ and $\{\alpha_2(t)\}$ in the present paper) and variation of the known part must be zero, not arbitrary ($\{\delta\alpha_1(t)\}$ is arbitrary but $\{\delta\alpha_2(t)\}$ must be zero in the present paper), which was not performed.

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