A STATISTICAL APPROACH TO THE MECHANICS OF GRANULAR MATERIALS

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INTRODUCTION

In the mechanics of granular materials, the calculations are usually made on the assumption that the Coulomb's frictional law is valid. However, when we have to concern with the transient phenomena for example, the formation of sliding surface, phenomena which occur during the shearing test of sand and so on, the traditional technique is powerless. Recently research on such materials from the other standpoints other than the traditional one has been performed. Statistical attack on the deformation of sand mass has years been made by Professor Litwiniszyn, but his studies concerns mainly with kinematics of the materials. Professor Rowe and others tried to construct an explanation for the shearing resistance of granular materials with the aid of model of regularly placed pattern of equal balls and he extended his results to much more complicated general cases.

A statistical approach to this problem was tried by the present author in 1949, however it contained imperfections so that he attempted to improve his theory from time to time but failed.

Recently he had an opportunity of having correspondence with Professor Litwiniszyn and also an opportunity of reading an above cited paper by Professor Rowe. These two chances were the exciters for the present author and he tackled again to his own problem.

The void ratio is recognized important when the mechanical properties of granular materials are the problem. However, it can easily be shown that the void ratio alone cannot be sufficient to describe the property.

Fig. 1. Schematic views of assemblies of granular material having equal void ratio, but not having different distribution of void ratio.

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For example, in the Fig. 1 a) and b), two boxes are shown in which equal number of grains are contained, but grain distribution is different. It might be a matter of common sense that these two assemblages of grains do not behave in the same way, even though they have the same void ratio.

This thinking experiment tells us that the distribution of void ratio must be added as another factor, the second important factor in the discussion of mechanical property of granular materials.

DESCRIPTION OF THE DISTRIBUTION OF VOID IN THE MATERIAL

The Fig. 2 shows an element of a volume of granular material which surrounds a void. If we imagine a surface through the centers of grains, this surface contains the void and a half of total volume of grains which are located around it. The void ratio of this volume element is V_v/V_g in which V_v signifies the volume of void and V_g is the total volume of the part of grains contained in this surface. In such a way total volume of the assemblage is separated into many elements which contain voids in them.

Fig. 2. The Volume Element

If we classify such elements according to their void ratios and the volumes of elements which have the same void ratio are summed up, we can get the following table.

Table 1.

$e_1, e_2, e_3, \dots, e_n$
$V_1, V_2, V_3, \dots, V_n$
$N_1, N_2, N_3, \dots, N_n$

In the above table, $e_1, e_2, \dots, e_n$ are void ratios arranged in ascending order, $V_i (i = 1, 2, \dots, n)$ are the total volume of elements which have the void ratio of e_i and $N_i (i = 1, 2, \dots, n)$ are the total volume of grains which are contained in elements which have the void ratio e_i calculated in terms of the number of grains.

The table shows the distribution of grains in the total volume. Any element which is expressed by the members in a column in the above table is separated from other elements which correspond to other columns and elements which are shown by all columns cover the total volume, hence

$$\left. \begin{aligned} N_1 + N_2 + \dots + N_n &= N \\ V_1 + V_2 + \dots + V_n &= V \\ N_i v_0(1 + e_i) &= V_i, \quad (i = 1, 2, \dots, n) \end{aligned} \right\} \dots \dots \dots (1)$$

where V is the total volume, N the total number of grains in the volume V , v_0 is the volume of each grain, assumed equal.

PROBABILITY OF THE STATE

The actual state of grains in the volume V is the one of all possible states which can be shown by the Table 1. We can put N grains of equal volume v_0 into the volume V and the distribution of void ratio can be adjusted to that shown in the table. As shown in the following, the number of ways of getting such configuration can be calculated.

At first, we calculate the number of ways of separation of the total volume V into volumes $V_1, V_2, \dots, V_n$. If we choose a sufficiently small volume v_u as an unit and $V_1, V_2, \dots, V_n, V$ are divided by this unit, these volumes are represented by integral numbers approximately, the degree of approximation can be raised up to the order as high as one wants if the unit volume v_u is sufficiently small.

We suppose that V, V_i themselves are integral values obtained in such a way for the sake of simplicity and these values be represented by marks along a straight line. We draw lines at both ends and between V_i and V_{i+1} ($i = 1, 2, \dots, n$) the marks and so on, a line is separated into n segments which contain T_i marks in the i -th segment.

Fig. 3. Illustration of a Mode of Distribution of Volume Elements

Hence the number of ways of separation of total volume into n elemental volume is the number of ways of separation of V marks by drawing $n-1$ bars between end marks.

So that the wanted number of separation of volume is

$${}_{v+n+1-2}C_n = \frac{(V+n-1)!}{n!(V-1)!} \dots\dots\dots(2)$$

Secondly the number of ways of putting N_i grains into the volume v_i is calculated as

$$\frac{V_i}{v_0} C_{N_i} = \frac{\{(1+e_i)N_i\}!}{N_i!(e_i N_i)!} \dots\dots\dots(3)$$

Finally the number of ways of getting the configuration shown in the Table 1 is obtained as

$$Z = \frac{\{(1+e)N+n-1\}!}{n!\{(1+e)N-1\}!} \prod_{i=1}^n \frac{\{(1+e_i)N_i\}!}{N_i!(e_i N_i)!} \dots\dots\dots(4)$$

For convenience of calculation, we put

$$\bar{e} = \frac{1}{n} \sum e_i \dots\dots\dots(5)$$

Then

$$e_i = \bar{e} + \epsilon_i, \quad \sum \epsilon_i = 0 \quad \dots\dots\dots(6)$$

And we put

$$\frac{1}{n} \sum \epsilon_i^2 = \delta, \quad \frac{1}{N} \sum N_i \epsilon_i^2 = \sigma \quad \dots\dots\dots(7)$$

In addition to them, we assume that

$$\sum \epsilon_i^{2k} = 0, \quad \sum N_i \epsilon_i^{2k-1} = 0 \quad (k \geq 2) \quad \dots\dots\dots(8)$$

Applying the Stirling's formula

$$m! \doteq \sqrt{2\pi} m^{m+\frac{1}{2}} e^{-m} \quad \dots\dots\dots(9)$$

to the right hand terms of the expression (4), and after some calculations with the conditions (8) we can get the probability W of the state of the granular materials as

$$\begin{aligned} \log W &= \log \frac{1}{Z} \\ &= (n+1) \log \sqrt{2\pi} - n \log N + \left(n + \frac{1}{2}\right) \log n - \left(N + n - \frac{1}{2} + eN\right) \log \left(1 + \frac{n-1}{N} + e\right) \\ &\quad + \left(N - \frac{1}{2} + eN\right) \log \left(1 + e - \frac{1}{N}\right) - N \log (1 + \bar{e}) - \left(Ne + \frac{n}{2}\right) \log \frac{1 + \bar{e}}{\bar{e}} \\ &\quad - \frac{1}{4} \left\{ \frac{1}{\bar{e}^2} - \frac{1}{(1 + \bar{e})^2} \right\} n \delta + \frac{1}{2} N (\sigma - \bar{e}^2) \left(\frac{1}{\bar{e}} - \frac{1}{1 + \bar{e}} \right) - \frac{N(e - \bar{e})}{1 + \bar{e}} + \frac{1}{2} \sum_{i=1} \log N_i \\ &= A + \frac{1}{2} \sum \log N_i \quad \dots\dots\dots(10) \end{aligned}$$

This probability depends on the distribution N_i , as stated before the actual configuration would be such that this probability be the maximum.

When we put

$$N_i = \frac{N}{n} + \alpha_i \quad \dots\dots\dots(11)$$

the α_i must satisfy the condition,

$$\sum \alpha_i = 0 \quad \dots\dots\dots(12)$$

because of the condition $\sum N_i = N$. Putting (11) into (10)

$$\log W = A + \frac{1}{2} n \log \frac{N}{n} - \frac{1}{4} \frac{n^2}{N^2} \sum \alpha_i^2 \quad \dots\dots\dots(13)$$

so that the probability W is maximum when the distribution of N_i is given as

$$N_i = \frac{N}{n} \quad \dots\dots\dots(14)$$

And in this case

$$\left. \begin{aligned} Ne &= \sum N_i e_i = \frac{N}{n} \sum e_i = N\bar{e} & \therefore e &= \bar{e} \\ N\sigma &= \sum N_i e_i^2 = \frac{N}{n} \sum e_i^2 = N\bar{e}^2 + N\delta & \therefore \delta &= \sigma - \bar{e}^2 = \sigma - e^2 \end{aligned} \right\} \dots\dots\dots(15)$$

putting these values in (10), we can get finally the probability of the state of the granular material defined by the Table 1 is

$$\begin{aligned} \log W &= (n+1) \log \sqrt{2\pi} - \frac{n}{2} \log N + \frac{1}{2}(n+1) \log n - \left(N + eN + \frac{n}{2}\right) \log (1+e) \\ &+ \left(eN + \frac{n}{2}\right) \log e - \left(N + eN + n - \frac{1}{2}\right) \log \left(1 + \frac{n-1}{N} + e\right) + \left(N + eN - \frac{1}{2}\right) \\ &\times \log \left(1 - \frac{1}{N} + e\right) + \frac{1}{2} \left(\frac{1}{e} - \frac{1}{e+1}\right) \left\{ N - \frac{n}{2} \left(\frac{1}{e} + \frac{1}{1+e}\right) \right\} \delta \dots\dots\dots(16) \end{aligned}$$

MEANING OF THE VOID RATIO e AND THE QUANTITY δ

The void ratios in the material arranged in ascending order as shown in the Table 1 are $e_1, e_2, \dots, e_n$, the i -th void ratio e_i can be expressed as

$$e_i = e_1 + \frac{i-1}{n-1}(e_n - e_1) \dots\dots\dots(17)$$

with the expressions (14) and (15), we have

$$\left. \begin{aligned} Ne &= \sum N_i e_i = \frac{N}{n} \sum \left[e_1 + \frac{i-1}{n-1}(e_n - e_1) \right] & \therefore e &= \frac{e_n + e_1}{2} \\ N\sigma &= \sum N_i e_i^2 = \frac{N}{n} \sum \left[e_1 + \frac{i-1}{n-1}(e_n - e_1) \right]^2 & \therefore \delta &= \sigma - e^2 = \frac{1}{12}(e_n - e_1)^2 \frac{n+1}{n-1} \end{aligned} \right\} \dots\dots(18)$$

As shown by the expression (16), the probability of the state is determined by e , the void ratio as a whole, and δ , the deviation of void ratios from its mean, and these values are expressed by the largest and smallest void ratios in the volume. The former, e , is the mean, and the latter, δ , shows the width of the distribution of the void ratio.

EXPERIMENTAL EVIDENCES, ABRIDGED

Steel balls of equal size are placed in one layer randomly in a flat box, the bottom of the box is consisted of a glass plate, under which several photo cells are placed in order to measure the amount of light which is emitted from electric lamps above the box. The light is obstructed by steel balls, so that the amount of light received by the photo cells indicates the voids in the mass of steel balls. The more the steel balls is crowded, the less the light is received.

One side wall of the box is pushed in, the two-dimensional uniaxial compression on

Fig. 4. Relation between the average void ratio $\bar{e}$, the deviation δ of void ratio and the displacement of the wall which is pushed in, started from the looser state

Fig. 5. Relation between the average void ratio $\bar{e}$, the deviation δ of void ratio and the displacement of the wall which is pushed in, started from the denser state

this assemblage of steel balls is performed.

The box is placed horizontally to avoid the influence of gravitational force. When the wall is pushed in, the steel balls near the wall start to move and gather, while those at a remote place are kept indifferent.

As the displacement of the wall proceeded, motion of balls is propagated toward the opposite side of the wall.

The mean void ratio and the deviation obtained from measurements by photo cells placed at several places are shown in the Figures 4 and 5.

The deviation of voids from the mean increases with the displacement of the wall and the second stage comes in which the deviation reaches its maximum, after that it gradually decreases, that is, the distribution of voids tends to uniform.

When the side walls, which are parallel to the displacement of the wall and are prevented from outward movement by the friction due to their own weight start to move outward when the displacement of the pushed in wall reaches to such an extent as the distribution of void ratio is uniform, and when the outward movements of side walls begin, deviation of void ratio sometimes increases or fluctuates.

These experimental evidences indicate that the void ratio as a whole concerns mainly with the volume change and the deviation of voids relates to the propagation of motion of balls or the shearing deformation.

RELATIONSHIP BETWEEN PRINCIPAL STRESSES WHEN THE SHEARING DEFORMATION OCCURS IN GRANULAR MATERIAL

The experimental evidences suggest that the void ratio becomes the least, and its distribution becomes uniform just before the shearing deformation starts. At this stage, the number n in the Table 1 can be considered very small, so that the probability of the state can be written in a simpler form like the following, neglecting smaller terms,

$$\begin{aligned}\log W &= -\log N + eN \log e - N(1+e) \log (1+e) + \frac{N}{2} \frac{\delta}{e(1+e)} \\ &= A + eN \log e - N(1+e) \log (1+e) + \frac{N}{2} \frac{\delta}{e(1+e)} \dots\dots\dots(19)\end{aligned}$$

where A is a constant which does not depend on e or δ .

Consider a rectangular plate of thickness 1 and its sides are a and b . This plate is subjected to normal stresses as shown in the figure, the sign of which is positive for tension.

The strains along these sides are ϵ_1 and ϵ_2 the signs of which are positive for elongation.

The work done on this plate, per unit volume is

$$\frac{\sigma_1 b(-\epsilon_1 a) - \sigma_2 a(\epsilon_2 b)}{ab} = -\sigma_1 \epsilon_1 - \sigma_2 \epsilon_2 \dots\dots\dots(20)$$

The volume dilatation is given by

Fig. 6.

$$\frac{\Delta V}{V} = \frac{\Delta e}{1+e} = \epsilon_1 + \epsilon_2 \dots\dots\dots(21)$$

If we put

$$\left. \begin{aligned} \sigma_1 &= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} = p + \tau \\ \sigma_2 &= \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2} = p - \tau \\ p &= \frac{\sigma_1 + \sigma_2}{2}, \quad \tau = \frac{\sigma_1 - \sigma_2}{2} \end{aligned} \right\} \dots\dots\dots(22)$$

where

As $\epsilon_1 - \epsilon_2$ is the maximum shearing strain, so that we write $\epsilon_1 - \epsilon_2 = -\Delta\gamma$, then the expression (20) can be written as

$$-\sigma_1 \epsilon_1 - \sigma_2 \epsilon_2 = -p \frac{\Delta e}{1+e} + \tau \Delta\gamma \dots\dots\dots(23)$$

Now, we have to get the $\log W$ in the expression (19), in term of e, γ instead of e and δ . Considering that $-\sigma_1 \epsilon_1 - \sigma_2 \epsilon_2$ is to be derived from $\Delta \log W$ by aid of thermodynamical principle, hence the mean pressure and the shearing stress must be connected with $\log W$ in the similar manner, so we can put

$$\log W = F(e - k\gamma) \dots\dots\dots(24)$$

the form of the function F will shortly be determined by the condition that $\gamma = 0$ for $\delta = 0$.

This condition comes from the consideration that the shearing strain is closely related to the deviation of void ratio.

Applying (19) into (24), we have

$$-\log N + eN \log e - N(1+e) \log (1+e) + \frac{N}{2} \frac{\delta}{e(1+e)} = F(e-k\gamma) \dots\dots\dots(25)$$

Putting $\gamma = 0$, and $\delta = 0$

$$-\log N + eN \log e - N(1+e) \log (1+e) = F(e)$$

$$\begin{aligned} \therefore \log W &= F(e-k\gamma) \\ &= -\log N + N(e-k\gamma) \log (e-k\gamma) - N(1+e-k\gamma) \log (1+e-k\gamma) \dots\dots\dots(26) \end{aligned}$$

If we put

$$\begin{aligned} S &= \frac{K}{N} \log W \\ &= K \left[-\frac{\log N}{N} + (e-k\gamma) \log (e-k\gamma) - (1+e-k\gamma) \log (1+e-k\gamma) \right], \dots\dots\dots(27) \end{aligned}$$

S is a quantity which has the similar character, as the entropy in thermodynamics.

We get from (23) or (22)

$$\begin{aligned} -p \frac{\Delta e}{1+e} + \tau \Delta \gamma &= \Delta S \\ &= K [\{\log (e-k\gamma) - \log (1+e-k\gamma)\} \Delta e + k \{\log (1+e-k\gamma) - \log (e-k\gamma)\} \Delta \gamma] \dots\dots\dots(28) \end{aligned}$$

then we get

$$\left. \begin{aligned} p &= K(1+e) \log \frac{1+e-k\gamma}{e-k\gamma} \\ \tau &= Kk \log \frac{1+e-k\gamma}{e-k\gamma} \end{aligned} \right\} \dots\dots\dots(29)$$

$$\therefore \left. \begin{aligned} \sigma_1 &= \frac{K}{2} \{(1+e)+k\} \log \frac{1+e-k\gamma}{e-k\gamma} \\ \sigma_2 &= \frac{K}{2} \{(1+e)-k\} \log \frac{1+e-k\gamma}{e-k\gamma} \end{aligned} \right\} \dots\dots\dots(30)$$

From the expression (30), we can get the relationship between two principal stresses,

$$\frac{\sigma_2}{\sigma_1} = \frac{1 - \frac{k}{1+e}}{1 + \frac{k}{1+e}} \dots\dots\dots(31)$$

The relationship between the deviation of void ratio δ and the void ratio e as a whole is from (25) or (27) as

$$\delta = 2e(1+e) \left[e \log \left(1 - \frac{k\gamma}{e} \right) - (1+e) \log \left(1 - \frac{k\gamma}{1+e} \right) + k\gamma \log \left(1 + \frac{1}{e-k\gamma} \right) \right] \dots\dots(32)$$

When γ is very small, we can write

$$\delta = 2k\gamma e(1+e) \log \frac{1+e}{e} \dots\dots\dots(33)$$

Hence the relationship (33), strictly speaking (32), should be satisfied, to get the expression (31). This relationship (32) might be difficult to be confirmed by experiment, however, its validity can be known by intuition and some simple calculations.

The expression (31) shows that the less is the void ratio, the larger is the angle of internal friction and that the deviation of void ratio increases with shearing strain and its rate is larger when the void ratio is large. This does not contradict to what intuition teaches.

If the expression (31) is compared with the well known formula in soil mechanics, we get

$$\frac{k}{1+e} = \sin \phi \dots\dots\dots(34)$$

where ϕ is the angle of internal friction.

As the value of e may be between 0.2–0.7 probably about 0.3 and ϕ is about 30° , k may be about 0.6–1.5, so that the value of δ , given by (34) is about 0.01–0.07 for γ of 0° – 3° .

These values do not seem unreasonable.

ANGLE OF INTERNAL FRICTION

If the relationship (34) between the angle of internal friction and the void ratio is valid, it can be said that the angle of internal friction is a measure of crowdedness of grains in a material. If grains are crowded in a body, the movement of grains is prevented by their existence, mutual interference, and the coefficient k in the expression (34) is defined by (32) or (33) as a measure of production of deviation of void ratio due to shearing deformation and hence the expression (34) tells us that the angle of internal friction has such a character that it gives an indication of degree of easiness for making disturbance in the distribution of void ratio by shearing deformation.

As stated before, the deviation of void is produced also in propagation of motion even though it does not accompany the shearing motion. Anyway both the shearing deformation and the propagating motion are transient phenomena, the former leads the material to failure, the latter transmits phase in material from one place to another.

It might be imagined that the most stable state of the material would be that in which the factors which signify its state are uniformly distributed, hence the transient state would be a disturbed state between stable ones.

By such a consideration, it seems natural that the angle of internal friction is closely related to void ratio and the coefficient k .

Of course, it would have relation to the true frictional angle between grains, physical frictional angle, the author thinks that this fact is expressed implicitly by the coefficient k , though its explicit expression cannot be obtained now.

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