ON THE FAILURE OF THE GRANULAR MATERIAL

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SYNOPSIS

Notwithstanding the statistical theory of mechanics of granular material proposed by the senior author, was established basing on experimental evidences, still it contained several uncertain assumptions. By the bi-axial compression tests on assemblage of equal balls, almost all of assumptions were ascertained. Observing the failure phenomena, some interesting and suggestive findings were added. When the assemblage is compressed, loosely packed assemblage grows into denser one until its deviation of void ratio s approaches to a certain value s_0 before failure. The value k which was proposed before as a parameter connecting void ratio, deviation of void ratio and shearing strain was found to be almost constant during failure independently of the initial arrangement of balls, whereas the above explained s_0 depends thereon. The expressions connecting void ratio and angle of internal friction obtained before are still valid even if non-zero constant s_0 was introduced so as to fit in with experimental evidences.

INTRODUCTION

A statistical theory of mechanics of granular material was proposed by the senior author basing on experiments on an assemblage of steel balls of equal size (Mogami, 1965 a, b). At first the theory was presented for the material composed of particles of equal size but it was later extended to the material composed of particles of various size (Mogami, 1967).

The formulae connecting the void ratio and the angle of internal friction derived by the theory were examined by the shearing test data given by several authors (Mogami, 1966 a). It was somewhat curious that the results derived by the theory for the material composed of particles of equal size could satisfactorily explain the results obtained by actual shearing tests on sands composed of grains of various sizes.

Notwithstanding the present theory contains some hypotheses which cannot be verified directly, the theory could favourably explain the experimental results as far as the relationship between the void ratio and the angle of internal friction is concerned. However, it is needless to say that further studies, either experimental or theoretical,

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will be helpful to confirm the limit of validity of the present theory. If the theory is confirmed by further studies, the certainty of the assumptions or hypotheses which could not be verified directly, will more and more be established and this would increase an understanding of the nature of the granular materials. To such purposes, further experiments were performed, which were bi-axial compression tests on an assemblage of steel balls of equal size, and similar to those explained in previous papers (Mogami, 1965 a, b).

In conclusion, the tests were not only helpful to strengthen the basis of the present theory but also they furnished a deeper understanding of failure phenomenon of the material.

EXPERIMENTS

A flat box composed of four side walls with bottom glass plate was used. The box

Fig. 1. Equipment of experiment

Fig. 2. Equipment of experiment

Fig. 3. Cross section of box

is shown in Figs. 1 and 2. Side walls a and b can be moved by two screws in the opposite directions simultaneously, whereas side walls c and d are rested still on the bottom glass plate until the horizontal forces exerted by particles exceed the resisting forces caused by friction due to the weight of walls.

Steel balls of equal size, 2.388 mm in diameter, are placed in the box. Three types of initial arrangement of balls are adopted. The type A is the densest packing of balls. The type C is a random packing of balls, in which almost all balls have gaps between surrounding balls, but this arrangement is denser than the packing which had been adopted in the experiments explained in the previous papers (Mogami, 1965 a, b). The type B has parts of densest packing bounded on balls of looser packing.

A glass plate is placed above the balls to prevent the upheaval of balls when the walls a and b are pushed in. After balls are placed in the box, the walls a and b are pushed in and void ratios at various places are measured by 31 photo-cells placed beneath the bottom plate of the box, at each stage of displacement of walls (Fig. 3). As the arrangement of balls grows denser, forces exerted on walls c and d by balls increase and at a certain stage these forces exceed the frictional resisting forces of walls, then the walls begin to move outwards. Displacements of walls a and b and of c and d are measured. Such bi-axial compression tests of an assemblage of balls are carried out 10 times for every type of initial packing. For each type of packing all data obtained are collected altogether, and the mean void ratio e and the deviation of void ratio s defined by the following expressions are calculated.

$$e = \sum_i n_i \frac{e_i + e_{i+1}}{2} \dots\dots\dots (1)$$

$$s = \frac{1}{\sum_i n_i} \sum_i \left[e - \frac{e_i + e_{i+1}}{2} \right]^2 n_i \dots\dots\dots (2)$$

where

n_i : the number of values of void ratio which fall in the interval e_i and e_{i+1}

e_i : the i -th void ratio when the total range of void ratio is divided into sub-intervals of equal width $\Delta e (= 0.008)$.

In another word $e_{i+1} - e_i = \Delta e = 0.008$. Actually $\sum_i n_i = 280$.

Increment of shearing strain $\Delta\gamma$ at each stage of displacement can be calculated by the expression.*

$$\Delta\gamma = -\frac{\Delta l_1}{l_1} + \frac{\Delta l_2}{l_2} = -\epsilon_1 + \epsilon_2 \dots\dots\dots (3)$$

where l_1 and l_2 are distances between walls a and b and walls c and d respectively at this stage and Δl_1 , Δl_2 are their increments. Intergrating equation (3), with the initial condition that $l_1 = l_{10}$, $l_2 = l_{20}$ when $\gamma = 0$, we get,

* Exactly speaking $\Delta\gamma$ is twice of the increment of shearing strain

Fig. 4. Relationship between strain-ratio and shearing strain

Fig. 5. Relationship between shearing strain and axial compression

Fig. 6. Relationship between shearing strain increment and axial compression

$$\gamma = \log \left\{ \left(\frac{l_2}{l_1} \right) / \left(\frac{l_{20}}{l_{10}} \right) \right\} \dots\dots\dots (4)$$

When the assemblage of balls is compressed, at first it is compressed uniaxially so that the shearing deformation would be zero. But in calculating shearing strain in this case, the shearing strain γ was obtained by the equation (4), hence even when the lateral displacements of walls c and d were negligibly small, the shearing strain was

calculated as finite one. In Fig. 4 the value $|\epsilon_2/\epsilon_1|$ is plotted against γ calculated by the equation (4). As shown in the figure, $|\epsilon_2/\epsilon_1|$ in type A is large at the initial stages whereas $|\epsilon_2/\epsilon_1|$ in type C is small in these stages. This means that in type A the shearing deformation develops from the beginning, but in type C the deformation is rather compressive at the beginning stages.

RESULTS OF EXPERIMENTS

a) *Shearing strain and displacements of walls a and b*

Let δ designate the decrease of distance between these walls, that is, the axial compression of the assemblage. In Fig. 5 the shearing strain calculated by the equation (4) is plotted against the axial compression δ . The difference in dependence of the $\delta-\gamma$ relationship on the initial arrangement of balls will be better shown in Fig. 6, where the increment of shearing strain is plotted against the axial compression. It will be seen that at larger axial compression, the increment of shearing strain is almost constant and independent of the type of initial arrangement of balls.

b) *Relationship between mean void ratio and shearing strain*

The relationship between mean void ratio and shearing strain is shown in Fig. 7. In the case of type A, the void ratio increases initially with the shearing strain, which corresponds to the property of dilatancy of granular material, but for larger shearing strain the void ratio remains almost constant. For types B and C, the trend is a little different. For type B, the void ratio remains almost constant in the whole range of shearing strain, whereas for type C the void ratio initially decreases but approaches to the value for type B when the shearing strain is larger.

c) *Relationship between deviation of void ratio and shearing strain*

The relationship between the deviation of void ratio and shearing strains is shown in Fig. 8. In case of the experiments explained in the previous papers (Mogami, 1965

Fig. 7. Relationship between mean void ratio and shearing strain

Fig. 8. Relationship between deviation of void ratio and shearing strain

a, b), the arrangement of balls was rather like that of type C but looser than that of type C, hence as written in the previous papers the arrangement of balls grew into the type A as the compression proceeded. In the previous papers it was also shown that the deviation of void ratio increased at the first stages until it reached maximum and decreased to zero when the initial arrangement of balls converted into the type A and it increased again. The last trend is just the same as shown by the curve A in Fig. 8. Contrary to the former cases, the deviations of void ratio in cases of type B and type C are not zero at the first stages and do not fall to zero, within the range of shearing strain in the present experiments. In the course of establishment of the present theory of mechanics of granular material, the initial condition that the deviation of void ratio s is zero when the shearing strain γ is zero,* was assumed, because the former experiments were performed on such assemblage of balls as stated above. In the present experiments it was found that this condition is not always satisfied. Hence some correction should be made to the present theory in this regard. This will be shown later.

CHANGE IN CONFIGURATION OF BALLS DURING COMPRESSION

The changes in configuration of balls during compression are shown in Figs. 9, 10, and 11.

a) Type A (Fig. 9)

From the first step to the fifth step, faint fissures are observed, and at the fifth to

* It would be needless to say that the shearing strain was measured from the instant of start of failure, hence at this moment γ is zero.

Fig. 9. Change in configuration of balls during compression (Type A)

Fig. 9. Change in configuration of balls during compression (Type A)

Fig. 10. Change in configuration of balls during compression (Type B)

Fig. 10. Change in configuration of balls during compression (Type B)

Fig. 11. Change in configuration of balls during compression (Type C)

Fig. 11. Change in configuration of balls during compression (Type C)

sixth steps several distinct fissures with definite directions (sliding lines) appear.

b) Type B (Fig. 10)

At earlier stages, distinct sliding lines can be observed in several blocks, but they are not continued into their neighbouring blocks. The directions of these sliding lines seem to be randomly distributed, but the direction inclined by 30 degrees to the axis of compression is predominant and this tendency becomes more remarkable at later stages. As the compression of the assemblage proceeds, the smaller blocks are absorbed by neighbouring ones and grow into larger blocks and the whole assemblage becomes to be composed of coarser blocks than it was at the earlier stages.

c) Type C (Fig. 11)

At first stage, the assemblage is composed of independently located particles, but as the compression proceeds particles begin to flock together particularly at places near the moving walls *a* and *b* and they form block pattern as seen in the type B. This formation of blocks propagates to the central part of the assemblage and at sixth or seventh step, for example, the pattern of block is just like that seen in the type B. Since all phenomena observed after these stages are quite similar to those in the type B, it can be said that the type B is the continuation of the type C with respect to the behaviour of particles in compression. As mentioned earlier, the similar bi-axial compression of assemblage of balls in the previous experiment was performed on assemblage much more looser than the type C, hence at that time the change in pattern of the assemblage continued not to the type B but to the type A, in other words, the assemblage of blocks continued to convert to the extreme state in which the whole assemblage is composed of only one block, i.e. the type A.

CONTINUATION OF TYPES

As explained in the previous section, the type B would be a continuation of the type C. In another word, the state at the earlier stage of compression of assemblage of type B would be considered to be the state at the later stage of type C. For larger shearing strain the values of void ratio *e* and deviation of void ratio for types B and C are so close to each other (Figs. 7 and 8). If the compression would start from the state much looser than the type C as in the previous experiments (Mogami, 1966 a), the void ratio and deviation of void ratio for such an arrangement would be close to those of type A, because in such a case the configuration of balls reaches the initial state of the case of type A.

SLIDING LINES

If the sliding lines are defined as the distinct fissures observed in the assemblage, one side of them moves along this fissure relatively to the other side. In case of type A, faint fissures observed in the beginning several stages are not the sliding lines as defined above. As stated before in the type A, it can be said that shearing deformation begins to develop from the very early state of compression, but some compressive deformation would be needed to form the sliding lines. In case of type B, the sliding lines do not go through the whole material, but local distinct sliding lines appear. In

Fig. 12. Sliding lines developed in assemblage (Type D)

Table 1.*

	Type A		Type B		Type C	
	$s/\frac{1}{n} \sum \epsilon^3$	$s/\frac{1}{n} \sum \epsilon^4$	$s/\frac{1}{n} \sum \epsilon^3$	$s/\frac{1}{n} \sum \epsilon^4$	$s/\frac{1}{n} \sum \epsilon^3$	$s/\frac{1}{n} \sum \epsilon^4$
1	1530	65200	513	528	142	511
2	931	24500	991	675	899	414
3	683	22300	1620	559	236	385
4	380	10200	21200	605	191	388
5	454	11000	603	619	109000	536
6	183	44200	225	554	196	552
7	2080	3740	271	538	147	535
8	169	1760	2155	511	825	550
9	85	1210	804	533	341	282
10	68	805	394	493	232	575
11	99	1380	141	469	187	414
12	56	843	308	548	217	508
13	87	1550	334	539	185	566
14	111	1450	422	561	436	509
15	89	1600	788	546	4840	560
16			445	559	1750	514
17			183	510	375	491

* Tabulated numbers are shown by their absolute values.

case of type C, similar sliding lines are developed in rather later stages than in type B. The sliding lines are not stationary in every type of initial packing, i.e. some sliding lines happen to vanish and new sliding lines happen to grow up. This situation can be seen in Figs. 9, 10 and 11.

The direction of sliding lines seems to depend on the geometry of the arrangement of balls. In type A, sliding lines incline to the axis of compression by 30 degrees, because such sliding lines are easier to develop. When the arrangement of balls is of the densest packing like the type A but the weakest direction is not same as that in the type A, the sliding lines incline to the axis of compression by 60 degrees instead of 30 degrees as shown in Fig. 12.

In this case the force needed to make sliding seems to be larger than that in case of type A. As for the direction of sliding lines in cases of types B and C explanation was already given in the section 4.

DISTRIBUTION OF VOID RATIO IN THE ASSEMBLAGE

Moments of higher order of the distribution of void ratio were calculated with the data obtained by the present experiments. The results are presented in Table 1 in which the ratios of moments of the second order to moments of higher order are tabulated.

It can be said that the moments of higher order than two are negligibly small compared with the moment of the second order.

In addition to above, the distribution of number of balls allocated to the sub-intervals of void ratio was examined.

Referring to the equation (16) in the previous paper (Mogami, 1965 b), the following calculations are made,

Table 2.

Type A				Type B			
	e	s	$\frac{1}{n} \sum \alpha_i^2 / \bar{N}$		e	s	$\frac{1}{n} \sum \alpha_i^2 / \bar{N}$
1	0.120	0.998×10^{-4}	0.692×10^{-4}	8	0.178	7.76×10^{-4}	1.02×10^{-3}
7	0.147	0.874×10^{-4}	0.843×10^{-4}				
14	0.137	2.20×10^{-4}	1.90×10^{-4}				

Type C			
	e	s	$\frac{1}{n} \sum \alpha_i^2 / \bar{N}$
1	0.255	6.18×10^{-4}	1.37×10^{-3}
3	0.236	8.65×10^{-4}	1.72×10^{-3}
6	0.210	5.41×10^{-4}	0.91×10^{-3}
13	0.189	6.85×10^{-4}	0.98×10^{-3}

$$N = \sum_i n_i N_i$$

$$N_i = \frac{N}{n} + \alpha_i, \quad n = \sum_i n_i, \quad \bar{N} = \frac{N}{n} = \frac{67.37}{1+e}, \quad \frac{1}{n} \sum \alpha_i^2 = \frac{67.37e^2}{(1+e)^4} s$$

Hence we have

$$\frac{1}{n} \sum \alpha_i^2 / \bar{N} = \frac{67.37e^2}{(1+e)^3} s \quad \dots\dots\dots(5)$$

This last term of the equation (5) is calculated with the data obtained by the experiments and the results are shown in Table 2.

From Table 2, it can be concluded that the hypothesis adopted in the present theory is reasonable. The hypothesis was that the actual distribution of N_i is such that makes the probability of the state stationary.

VALUE OF k IN THE TRANSFORMATION $\log W = F(e - k\gamma)$

In the previous paper (Mogami, 1965 b), after getting the concrete expression of $\log W$, it was wanted to convert variables e and s to e and γ , because notwithstanding the deviation of void ratio s is very important theoretically, but it is not so convenient from the practical point of view. Hence, putting $\log W = F(e - k\gamma)$, the functional form of F was determined by the condition that $\gamma = 0$ for $s = 0$ which was found by experiments.

As stated before, in the present experiments the condition shown above is not satisfied generally but the condition $\gamma = 0$ for $s = s_0$ is the general condition to be satisfied. Applying this condition to the equation (30) in the previous paper (Mogami, 1965 b) we get.

$$-3 \log \sqrt{2\pi} + \log N - \frac{3}{2} \log 2 - Ne \log e + N(1+e) \log (1+e) - \frac{N}{2} \frac{s_0}{e(1+e)} = F(e) \dots\dots(6)$$

$$\begin{aligned} \log W &= F(e - k\gamma) \\ &= -3 \log \sqrt{2\pi} + \log N - \frac{3}{2} \log 2 - N(e - k\gamma) \log (e - k\gamma) \\ &\quad + N(1+e - k\gamma) \log (1+e - k\gamma) - \frac{N}{2} \frac{s_0}{(e - k\gamma)(1+e - k\gamma)} \dots\dots\dots(7) \end{aligned}$$

$$\begin{aligned} S &= \frac{K}{N} \log W \\ &= K \left[-\frac{3 \log \sqrt{2\pi} + \log N - \frac{3}{2} \log 2}{N} - (e - k\gamma) \log (e - k\gamma) \right. \\ &\quad \left. + (1+e - k\gamma) \log (1+e - k\gamma) - \frac{1}{2} \frac{s_0}{(e - k\gamma)(1+e - k\gamma)} \right] \dots\dots\dots(8) \end{aligned}$$

Hence we have

$$\begin{aligned}
 \log W &= A - Ne \log e + N(1+e) \log (1+e) - \frac{N}{2} \frac{s}{e(1+e)} \\
 &= A - N(e - k\gamma) \log (e - k\gamma) + N(1+e - k\gamma) \log (1+e - k\gamma) \\
 &\quad - \frac{N}{2} \frac{s_0}{(e - k\gamma)(1+e - k\gamma)} \dots\dots\dots (9)
 \end{aligned}$$

Then we have

$$\frac{1}{2e(1+e)} \left\{ s - \frac{s_0 e(1+e)}{(e - k\gamma)(1+e - k\gamma)} \right\} = e \log \frac{e - k\gamma}{e} - (1+e) \log \frac{1+e - k\gamma}{1+e} + k\gamma \log \frac{1+e - k\gamma}{e - k\gamma}$$

If the condition $k\gamma \ll e$ is satisfied, we have

$$s - s_0 = 2k\gamma e(1+e) \log \frac{1+e}{e} \dots\dots\dots (10)$$

This equation is reduced to the equation (38) in the previous paper (Mogami, 1965 b) if $s_0 = 0$. Inserting the observed values of s_0 , γ , e and s , the value of k can be calculated by the equation (10), which is plotted against ν as shown in Fig. 13.

Fig. 13. Relationship between value of k and shearing strain

The value of k is almost constant except when the shearing strain is small, and moreover the value is independent of the type which the assemblage had at its initial stage of compression. Referring to Fig. 13, an interpretation of the expression (10) can be made. The experimental fact that the value of k is independent of type of initial packing, means that this value is a characteristic constant of particles of which an assemblage is composed, whereas the value s_0 is dependent of the initial type of packing. Theoretically speaking an assemblage of densest packing cannot be compressed, whereas a looser assemblage can be compressed without shearing before failure. When the packing of balls is very loose the assemblage can be compressed without failure until it reaches the densest packing. On the other hand, when the initial packing was loose but not very loose, the assemblage cannot be compressed to such an extent

that the arrangement of balls reaches the densest. The difference in the final degree of packing to which the assemblage can be compressed without failure can be measured by the value s_0 . In case when the initial packing is densest, for example, the value s_0 is zero.

The transformation $\log W = F(e - k\gamma)$ was thought out basing on the consideration that there would exist a function whose gradient determines stresses in the material, that is, the plastic potential. The $\log W$ is one of such functions from the thermodynamical point of view, this was explained in the previous paper (Mogami, 1965 b). It is an intuition that the roll of the shearing strain ν in the function would be of the same rank with the deviation of void ratio, because the void ratio concerns with the volumetric change whereas the shearing strain with the change in shape, these two kinds of deformation of the material are the essential types of deformation of the deformable bodies.

However, the justification whether the relation $\log W = F(e - k\gamma)$ is correct or not should be given by comparison of the theoretically deduced characteristic values and corresponding ones obtained by experiments. The constancy of the value k obtained by experiments is one of the proofs which the authors want to have.

RELATIONSHIP BETWEEN VOID RATIO AND ANGLE OF INTERNAL FRICTION

By the present experiments, it was suggested that the initial condition, which is satisfied in general at the initial moment of failure, should be modified as $s = s_0$ for $\gamma = 0$ instead of $s = 0$ for $\gamma = 0$. If this modified initial condition is adopted, it is anticipated that some modification would be needed for the relationships between void ratio and angle of internal friction obtained in the previous papers (Mogami, 1966 b).

However, when calculations similar to those in the previous papers are conducted, it is found that no modification of the formulae stated above is needed. For example, in case $\sigma_3 = 0$, from the equation (8), we have

$$\left. \begin{aligned} \frac{\partial S}{\partial e} &= K \left[\log \frac{1+e-k\gamma}{e-k\gamma} + \frac{s_0}{2} \frac{1+2e-2k\gamma}{(e-k\gamma)^2(1+e-k\gamma)^2} \right] \\ \frac{\partial S}{\partial \gamma} &= -Kk \left[\log \frac{1+e-k\gamma}{e-k\gamma} + \frac{s_0}{2} \frac{1+2e-2k\gamma}{(e-k\gamma)^2(1+e-k\gamma)^2} \right] \end{aligned} \right\} \dots\dots\dots(11)$$

Hence we have,

$$\left. \begin{aligned} p = \frac{\sigma_1 + \sigma_2}{2} &= -K(1+e) \left[\log \frac{1+e-k\gamma}{e-k\gamma} + \frac{s_0}{2} \frac{1+2e-2k\gamma}{(e-k\gamma)^2(1+e-k\gamma)^2} \right] \\ \tau = \frac{\sigma_1 - \sigma_2}{2} &= -Kk \left[\log \frac{1+e-k\gamma}{e-k\gamma} + \frac{s_0}{2} \frac{1+2e-2k\gamma}{(e-k\gamma)^2(1+e-k\gamma)^2} \right] \end{aligned} \right\} \dots\dots\dots(12)$$

Therefore we have,

$$\sin \phi = \frac{k}{1+e} \dots\dots\dots(13)$$

This relationship is the same as the expression (39) in the previous paper (Mogami, 1965 a). Other similar relationships can be obtained in like manner.

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(Received : April 15, 1967)