ON THE CRITICAL STATE LINE PROPOSED BY ROSCOE ET AL.

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1. THE CRITICAL STATE LINE

To explain the change of state of soil during deformation, Roscoe et al. proposed a space representation of the state⁽¹⁾. Considering that the fundamental quantities to represent the state are the void ratio, the mean pressure and the shearing stress on slip surface, they adopted a three-dimensional space, taking the above three quantities as the three co-ordinate axes (Figure 1). A point in the space corresponds to a particular state of soil. When the state of soil changes under an action from outside, for example, the load, the representative point moves along a space curve. With experimental data, they found that this point in space approaches to a point lying on a space curve, called the critical void ratio line, when the soil is at failure. This line was renamed later as the critical state line. Investigators including Roscoe have developed their opinion based on this representation and many interesting results have been obtained.

It is emphasized here that their proposal of the critical state line is based on experimental evidences and was not deduced theoretically.

2. THE AUTHOR'S THEORY OF MECHANICS OF GRANULAR MATERIALS AND ITS RELATION TO THE CRITICAL STATE LINE

When the author's theory of mechanics of granular materials is applied to the failure problem in case of axial symmetry, it gives the relationships between void ratio, mean pressure and shearing stress at failure, they are,

$$\left. \begin{aligned} p &= \frac{\sigma_1 + 2\sigma_3}{2} = -K(1+e) \left\{ \log \frac{1+e}{e} + \frac{s_0}{2} \frac{1+2e}{e^2(1+e)^2} \right\} \quad (1.1) \\ \tau &= \frac{\sigma_1 - \sigma_3}{2} = -\frac{3}{2} Kk \left\{ \log \frac{1+e}{e} + \frac{s_0}{2} \frac{1+2e}{e^2(1+e)^2} \right\} \quad (1.2) \end{aligned} \right\} (1)$$

where p is the mean pressure, τ is the shearing stress on slip plane, K, k and s_0 are constants and the other notations are those in common usage.

Above relations are the equations of a space curve in three-dimensional space, explained in the former paragraph. If the author's theory is valid,

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Figure 1. Space representation of the state of soil
 $P_1P_2P_3$: the critical state line

the curve explained above must be the critical state line due to Roscoe et al. This theory was verified by comparing experimental data with the theoretically derived expression such as

$$\sin \phi = \frac{k}{1+e} \quad (2)$$

where ϕ is the angle of internal friction and k is a constant. However, such relations are obtained from the ratio of p and τ and hence by such a procedure, the validity of the term

$$\tau^* = \log \frac{1+e}{e} + \frac{s_0}{2} \frac{1+2e}{e^2(1+e)^2} \quad (3)$$

could not be made clear.

Generally speaking, any new theory allows some assumptions which cannot be verified directly by experiments and the author's theory is not an exception.

The critical state line proposed by Roscoe et al. is based on experimental data. Therefore, if it is shown that the space curve expressed by equation (1) is not anything other than the critical state line, it gives a strong support for the author's theory.

3. PROJECTION OF THE CRITICAL STATE LINE ON THE $(e-p)$ PLANE

Equation (1.1) would describe a curve projected on the $e-p$ plane. For

Figure 2. Relationship between p^* and e , for several values of s_0 .

the convenience of plotting, the following expression is adopted.

$$p^* = -\frac{P}{K} = (1+e)\left\{\log \frac{1+e}{e} + \frac{s_0}{2} \frac{1+2e}{e^2(1+e)^2}\right\} \quad (4)$$

Figure 2 shows the relationship between p^* and e for several values of s_0 . In these calculations, s_0 was assumed to be 0 or some value close to zero, because s_0 is the deviation of void ratio at failure. When the range of the void ratio is large, the relationship is not linear, however, when the range is small, for example, between 0.5 and 0.7, the relationship between the void ratio and the logarithm of p^* is fairly linear as shown in Figure 3. This agrees with what Roscoe et al. state. The inclination of the line depends on the value of s_0 .

Figure 3. Relationship between e and $\log p^*$, showing the linear relation for narrow range of e .

4. PROJECTION OF THE CRITICAL STATE LINE ON THE τ - p PLANE

The relationship between the shearing stress on the slip plane and the mean pressure p given by Eqs. (1) is shown in Figure 4.

All the curves are nearly straight lines, but near the point (1, 0) they are not straight. The fact that the curves pass through the point (1, 0) is shown by the relations:

$$\lim_{e \rightarrow \infty} (1+e) \log \frac{1+e}{e} = 1, \quad \lim_{e \rightarrow \infty} \frac{1+2e}{e^2(1+e)} = 0 \quad (5)$$

Hence, the curves expressed by Eqs. (1) have an asymptotic straight line $p^* = 1$ situated on the plane p^*, e . However, it is absurd to consider a state of infinite or zero void ratio for the granular material, since the void ratio lies usually between 0.4 and 0.8. Therefore, the parts of the curves shown in Figure 4 close to the point (1, 0) should be excluded.

From Eqs. (3) and (4), we get

$$dp^* = \tau^* de + (1+e) \frac{d\tau^*}{de} de \quad (6)$$

hence, we have

$$\frac{dp^*}{d\tau^*} = 1 + e + \tau^* / (d\tau^* / de) \quad (7)$$

Figure 4. Relationship between p^* and τ^*

and

$$\begin{aligned}\frac{\tau^*}{(d\tau^*/de)} &= \left\{ \log \frac{1+e}{e} + \frac{s_0}{2} \frac{1+2e}{e^2(1+e)^2} \right\} / \left[-\frac{1}{e(1+e)} + s_0 \left\{ -\frac{1}{e^3} + \frac{1}{(1+e)^3} \right\} \right] \\ &= (1+e) \left\{ \log \frac{1+e}{e} + \frac{s_0}{2} \frac{1+2e}{e^2(1+e)^2} \right\} / \left[-\frac{1}{e} + s_0 \left\{ -\frac{1}{e^3} - \frac{1}{e^2} + \frac{1}{(1+e)^2} \right\} \right]\end{aligned}\quad (8)$$

Hence, we get

$$\begin{aligned}\frac{dp^*}{d\tau^*} &= (1+e) \left[1 - \left\{ \log \frac{1+e}{e} + \frac{s_0(1+2e)}{2e^2(1+e)^2} \right\} / \left[\frac{1}{e} + s_0 \left\{ \frac{1}{e^3} + \frac{1}{e^2} - \frac{1}{(1+e)^2} \right\} \right] \right] \\ &= (1+e)[1-C]\end{aligned}\quad (9)$$

where

$$C = \frac{\log \frac{1+e}{e} + \frac{s_0(1+2e)}{2e^2(1+e)^2}}{\frac{1}{e} + s_0 \left\{ \frac{1}{e^3} + \frac{1}{e^2} - \frac{1}{(1+e)^2} \right\}} \quad (10)$$

Table 1

1/(1-C)									
$\begin{matrix} s_0 \\ e \end{matrix}$	0,0	0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0
0.20	1.558	1.178	1.138	1.122	1.115	1.110	1.106	1.103	1.009
0.24	1.658	1.238	1.181	1.157	1.145	1.138	1.133	1.127	1.121
0.28	1.742	1.295	1.221	1.190	1.174	1.164	1.156	1.147	1.142
0.30	1.786	1.326	1.244	1.209	1.190	1.178	1.170	1.159	1.152
0.34	1.873	1.391	1.290	1.248	1.222	1.206	1.196	1.182	1.174
0.38	1.961	1.462	1.344	1.290	1.259	1.241	1.227	1.209	1.198
0.40	2.004	1.497	1.368	1.311	1.277	1.255	1.241	1.221	1.208
0.45	2.114	1.587	1.437	1.366	1.325	1.299	1.279	1.253	1.239
0.50	2.222	1.681	1.511	1.427	1.376	1.344	1.321	1.289	1.269
0.55	2.326	1.767	1.582	1.486	1.427	1.389	1.361	1.325	1.302
0.60	2.427	1.866	1.658	1.550	1.484	1.439	1.404	1.361	1.333
0.65	2.532	1.961	1.736	1.613	1.538	1.489	1.449	1.399	1.368
0.70	2.639	2.058	1.828	1.695	1.613	1.541	1.513	1.453	1.403
0.75	2.740	2.146	1.898	1.751	1.658	1.595	1.546	1.479	1.439
0.80	2.849	2.252	1.976	1.822	1.718	1.651	1.595	1.522	1.477

Equations (3), (4), (9), and (10) give

$$\begin{aligned}\frac{d\tau}{dp} &= \frac{3k}{2} \frac{d\tau^*}{dp^*} \\ &= \frac{3}{2} \frac{k}{1+e} \frac{1}{1-C}\end{aligned}\quad (11)$$

The values of $1/(1-C)$ are given in Table 1. The value $1/(1-C)$ changes with the void ratio for a constant value of s_0 . However, the amplitude of the value $1/(1-C)$ is not so large for a narrow range of void ratio. Hence one can conclude that this value is almost a constant for a narrow range of void ratio, or one might allow a slight change in the value of s for different values of void ratio in order to keep the value $(1-C)$ a constant.

In Figure 4, it can be seen that the linearity of the curves is violated for the values of p^* smaller than about 2. The deduction of the state function $\log W$, which is of fundamental importance in the author's theory, is made on the assumption that every particle has some room around it and therefore, this theory should be applied only to an assemblage of particles satisfying such a condition.

From this theory it can be shown that the relation between τ^* and p^* is linear. Some physical interpretation about the exclusion of the part of curves close to the point (1, 0) is explained. To bring particles into an assemblage in such a way that they can be regarded as one particulated mass, some stresses should be applied from outside, although such stresses are not usually shown explicitly. Hence the total stress can be expressed by

$$p = p' + p_0, \quad \tau = \tau' + \tau_0 \quad (12)$$

where p_0 and τ_0 are the stresses explained above and p' and τ' are those induced in the material which are given by Eqs. (1).

From Eqs. (12), we have

$$p' = p - p_0, \quad \tau' = \tau - \tau_0 \quad (13)$$

They can also be expressed in terms of modified stresses superscribed with asteriks:

$$p'^* = p^* - p_0^{**}, \quad \tau'^* = \tau^{**} - \tau_0^*; \quad p_0^* = p^{**} \text{ and } \tau_0^* = \tau^{**} \quad (14)$$

This is a physical interpretation of the necessary translation of the co-ordinate axes shown in Figure 4. According to this translation, the parts of curves near the point (1, 0) are excluded and the almost linear relationship between the shearing stress on the slip surface and the mean pressure is explained theoretically.

Roscoe et al. showed that the projection of the critical state line on the

Figure 5. Relation between τ^* and e

plane (p, τ) is a straight line. This experimental fact is explained by the authors's theory.

5. PROJECTION OF THE CRITICAL STATE LINE ON THE (τ, e) PLANE

The theoretical curves obtained by Eq. (1.2) are shown in Fig. 5.

6. CONCLUSION

It can be concluded from these studies that the critical state line proposed by Roscoe et al. can be derived from the author's theory.

The problem of determining the values of k , s_0 , K , p_c^{**} , and τ_0^{**} which agree with experimental data still remains unsolved. At present the author cannot elaborate on this problem due to lack of experimental data.

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