LIMIT ANALYSIS SOLUTIONS OF EARTH PRESSURE PROBLEMS

W. F. CHEN* and J. L. ROSENFARB**

ABSTRACT

The upper bound technique of limit analysis is used to obtain rationally founded solutions to the active and passive limit earth pressures for a cohesionless soil retained by a rigid wall of varying roughness. These results are found to favorably agree with known solutions including those obtained by slip-line methods. The formulation is based on a treatment of the soil as a perfectly plastic medium obeying the Mohr-Coulomb yield condition. Several failure mechanisms composed of rigid bodies and radial shearing zones are studied. A new circular shearing zone for finite soil friction is also developed. Considerable solution improvement for passive pressures on rough walls is obtained. The necessary formulation for cohesive soils and surcharge loading is also given.

Key words: active earth pressure, design, earth pressure, passive earth pressure, plasticity, retaining wall, sandy soil, slope stability

IGC: E5/H2

INTRODUCTION

The problem of the active and passive earth pressures acting on a rigid retaining wall has been studied ever since Coulomb formulated the limit equilibrium solutions in 1776 by assuming a simple straight failure line (Fig. 1 (a)). It has long been recognized that such solutions greatly overestimate the passive pressure exerted on a relatively rough wall for high soil friction angles, e.g. $\phi=30$ to 40 degrees. Indeed for these situations the actual failure surface is far from straight but is curved. With the formulation of Kötter's curvilinear equilibrium equations came the complicated, numerically integrated, slipline solutions of Sokolovskii (1965). Although such solutions have been considered as "exact", it should be remembered that nowhere in the formulation has the soil deformation been considered, i.e., the soil stress-strain relationship.

With the development of the plastic limit theorems of perfect plasticity and their adaptation to the field of soil mechanics (Drucker and Prager, 1952), many problems have been solved on a much more logical and simple basis through the concept of a flow rule or normality. This adaptation, called the limit analysis technique, has been successfully applied in obtaining solutions to the problems of slope stability (Chen and

* Associate Professor of Civil Engineering, Fritz Engineering Laboratory, Lehigh University, Bethlehem, Pa., U.S.A.

** Research Assistant, Department of Civil Engineering, Northwestern University, Evanston, Ill., U.S.A.

Written discussions on this paper should be submitted before October 1, 1974.

Fig. 1. Failure mechanisms

and 1 (f). Although the general formulation for a $c-\phi$ soil is presented, results are primarily discussed for a cohesionless soil with no surcharge on the backfill. The necessary additional equations for the inclusion of both cohesion and surcharge are however included in Appendix I. As in Ref. 2) the soil is idealized as a perfectly plastic material which obeys the Coulomb yield condition and its associated flow rule.

RADIAL SHEAR ZONE ($\phi \neq 0$)

Velocity Field

A circular shearing zone for a soil with finite internal friction can be developed in a manner similar to that which was used in Ref. 2) for a log-spiral radial shearing zone. Consider a sector of a circle with central angle $\bar{\theta}$ to be composed of a series of n rigid triangles each of angle $\Delta\theta$, as shown in Fig. 2 (a). The velocity vector of each triangle is directed at an angle ϕ to the discontinuous rigid boundary A-B-C-D-E as required by the associated flow rule idealization. Fig. 2 (b) shows the compatible velocity diagram for triangles AOB and BOC. It should be noted that the discontinuous velocity vector V_{01} also makes an angle ϕ with the line OB. This vector is shown to be composed of components δu and δv parallel and perpendicular to the discontinuity OB. Thus δu can be considered as a simple slip velocity; while δv , a separation velocity. Assuming the central angle $\Delta\theta$ is sufficiently small, the velocity in the n th triangle can be expressed as

Giger, 1971; Chen et al., 1969; Chen et al., 1972) and bearing capacity (Chen and Davidson, 1973).

The limit analysis technique has also been applied to obtain the active and passive earth pressures acting on a rigid retaining wall. Finn (1967), and Chen and Scawthorn (1970) have investigated the problem using the classical Coulomb straight line failure mechanism (Fig. 1 (a)) and simple discontinuous stress fields. A slightly more complicated mechanism consisting of two rigid sliding blocks (Fig. 1 (b)) has briefly been discussed by Davis (1968). Some lower bound, straight line solutions, have been obtained by Lysmer (1970) with the use of a computer technique somewhat related to the finite element method.

In the following work the upper bound technique of limit analysis is applied to obtain the upper bounds for the active and passive earth pressures acting on a rigid wall using various failure mechanisms (Fig. 1). A new circular shearing zone is also developed and is used in two new mechanisms as shown in Figs. 1 (d)

$$V_n = \frac{V_{n-1} \cos\left(\frac{\Delta\theta}{2} - 2\varphi\right)}{\cos\left(\frac{\Delta\theta}{2} + 2\varphi\right)} \quad (n=1, 2, \dots, n) \quad (1)$$

which reduces to

$$V_n = V_0 \left[\frac{\cos\left(\frac{\Delta\theta}{2} - 2\varphi\right)}{\cos\left(\frac{\Delta\theta}{2} + 2\varphi\right)} \right]^n \quad (2)$$

where V_0 is the initial zone velocity. The circular radial shearing zone will be obtained in the limit as the number of triangles grows to infinity. Eq. (2) can be written as

$$\begin{aligned} V_0 \left[\frac{\cos\left(\frac{\Delta\theta}{2} - 2\varphi\right)}{\cos\left(\frac{\Delta\theta}{2} + 2\varphi\right)} \right]^n &= V_0 \left[\frac{\cos\left(\frac{\bar{\theta}}{2n} - 2\varphi\right)}{\cos\left(\frac{\bar{\theta}}{2n} + 2\varphi\right)} \right]^n \\ &= V_0 \left[\frac{1 + \tan \frac{\bar{\theta}}{2n} \tan 2\varphi}{1 - \tan \frac{\bar{\theta}}{2n} \tan 2\varphi} \right]^n \end{aligned}$$

Now if $n \rightarrow \infty$ we obtain the limit

$$\lim_{n \rightarrow \infty} \left[1 + \frac{\bar{\theta} \tan 2\varphi}{n} \right]^n \rightarrow \exp(\bar{\theta} \tan 2\varphi)$$

or

$$V = V_0 \exp(\bar{\theta} \tan 2\varphi) \quad (3)$$

where V is the velocity at any location θ along the circular arc. Eq. (3) is similar to the one derived for a log-spiral zone:

$$V = V_0 \exp(\theta \tan \varphi) \quad (4)$$

Rate of Dissipation of Energy

The general formulation for the energy dissipation due to shearing for a Coulomb material has previously been developed (Chen, 1968, 1969). In general, energy is dissipated along velocity discontinuities (narrow transition zones) and in the circular shearing zone. From Fig. 2 (a) it is clear that this energy will be dissipated along radial and boundary surfaces. The rate of energy dissipation along a typical radial line, say OE, can be found by multiplying the cohesion C , discontinuity length r_0 , and discontinuous tangential velocity $\delta u = V_{n,n+1} \cos \varphi$:

$$Cr_0 V_{n,n+1} \cos \varphi \quad (5)$$

Using Fig. 2 (b) and assuming $\Delta\theta$ to be small, $\delta u = V_n \Delta\theta \cos \varphi / \cos 2\varphi$; and Eq. (5) can be written as:

$$\frac{Cr_0 V_n \Delta\theta \cos \varphi}{\cos 2\varphi} = \frac{Cr_0 V_0 \Delta\theta \cos \varphi \exp(\bar{\theta} \tan 2\varphi)}{\cos 2\varphi} \quad (6)$$

Integrating over the total circular radial shearing zone $\bar{\theta}$:

$$\frac{Cr_0 V_0 \cos \varphi}{\sin 2\varphi} [\exp(\bar{\theta} \tan 2\varphi) - 1] \quad (7)$$

Likewise, the dissipation along a typical boundary surface, DE, is given by

(a) Division of Zone Into n Rigid Triangles

$$C \left[2r_0 \sin \frac{\Delta\theta}{2} \right] V_n \cos \varphi \quad (8)$$

which if $\Delta\theta$ is small becomes:

$$Cr_0 V_n \Delta\theta \cos \varphi = Cr_0 V_0 \Delta\theta \cos \varphi \exp(\theta \tan 2\varphi) \quad (9)$$

Integrating over the total length A-B-C-D-E:

$$\frac{Cr_0 V_0 \cos \varphi}{\tan 2\varphi} [\exp(\bar{\theta} \tan 2\varphi - 1)] \quad (10)$$

The corresponding total radial and boundary dissipation expressions for the log-spiral shearing zone are both equal to:

$$\frac{1}{2} Cr_0 V_0 \cot \varphi [\exp(2\bar{\theta} \tan \varphi) - 1] \quad (11)$$

It is noted that for the circular zone this equality does not occur.

Rate of External Work

The external rate of work done by the soil weight in the circular zone can be computed by summing over the region $\bar{\theta}$ the products of each triangle's component of vertical velocity with its weight. Using Fig. 2 (a) this can be expressed in integral form as:

$$\pm \frac{1}{2} \gamma V_0 r_0^2 \int_0^{\bar{\theta}} \exp(\mp \theta \tan 2\varphi) \sin\left(\xi + \theta \mp \varphi - \frac{\pi}{2}\right) d\theta \quad (12)$$

where the upper and lower signs signify the active and passive states respectively; and ξ denotes the angular inclination of the zone from the horizontal. The corresponding expression for the log-spiral zone is

$$\pm \frac{1}{2} \gamma V_0 r_0^2 \int_0^{\bar{\theta}} \exp(\mp 3\theta \tan \varphi) \sin(\pi - \theta - \xi) d\theta \quad (13)$$

LOG-SANDWICH MECHANISM

The upper bound theorem of limit analysis states that a soil mass will collapse if there is any compatible pattern of plastic deformation (velocity field or "mechanism") for which the rate of work of the external loads exceeds the rate of internal energy dissipation due to shearing of the soil.

A total of six different failure mechanisms, as shown in Fig. 1, are considered herein. Since the solution procedure for all cases is identical a detailed formulation will only be given for the log-sandwich mechanism and a cohesionless soil. The necessary equations for the inclusion of cohesion and surcharge loading are given in Appendix I while the two-triangle mechanism is discussed in Appendix II.

Fig. 3 shows a logarithmic spiral shearing zone, OBC, sandwiched between two rigid blocks, OAB and OCD. Since the velocities V_1 and V_3 for the rigid triangles OAB and OCD are assumed perpendicular to the radial lines OB and OC, two angular parameters ρ and ψ describe the mechanism completely. It will be shown later that for certain limited boundary situations only one parameter need be considered. This simplification reduces the complexity of the solution process. The compatible velocity diagrams corresponding to the passive pressure case are given in Figs. 3 (b) and 3 (c) for the smooth ($\delta < \varphi$) and rough ($\delta = \varphi$) wall conditions respectively. Fig. 4 shows the corresponding diagrams for the active case. The wall is assumed to translate horizontally with a velocity V_0 . All other velocities in the mechanism can then be expressed in terms of V_0 .

Fig. 3. Passive log-sandwich mechanism

Fig. 4. Active log-sandwich mechanism

Rate of External Work

For a cohesionless soil with no surcharge loading, the rate of external work due to self-weight in any region is simply the vertical component of velocity in that region multiplied by the weight of the region: (Note: for passive case use lower signs)

Triangular region OAB:

$$\frac{\pm \frac{1}{2} \gamma H^2 V_1 \sin \rho \cos(\rho \pm \phi) \cos(\alpha - \rho)}{\sin^2 \alpha \cos \phi} \quad (14)$$

Log-spiral region OBC: (see Eq. (13))

$$\frac{\pm \frac{1}{2} \gamma H^2 V_1 \cos^2(\rho \pm \phi)}{\sin^2 \alpha \cos^3 \phi (1 + 9 \tan^2 \phi)} \{ \cos(\alpha - \rho) [\pm 3 \tan \phi + (\mp 3 \tan \phi \cos \psi + \sin \psi) \exp(\mp 3 \psi \tan \phi)] \\ + \sin(\alpha - \rho) [1 + (\mp 3 \tan \phi \sin \psi - \cos \psi) \exp(\mp 3 \psi \tan \phi)] \} \quad (15)$$

Triangular region OCD:

$$\frac{\pm \frac{1}{2} \gamma H^2 V_1 \cos^2(\rho \pm \phi) \sin(\alpha - \rho - \psi + \beta) \cos(\alpha - \rho - \psi) \exp(\mp 3 \psi \tan \phi)}{\sin^2 \alpha \cos \phi \cos(\alpha - \rho - \psi + \beta \mp \phi)} \quad (16)$$

External work is also done by the components of the resultant wall load P moving in the horizontal direction with velocity V_0 :

$$\left\{ \begin{matrix} P_{aN} \\ P_{pN} \end{matrix} \right\} V_0 [\mp \sin \alpha + \tan \delta \cos \alpha] \quad (17)$$

where P_{aN} and P_{pN} are the normal to the wall components of the active and passive states respectively.

Rate of Internal Energy Dissipation

Since a cohesionless soil is being considered, the only dissipation occurs at the soil-wall interface. For a smooth wall ($\delta < \varphi$) the dissipation by sliding friction is given by

$$\left\{ \begin{matrix} P_{aN} \\ P_{pN} \end{matrix} \right\} \tan \delta V_{01} \quad (18)$$

For a rough wall ($\delta = \varphi$) the internal dissipation of energy is 0 since $C=0$ (see Eq. (5)). Using the velocity diagrams, Figs. 3 (b) and 3 (c), or 4 (b) and 4 (c), the velocities V_1 and V_{01} can be expressed in terms of the translational wall velocity V_0 for both smooth and rough cases in all the previous expressions. Equating the rate of external work to the rate of internal energy dissipation, a closed form expression for the resultant coefficients of active and passive pressures is obtained:

For a smooth wall:

$$\begin{aligned} \left\{ \begin{matrix} K_{ar} \\ K_{pr} \end{matrix} \right\} = & \frac{\mp \sec \delta}{\mp \sin \alpha + \tan \delta \cos \alpha - \frac{\tan \delta \cos (\alpha - \rho)}{\cos \rho}} \left\{ \frac{\tan \rho \cos (\rho \pm \varphi) \cos (\alpha - \rho)}{\sin \alpha \cos \varphi} \right. \\ & + \frac{\cos^2 (\rho \pm \varphi)}{\cos \rho \sin \alpha \cos^2 \varphi (1 + 9 \tan^2 \varphi)} [\cos (\alpha - \rho) [\pm 3 \tan \varphi + (\mp 3 \tan \varphi \cos \Psi + \sin \Psi) \\ & \times \exp (\mp 3 \Psi \tan \varphi)] + \sin (\alpha - \rho) [1 + (\mp 3 \tan \varphi \sin \Psi - \cos \Psi) \exp (\mp 3 \Psi \tan \varphi)] \\ & \left. + \frac{\cos^2 (\rho \pm \varphi) \sin (\alpha - \rho - \Psi + \beta) \cos (\alpha - \rho - \Psi) \exp (\mp 3 \Psi \tan \varphi)}{\cos \varphi \sin \alpha \cos (\alpha - \rho - \Psi \mp \varphi + \beta) \cos \rho} \right\} \quad (19) \end{aligned}$$

For a rough wall:

$$\begin{aligned} \left\{ \begin{matrix} K_{ar} \\ K_{pr} \end{matrix} \right\} = & \frac{\mp \sec \delta}{\mp \sin \alpha + \tan \delta \cos \alpha} \left\{ \frac{\sin^2 \rho \cos (\rho \pm \varphi) \cos (\alpha - \rho) \sin (\alpha \mp \varphi)}{\sin^2 \alpha \cos \varphi \cos (\rho \mp \varphi)} \right. \\ & \mp \frac{\cos^2 (\rho \pm \varphi) \sin (\alpha \mp \varphi)}{\sin^2 \alpha \cos^2 \varphi (1 + 9 \tan^2 \varphi) \cos (\rho \mp \varphi)} [\cos (\alpha - \rho) [\pm 3 \tan \varphi + (\mp 3 \tan \varphi \cos \Psi + \sin \Psi) \\ & \times \exp (\mp 3 \Psi \tan \varphi)] + \sin (\alpha - \rho) [1 + (\mp 3 \tan \varphi \sin \Psi - \cos \Psi) \exp (\mp 3 \Psi \tan \varphi)] \\ & \left. + \frac{\cos^2 (\rho \pm \varphi) \sin (\alpha - \rho - \Psi + \beta) \cos (\alpha - \rho - \Psi) \sin (\alpha \mp \varphi) \exp (\mp 3 \Psi \tan \varphi)}{\sin^2 \alpha \cos \varphi \cos (\alpha - \rho - \Psi + \beta \mp \varphi) \cos (\rho \mp \varphi)} \right\} \quad (20) \end{aligned}$$

In order to obtain the critical active and passive wall loads, expressions (19) and (20) must be either maximized or minimized respectively, with regard to the mechanism parameters ρ and Ψ . This was accomplished with the aid of an iterative technique incorporating the method of steepest descent. Solutions were obtained on the CDC 6400 computer. A typical solution required at most 15 iterations or 0.1 seconds.

DISCUSSION OF RESULTS

Passive Pressure

Table 1 shows the resultant passive pressure coefficients K_{pr} for three different wall inclinations α and a horizontal backfill ($\beta=0$). These solutions were obtained by using

Table 1. Passive pressure coefficients K_{pr} ($\beta=0$)

φ	δ	Mechanism					Sokolovskii	
		(1)	(2)	(3)	(4)	(5)	(6)	
$\alpha = 70$	10	0	1.36	1.36	1.52	1.47	1.36	1.36
		5	1.45	1.46	1.53	1.50	1.45	1.46
		10	1.55	1.54	1.54	1.54	1.54	1.53
	20	0	1.75	1.75	2.31	2.01	1.75	1.71
		10	2.08	2.08	2.36	2.18	2.08	2.08
		20	2.49	2.44	2.46	2.47	2.44	2.42
	30	0	2.27	2.28	3.86	2.75	2.28	2.30
		15	3.16	3.16	4.06	3.36	3.16	3.18
		30	4.76	4.43	4.50	4.76	4.41	4.76
	40	0	3.02	3.02	7.76	3.52	3.02	3.27
		20	5.34	5.32	8.33	5.39	5.31	5.89
		40	12.80	10.00	10.10	15.50	9.88	--
$\alpha = 90$	10	0	1.42	1.42	1.68	1.60	1.42	1.42
		5	1.57	1.56	1.69	1.63	1.56	1.56
		10	1.73	1.68	1.71	1.68	1.68	1.67
	20	0	2.04	2.04	3.07	2.60	2.04	2.04
		10	2.64	2.58	3.12	2.82	2.58	2.61
		20	3.53	3.18	3.27	3.19	3.17	3.19
	30	0	3.00	3.00	6.38	4.80	3.00	3.01
		15	4.98	4.71	6.61	5.88	4.71	4.97
		30	10.10	7.24	7.37	8.31	7.10	8.31
	40	0	4.60	4.61	16.10	15.40	4.60	4.67
		20	11.80	10.10	17.70	23.60	10.10	12.50
		40	92.60	22.70	21.70	67.90	20.90	--
$\alpha = 110$	10	0	1.76	1.74	3.10	2.06	1.74	1.74
		5	1.90	1.83	3.12	1.97	1.96	1.96
		10	2.04	1.91	2.77	1.90	2.16	2.14
	20	0	2.98	2.91	6.41	4.03	2.91	2.93
		10	3.78	3.38	6.50	3.85	3.91	3.94
		20	4.81	3.92	5.22	3.79	5.04	4.95
	30	0	5.34	5.09	15.60	10.20	5.08	5.33
		15	9.22	6.99	16.10	10.10	8.93	10.20
		30	72.70	10.10	11.70	11.50	14.40	17.60
	40	0	10.70	9.73	50.30	127.00	9.71	11.40
		20	89.70	17.60	53.50	141.00	25.50	69.40
		40	77.40	--	34.90	298.00	56.60	--

the failure mechanisms given in Fig. 1. Mechanisms 3 and 4 failed to yield good solutions for the case of perfectly smooth walls ($\delta=0$). It should be noted that these mechanisms cannot physically reduce down to the classical straight line failure mechanisms which effectively model smooth wall behavior. Fig. 5 (a) shows this reduction for the two-triangle mechanism (Mechanism 2) while Fig. 5 (b) shows a similar reduction for the log-sandwich mechanism (Mechanism 5). In addition, for high soil friction angles ($\varphi=40^\circ$) the arc-triangle mechanism (Mechanism 4) yielded poor results. This was expected due to the fact that the velocity in the circular radial shearing region approaches infinity as the region extent $\bar{\theta}$ approaches 45° .

Solutions were found to be improved by the use of more elaborate mechanisms only for the case of rough walls retaining highly frictional backfills. The effects of wall

Fig. 5. Typical passive mechanism results ($\phi=30^\circ$)

Fig. 6. Effect of wall roughness on passive pressure

surface (Fig. 1). Ideally for a Rankine failure state such lines should make angles of $\pi/4 - \phi/2$ with the horizontal. When these constraints were employed the log-sandwich mechanism could be described by one independent parameter (Ψ); and the resulting

roughness is shown in Fig. 6. For a horizontal backfill it is seen that roughness is particularly important for walls angled into a backfill of high soil friction angle. For a sloping backfill ($\beta=20^\circ$) it is evident that roughness is important for all wall inclinations.

Mechanism 6 was identical to the log-sandwich mechanism except that the log-spiral region was replaced with the circular ($\phi \neq 0$) shearing region discussed in Section 2. As before, results between the two compared favorably until high soil friction angles were encountered.

From these observations it is evident that Mechanisms 2 and 5 yielded the best solutions, with Mechanism 5 improving those solutions for rough walls. Although Mechanism 5 was physically more complicated, it was fully described by only two independent parameters as opposed to three for Mechanism 2. In an attempt to further reduce the number of parameters and hence simplify the minimization scheme, a study was made of the inclination of the straight line portion of the upper rigid body as it intersected the unloaded horizontal backfill

solutions were only slightly in error.

For a backfill of varying inclination the Rankine state is defined by a line to the horizontal such that

$$\Omega = \frac{\pi}{4} - \frac{\varphi}{2} + \frac{\varepsilon}{2} - \frac{\beta}{2} \quad (21)$$

where

$$\sin \varepsilon = \frac{\sin \beta}{\sin \varphi} \quad (22)$$

and β is the backfill inclination from the horizontal. The usefulness of Eq. (21) was found to be limited only to the cases of soil friction angle φ less than 30° and backfill

Table 2. Active pressure coefficients K_{ax} ($\beta=0$)

		Mechanism					Sokolovskii	
φ	δ	(1)	(2)	(3)	(4)	(5)	(6)	
$\alpha = 70$	10	0	.833	.821	.774	.738	.832	.826
		5	.801	.800	.775	.778	.801	.794
		10	.786	.787	.786	.826	.787	.794
	20	0	.648	.616	.576	.447	.647	.656
		10	.615	.610	.582	.485	.615	.612
		20	.613	.614	.613	.549	.613	.612
	30	0	.498	.490	.434	.173	.497	.521
		15	.476	.473	.446	.187	.475	.487
		30	.501	.501	.501	.230	.501	.510
	40	0	.375	.320	.328	--	.375	.396
		20	.370	.303	.346	--	.368	.385
		40	.428	.417	.428	--	.428	.430
$\alpha = 90$	10	0	.704	.704	.622	.572	.704	.700
		5	.662	.664	.624	.566	.664	.670
		10	.635	.642	.631	.564	.642	.650
	20	0	.490	.490	.394	.236	.490	.490
		10	.447	.448	.400	.226	.448	.450
		20	.427	.434	.420	.222	.434	.440
	30	0	.333	.333	.250	--	.333	.330
		15	.301	.302	.259	--	.302	.300
		30	.297	.303	.289	--	.302	.310
	40	0	.217	.215	.155	--	.217	.220
		20	.199	.200	.165	--	.200	.200
		40	.210	.214	.202	--	.214	.220
$\alpha = 110$	10	0	.644	.649	1.09	.441	.649	.665
		5	.625	.639	1.09	.436	.601	.595
		10	.616	.649	1.26	.435	.569	.561
	20	0	.380	.387	.679	.015	.385	.382
		10	.371	.386	.689	--	.341	.353
		20	.378	.417	.943	--	.319	.307
	30	0	.212	.218	.434	--	.216	.212
		15	.215	.226	.449	--	.188	.181
		30	.237	.275	.767	--	.178	.168
	40	0	.106	.111	.280	--	.109	.106
		20	.115	.123	.298	--	.095	.090
		40	.146	.180	.687	--	.095	.088

angle β less than 15° . For cases outside these limits the one-parameter solutions over-estimated the corresponding two-parameter solutions by 5% to as much as 50%.

Active Pressure

Table 2 shows the resultant active pressure coefficients $K_{a\gamma}$ for three different wall inclinations and a horizontal backfill. These results correspond to the failure mecha-

Fig. 7. Typical active mechanism results ($\phi=30^\circ$)

Fig. 8. Effect of wall roughness on active pressure

nisms given in Fig. 1. As in the case of passive pressure, Mechanisms 3 and 4 yielded poor solutions. Figs. 7 (a) and 7 (b) show typical results for the two-triangle and log-sandwich mechanisms respectively. For the case of a vertical wall, $\alpha=90^\circ$, both mechanisms yielded nearly identical results. For walls angled into the backfill ($\alpha=110^\circ$), however, the log-sandwich mechanism underestimated the upper bound solution. Again the inclusion of a log-spiral zone in place of a circular shearing zone in the log-sandwich mechanism improves the solution.

With regard to active Rankine constraints, the solutions for horizontal backfills were even less sensitive than those for the passive case. These one-parameter mechanisms for sloping backfills were again however inferior. The effect of wall roughness on the active pressure coefficients is shown in Fig. 8. Unlike the passive case for horizontal

backfills, wall roughness has a much smaller effect and is important for walls angled out of the soil ($\alpha < 90^\circ$).

COMPARISON WITH KNOWN SOLUTIONS

Passive Pressure

To date the best solutions have been generated by Sokolovskii (1965) using the slip-line method or method of stress characteristics. For soils with weight, solutions are obtained by an approximate numerical integration of the characteristic stress equations of the plastic equilibrium field. The question of whether such a slip-line solution is either an upper or lower bound solution has already been discussed (Chen and Davidson, 1973). It is generally agreed, however, that such solutions give good estimates of the exact values. Table 1 shows a comparison of limit analysis solutions for horizontal backfills with those obtained by Sokolovskii. For the cases $\alpha = 70^\circ$ and $\alpha = 90^\circ$ (vertical wall) good agreement exists. It is also seen that the two-triangle and log-sandwich mechanisms control over the whole range of soil-wall conditions, especially for rough walls retaining soil of high friction angle ($\varphi = 30^\circ$ to 40°). For these cases the limit analysis method yields values which differ from the Sokolovskii solutions by no more than 15% ($\varphi = 40^\circ$, $\delta = 40^\circ$). For the case of the wall bearing into the backfill ($\alpha = 110^\circ$), considerably more disagreement occurred.

It should be noted that the Sokolovskii solutions are very limited and easily available only for horizontal backfills. In a recent study by Lee and Herington (1972) some slip-line solutions for perfectly rough walls with sloping backfills have been formulated for both associated and non-associated flow rule materials. As previously mentioned the power of the method of limit analysis lies in the capability of bounding the true solution. Although lower bound solutions are much more difficult to obtain and involve the formulation of statically admissible stress fields, some limited solutions are available. As an example, for the particular case of a vertical wall with $\varphi = 40^\circ$, $\delta = 20^\circ$; a lower bound solution of 8.97 has been obtained by Lysmer (1970). The upper bound solution for this case is 10.10. The corresponding Sokolovskii solution of 9.69 is seen to lie between these two bounds. If an upper bound can be obtained that agrees with the corresponding lower bound solution, then of course, the exact solution will be found.

Active Pressure

A comparison of the active limit analysis solutions with those obtained by Sokolovskii is given in Table 2. In general the limit analysis results underestimate the slip-line solutions, with the greatest deviation being 6%. As in the case of passive pressures the log-sandwich and two-triangle mechanisms controlled in most cases. The exception to this, however, occurred for the case of the wall sloping into the earth backfill (i.e., $\alpha = 110^\circ$). The two-triangle mechanism yielded results which greatly exceeded the slip-line solutions. This was particularly true for the rough wall conditions i.e. $\delta = \varphi$. For these particular cases, however, the log-sandwich mechanism yielded much better agreement.

Cohesion and Surcharge

In order to illustrate the case with which the effects of cohesion and surcharge are included in the analysis, two passive pressure example problems were solved using the expressions derived in Appendix I.

The problem of a vertical wall retaining a $c-\varphi$ soil was solved (Fig. 9 (a)). A comparison with the trial wedge method shows excellent agreement. The problem of a

Fig. 9. Inclusion of cohesion and surcharge loading

cohesive, loaded soil is given in Fig. 9 (b) which shows the resulting failure mechanisms for the effects of weight, cohesion, and surcharge. The final solution was obtained by superimposing these effects. As expected, the limit analysis method yield lower solutions for highly frictional walls.

SUMMARY AND CONCLUSIONS

It has been shown that the upper bound technique of limit analysis can yield rationally founded solutions that are in good agreement with the Sokolovskii slip-line results. In addition, these solutions are easily obtainable in a closed form. The formulation needed can be readily derived and has great physical appeal. A tabulation of active and passive pressure results for a cohesionless soil retained by a rigid wall of varying roughness δ and inclination α from the horizontal can be found in Ref. 13).

The investigation of several assumed failure mechanisms has shown that significant solution improvement can especially be realized for the case of rough walls. These improvements have basically resulted from the use of new sandwich mechanism which incorporates a logarithmic spiral shearing zone. The use of this mechanism is particularly convenient and desirable due to the fact that only two independent parameters are needed in its description. An investigation was also made to study the effects of shifting the pole of the log-spiral region away from its position at the top of the wall. The resulting general sandwich mechanisms are shown in Fig. 10. From the results obtained it was concluded that very little solution improvement, if any, could be ex-

Fig. 10. General passive log-sandwich mechanism with shifting pole

pected. Such improvement is out-weighed by the complexity of the optimization procedure needed, since it must now contend with various functional discontinuities resulting from unknown velocity directions V_1 and V_3 as shown in Figs. 10 (a) and 10 (b) respectively. Improvement beyond the present stage may perhaps be realized only for other types of simple failure mechanisms.

It has also been shown that the simplicity of the upper bound technique makes it possible to easily include the effects of cohesion and surcharge. Such problems have previously been solved using the graphical forms of limit equilibrium. The problems of retaining walls with broken backs as well as backfills with irregular slopes can also be solved when appropriate kinematically admissible failure mechanisms are constructed. With the inclusion of non-homogeneous, layered soils for the limit analysis of slope stability problems (Chen et al., 1972), the extension to earth pressure situations is also possible.

ACKNOWLEDGEMENTS

The research reported here was supported by the National Science Foundation under Grant GK-14274 to Lehigh University.

NOTATION

- ϕ =internal friction angle of soil
- γ =soil unit weight
- C =soil cohesion
- q =surcharge per unit width
- H =vertical height of wall
- α =angle of wall inclination
- β =angle of backfill surface
- δ =wall angle of friction
- K_γ =coefficient of earth pressure due to weight
- K_c =coefficient of earth pressure due to cohesion
- K_q =coefficient of earth pressure due to surcharge
- $\left. \begin{matrix} P_a \\ P_p \end{matrix} \right\}$ =active and passive resultant earth pressures
- $\left. \begin{matrix} P_{aN} \\ P_{pN} \end{matrix} \right\}$ =normal active and passive earth pressures
- ρ, Ψ, Ω, η =mechanism parameters
- V_0 =initial translational wall velocity
- δu =discontinuous tangential velocity
- r_0 =initial radius of radial shear zone

REFERENCES

- 1) Chen, W. F. (1968): "On the rate of dissipation of energy in soils," *Soils and Foundations*, Vol. 8, No. 4, pp. 48-51.
- 2) Chen, W. F. (1969): "Soil mechanics and theorems of limit analysis," *Journal of the Soil Mechanics and Foundations Division, ASCE*, Vol. 95, No. SM2, pp. 493-518.
- 3) Chen, W. F. and Davidson, H. L. (1973): "Bearing capacity determination by limit analysis," *Journal of Soil Mechanics and Foundations Division, ASCE*, Vol. 99, No. SM6, pp. 433-449.

- 4) Chen, W. F. and Giger, M. W. (1971): "Limit analysis of stability of slopes," Journal of the Soil Mechanics and Foundations Division, ASCE, Vol. 97, No. SM1, pp. 19-26.
- 5) Chen, W. F., Giger, M. W. and Fang, H. Y. (1969): "On the limit analysis of stability of slopes," Soils and Foundations, Vol. 9, No. 4, pp. 23-32.
- 6) Chen, W. F. and Scawthorn, C. R. (1970): "Limit analysis and limit equilibrium solutions in soil mechanics," Soils and Foundations, Vol. 10, No. 3, pp. 13-49.
- 7) Chen, W. F., Snitbhan, N. and Fang, H. Y. (1972): "Limit analysis of stability of slopes in anisotropic non-homogeneous soils," Fritz Laboratory Report No. 355.13, Lehigh University, Bethlehem, Pennsylvania.
- 8) Davis, E. H. (1968): Theories of Plasticity and the Failure of Soil Masses, Soil Mechanics—Selected Topics, I. K. Lee, ed., Chapter 6, American Elsevier, New York, pp. 341-380.
- 9) Drucker, D. C. and Prager, W. (1952): "Soil mechanics and plastic analysis or limit design," Quarterly of Applied Mathematics, Vol. 10, No. 2, pp. 157-165.
- 10) Finn, W. D. Liam (1967): "Applications of limit plasticity in soil mechanics," Journal of the Soil Mechanics and Foundations Division, ASCE, Vol. 93, No. SM5, Part 1, pp. 101-120.
- 11) Lee, I. K. and Herington, J. R. (1972): "A theoretical study of the pressures acting on a rigid wall by a sloping earth or rock fill," Geotechnique, Vol. 22, No. 1, pp. 1-26.
- 12) Lysmer, J. (1970): "Limit analysis of plane problems in soil mechanics," Journal of the Soil Mechanics and Foundations Division, ASCE, Vol. 96, No. SM4, pp. 1311-1334.
- 13) Rosenfarb, J. and Chen, W. F. (1972): "Limit analysis solutions of earth pressure problems," Fritz Laboratory Report No. 355.14, Lehigh University, Bethlehem, Pennsylvania.
- 14) Sokolovskii, V. V. (1965): Statistics of Granular Media, Pergamon Press, New York.

APPENDIX I

LOG-SANDWICH MECHANISM

Note: For a detailed derivation see Ref. 13); use lower signs for passive case.

Cohesive Soil

For a smooth wall:

$$\left\{ \begin{array}{l} K_{ac} \\ K_{pc} \end{array} \right\} = \frac{\sec \delta}{\mp \sin \alpha + \tan \delta \cos \alpha - \frac{\tan \delta \cos (\alpha - \rho)}{\cos \rho}} \times \left\{ \begin{array}{l} \tan \rho + \frac{\cos (\rho \pm \varphi) \sin (\alpha - \rho - \Psi + \beta) \exp (\mp 2\Psi \tan \varphi)}{\cos \rho \cos (\alpha - \rho - \Psi \mp \varphi + \beta)} \\ \mp \frac{\cos (\rho \pm \varphi) [\exp (\mp 2\Psi \tan \varphi) - 1]}{\sin \varphi \cos \rho} \end{array} \right\} \quad (23)$$

For a rough wall:

$$\left\{ \begin{array}{l} K_{ac} \\ K_{pc} \end{array} \right\} = \frac{\sec \delta}{\mp \sin \alpha + \tan \delta \cos \alpha} \left\{ \begin{array}{l} \frac{\cos \varphi \cos (\alpha - \rho)}{\sin \alpha \cos (\rho \mp \varphi)} + \frac{\sin \rho \sin (\alpha \mp \varphi)}{\sin \alpha \cos (\rho \mp \varphi)} \\ + \frac{\cos (\rho \pm \varphi) \sin (\alpha - \rho - \Psi + \beta) \sin (\alpha \mp \varphi) \exp (\mp 2\Psi \tan \varphi)}{\sin \alpha \cos (\alpha - \rho - \Psi \mp \varphi + \beta) \cos (\rho \mp \varphi)} \\ \mp \frac{\cos (\rho \pm \varphi) \sin (\alpha \mp \varphi) [\exp (\mp 2\Psi \tan \varphi) - 1]}{\sin \varphi \sin \alpha \cos (\rho \mp \varphi)} \end{array} \right\} \quad (24)$$

*Surcharge**For a smooth wall:*

$$\left\{ \begin{matrix} K_{aq} \\ K_{pq} \end{matrix} \right\} = \frac{\mp \sec \delta}{\mp \sin \alpha + \tan \delta \cos \alpha - \frac{\tan \delta \cos (\alpha - \rho)}{\cos \rho}} \left\{ \frac{\cos (\rho \pm \varphi) \cos (\rho + \phi - \alpha) \exp (\mp 2\psi \tan \varphi)}{\cos \rho \cos (\alpha - \rho - \psi \mp \varphi + \beta)} \right\} \quad (25)$$

For a rough wall:

$$\left\{ \begin{matrix} K_{aq} \\ K_{pq} \end{matrix} \right\} = \frac{\mp \sec \delta}{\mp \sin \alpha + \tan \delta \cos \alpha} \left\{ \frac{\cos (\rho \pm \varphi) \sin (\alpha \mp \varphi) \cos (\rho + \phi - \alpha) \exp (\mp 2\psi \tan \varphi)}{\sin \alpha \cos (\alpha - \rho - \psi \mp \varphi + \beta) \cos (\rho \mp \varphi)} \right\} \quad (26)$$

APPENDIX II

TWO-TRIANGLE MECHANISM

Note: For a detailed derivation see Ref. 13); use lower signs for passive case.

For a smooth wall:

$$\left\{ \begin{matrix} K_{ar} \\ K_{pr} \end{matrix} \right\} = \frac{\mp \sec \delta}{\mp \sin \alpha + \tan \delta \cos \alpha - \frac{\tan \delta \sin (\eta \mp \varphi)}{\sin (\eta \mp \varphi + \alpha)}} \left\{ \frac{\sin \rho \sin (\alpha + \eta) \sin (\eta \mp \varphi)}{\sin \alpha \sin (\alpha + \eta - \rho) \sin (\eta \mp \varphi + \alpha)} \right. \\ \left. + \frac{\sin^2 (\alpha + \eta) \sin (\alpha - \rho + \beta) \sin (\alpha - \rho + \Omega) \sin (\Omega \mp \varphi) \sin (\alpha + \eta - \rho \mp 2\varphi)}{\sin \alpha \sin^2 (\alpha + \eta - \rho) \sin (\Omega - \beta) \sin (\alpha + \eta \mp \varphi) \sin (\alpha - \rho + \Omega \mp 2\varphi)} \right\} \quad (27)$$

For a rough wall:

$$\left\{ \begin{matrix} K_{ar} \\ K_{pr} \end{matrix} \right\} = \frac{\mp \sec \delta}{\mp \sin \alpha + \tan \delta \cos \alpha} \left\{ \frac{\sin \rho \sin (\alpha + \eta) \sin (\eta \mp \varphi) \sin (\alpha \mp \varphi)}{\sin^2 \alpha \sin (\alpha + \eta - \rho) \sin (\eta + \alpha \mp 2\varphi)} \right. \\ \left. + \frac{\sin^2 (\alpha + \eta) \sin (\alpha - \rho + \beta) \sin (\alpha - \rho + \Omega) \sin (\Omega \mp \varphi) \sin (\alpha \mp \varphi) \sin (\alpha + \eta - \rho \mp 2\varphi)}{\sin^2 \alpha \sin^2 (\alpha + \eta - \rho) \sin (\Omega - \beta) \sin (\eta + \alpha \mp 2\varphi) \sin (\alpha - \rho + \Omega \mp 2\varphi)} \right\} \quad (28)$$

*Cohesive Soil**For a smooth wall:*

$$\left\{ \begin{matrix} K_{ac} \\ K_{pc} \end{matrix} \right\} = \frac{\sec \delta}{\mp \sin \alpha + \tan \delta \cos \alpha - \frac{\tan \delta \sin (\eta \mp \varphi)}{\sin (\eta \mp \varphi + \alpha)}} \\ \times \left\{ \frac{\cos \varphi \sin \rho}{\sin (\eta \mp \varphi + \alpha) \sin (\alpha + \eta - \rho)} + \frac{\cos \varphi \sin (\alpha + \eta) \sin (\Omega - \eta)}{\sin (\alpha + \eta - \rho) \sin (\eta \mp \varphi + \alpha) \sin (\alpha - \rho + \Omega \mp 2\varphi)} \right. \\ \left. + \frac{\cos \varphi \sin (\alpha + \eta) \sin (\alpha - \rho + \beta) \sin (\alpha + \eta - \rho \mp 2\varphi)}{\sin (\alpha + \eta - \rho) \sin (\Omega - \beta) \sin (\eta \mp \varphi + \alpha) \sin (\alpha - \rho + \Omega \mp 2\varphi)} \right\} \quad (29)$$

For a rough wall:

$$\left\{ \begin{matrix} K_{ac} \\ K_{pc} \end{matrix} \right\} = \frac{\sec \delta}{\mp \sin \alpha + \tan \delta \cos \alpha} \left\{ \frac{\cos \varphi \sin (\eta \mp \varphi)}{\sin \alpha \sin (\eta + \alpha \mp 2\varphi)} + \frac{\cos \varphi \sin \rho \sin (\alpha \mp \varphi)}{\sin \alpha \sin (\alpha + \eta - \rho) \sin (\eta + \alpha \mp 2\varphi)} \right. \\ \left. + \frac{\cos \varphi \sin (\alpha + \eta) \sin (\alpha \mp \varphi) \sin (\Omega - \eta)}{\sin \alpha \sin (\alpha + \eta - \rho) \sin (\eta + \alpha \mp 2\varphi) \sin (\alpha - \rho + \Omega \mp 2\varphi)} \right. \\ \left. + \frac{\cos \varphi \sin (\alpha + \eta) \sin (\alpha - \rho + \beta) \sin (\alpha \mp \varphi) \sin (\alpha + \eta - \rho \mp 2\varphi)}{\sin \alpha \sin (\alpha + \eta - \rho) \sin (\Omega - \beta) \sin (\eta + \alpha \mp 2\varphi) \sin (\alpha - \rho + \Omega \mp 2\varphi)} \right\} \quad (30)$$

*Surcharge Loading**For a smooth wall:*

$$\begin{aligned} \frac{\{K_{aq}\}}{\{K_{pq}\}} &= \frac{\mp \sec \delta}{\mp \sin \alpha + \tan \delta \cos \alpha - \frac{\tan \delta \sin (\eta \mp \varphi)}{\sin (\eta \mp \varphi + \alpha)}} \\ &\times \left\{ \frac{\sin (\alpha + \eta) \sin (\alpha - \rho + \Omega) \sin (\Omega \mp \varphi) \sin (\alpha + \eta - \rho \mp 2\varphi)}{\sin (\alpha + \eta - \rho) \sin (\Omega - \beta) \sin (\eta \mp \varphi + \alpha) \sin (\alpha - \rho + \Omega \mp 2\varphi)} \right\} \end{aligned} \quad (31)$$

For a rough wall:

$$\frac{\{K_{aq}\}}{\{K_{pq}\}} = \frac{\mp \sec \delta}{\mp \sin \alpha + \tan \delta \cos \alpha} \left\{ \frac{\sin (\alpha + \eta) \sin (\alpha \mp \varphi) \sin (\alpha - \rho + \Omega) \sin (\Omega \mp \varphi) \sin (\alpha + \eta - \rho \mp 2\varphi)}{\sin \alpha \sin (\alpha + \eta - \rho) \sin (\Omega - \beta) \sin (\eta + \alpha \mp 2\varphi) \sin (\alpha - \rho + \Omega \mp 2\varphi)} \right\} \quad (32)$$

(Received March 24, 1973)