A MECHANICAL AND STATISTICAL MODEL OF GRANULAR MATERIAL

MASANOBU ODA*

ABSTRACT

On the confirmed knowledges in regard to the fabric characters of granular material and their reconstructions during the compressional deformation, its mechanical and statistical model is proposed. The mean value of force acting on a contact is determined as a function of the state of principal stresses and the inclination angles of the normal to the principal stress axes. The relation between the mobilized principal stress ratio and the fabric index of granular material is determined by the consideration on the static equilibrium of forces at the contact. The probability of sliding at a contact can be calculated by considering that the forces at the contact are random variables having their mean values and standard deviations. The rates of strain in the principal directions are theoretically obtained in terms of the frequency and intensity of sliding and the fabric characters, and then the relations between the fabric character, the dilatancy index and the mobilized stress ratio are proposed. The relation between the void ratio and the distribution character of the normal which must be satisfied at the peak stress state is also discussed. The theoretical equations accord well with the experimental results by means of the microscope and thin section method.

Key words: compressive strength, dilatancy, friction, progressive failure, sand, soil structure, triaxial compression test

IGC: D 6

INTRODUCTION

Discrete nature among particles is one of the most important characters of granular material. Configuration relation among discrete particles has been represented by some statistical parameters (Smith, et al., 1929; Smith, 1932; Kallstenius, et al., 1961; Field, 1963; Small, 1964; Labeber, 1966; Oda, 1972; Marsal, 1973). Since mechanical quantities such as contact forces acting between two neighbouring particles are considered to be random variables having their means and standard deviations, they must be also represented by some statistical parameters. It must be an important contribution for the progress of science of soil mechanics to make clear the relations between the statistical parameters of mechanical quantities and those of the granular fabric.


* Research Assistant, Department of Foundation Engineering, Faculty of Science and Engineering, University of Saitama, Urawa, Saitama.

Written discussions on this paper should be submitted before January 1, 1975.
granular models used by them have not been established on the basis of well-confirmed knowledges about the configuration relation of particles and the deformation mechanism of microscopic scale.

The author has examined in detail the granular fabric of deformed sand to make clear the deformation mechanism of granular material by means of the microscope and thin section method (Oda, 1972a; Oda, 1972b; Oda, 1972c). In this paper, he will propose a new model of granular material on the basic assumption that slidings at contacts among grains are the main mechanism of microscopic deformation and rollings of grains have negligible effect on the mechanical behavior of granular material.

MEAN VALUES OF INTERPARTICLE FORCES AND THEIR RELATIONS

Oda (1972c) has shown that the mean values of resultant forces resolved in the principal stress axes X, Y, and Z (acting on a contact \( c_i \)) are given by:

\[
\begin{align*}
F_{x'i} &= k'_x \cdot \mathcal{A}S \cdot \sigma_{x'i} \cdot | \cos \alpha \cdot \sin \beta | \\
F_{y'i} &= k'_y \cdot \mathcal{A}S \cdot \sigma_{y'i} \cdot | \sin \alpha \cdot \sin \beta | \\
F_{z'i} &= k'_z \cdot \mathcal{A}S \cdot \sigma_{z'i} \cdot | \cos \beta |
\end{align*}
\]  

(1)

where \( \alpha \) and \( \beta \) are spherical coordinates to define a direction \( N_i \) normal to a tangential plane at a contact \( c_i \), \( \sigma_{x'i} \) and \( \sigma_{y'i} \) are axial stress and confining stress in a triaxial compression test respectively and \( \mathcal{A}S \) is the mean area of contact surface at the contact (see Fig. 1). The parameters \( k'_x, k'_y \) and \( k'_z \) have been given by:

Fig. 1. Contact forces resolved in the principal stress directions X, Y and Z

\[
\begin{align*}
k'_x &= \frac{1}{n \cdot \mathcal{A}S \cdot \int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} 2E(\alpha, \beta) \cos \alpha \cdot \sin^2 \beta \, d\alpha \, d\beta} \\
k'_y &= \frac{1}{n \cdot \mathcal{A}S \cdot \int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} 2E(\alpha, \beta) \sin \alpha \cdot \sin^2 \beta \, d\alpha \, d\beta} \\
k'_z &= \frac{1}{n \cdot \mathcal{A}S \cdot \int_0^{\pi/2} \int_{0}^{2\pi} E(\alpha, \beta) \sin 2\beta \, d\alpha \, d\beta}
\end{align*}
\]

where \( n \) = number of contacts and \( E(\alpha, \beta) \) = three dimensional probability density of \( N_i \) which is determined by the microscope and thin section method (Oda, 1972a) and must satisfy the following relations;

\[
E(\alpha, \beta) = E(\pi + \alpha, \pi - \beta), \quad \int_0^{\pi} \int_0^{2\pi} E(\alpha, \beta) \sin \beta \, d\alpha \, d\beta = 1
\]

Let us consider the static equilibrium of the resultant forces resolved in the principal stress directions at a contact (Fig. 2). If the relative movements between rigid particles are considered to be due to sliding rather than due to rolling, the sliding occurs along the intersection of the plane containing both of the directions \( N_i \) and Z-axis and the tangential plane when \( \sigma_{x'} = \sigma_{y'} \) (Horne, 1965). The axis \( X' \) is then selected in the plane containing both of the directions \( N_i \) and Z as shown in Fig. 2. The forces resolved in the axes \( X' \),
Y' and Z (F_{X'i}, F_{Y'i}, and F_{Z'i}) can be determined by the following equations:

\[ F_{X'i} = F_{Xi} \cdot \cos \alpha + F_{Yi} \cdot \sin \alpha = k' \cdot \Delta S \cdot \sigma_3 \cdot \sin \beta \]
\[ F_{Y'i} = -F_{Xi} \cdot \sin \alpha + F_{Yi} \cdot \cos \alpha = 0 \]
\[ F_{Z'i} = k'' \cdot \Delta S \cdot \sigma_1 \cdot \cos \beta \]

Oda (1972c) has also obtained the theoretical equation (Eq. 3) concerning the relationship between the fabric index of sand \( \frac{S_x}{S_y} \) and the mobilized principal stress ratio \( \frac{\sigma_1}{\sigma_3} \) by considering that when there are some critical contacts satisfying the following relation,

\[ \frac{F_{Z'i}}{F_{X'i}} = \tan \left( \frac{\pi}{2} + \phi_p - \beta \right) \]

the static stability of sand is entirely lost and the interparticle slidings must occur at the contacts to reconstruct the granular fabric and, as a result, to strengthen it.

\[ \frac{\sigma_1}{\sigma_3} = \frac{S_x}{S_y} \cdot \tan \left( \frac{\pi}{4} + \frac{1}{2} \phi_p \right) \]

where \( \phi_p \) = interparticle friction angle of sand. He has also shown that the experimental results for the two sands are well in accordance with the theoretical lines expected by Eq. (3).

A STATISTICAL MODEL OF GRANULAR MATERIAL

In the previous section, the author has discussed the mean values of resultant forces \( F_{X'i}, F_{Y'i}, \) and \( F_{Z'i} \) resolved in the principal stress directions and their relation at a sliding contact. However, resultant forces \( F_{X'i}, F_{Y'i}, \) and \( F_{Z'i} \) at a contact may be considered as random variables whose mean values are \( F_{X'i}, F_{Y'i}, \) and \( F_{Z'i} \) and standard deviations are \( \rho_1, \rho_2 \) and \( \rho_3 \) respectively. The probability density of \( F_{X'i}, F_{Y'i}, \) and \( F_{Z'i} \) are assumed to be represented by the following equations instead of Gaussian distribution function (Murayama, 1964; cf. Marsal, 1973);

\[ f(F_{X'i}) = \frac{1}{\rho_1 \cdot \pi A} \exp \left( \frac{(F_{X'i} - F_{X'i})^2}{2 \rho_1^2} \right) \]
\[ f(F_{X'i}) = \frac{1}{2A} \cos \left( \frac{F_{X'i} - F_{X'i}}{A} \right) \]

\[ f(F_{X'i}) = \frac{1}{\sqrt{2\pi} \cdot \rho_1} \exp \left( -\frac{(F_{X'i} - F_{X'i})^2}{2 \rho_1^2} \right) \]

\[ A = \rho_1 \sqrt{\frac{4 \pi^2 - 8}{4 \pi^2 - 8}} = 1 \]

Fig. 3. Probability density function for a random variable \( F_{X'i} \)
\[ f(F_{X't}) = \frac{1}{2A} \cos \frac{1}{A} (F_{X't} - \bar{F}_{X't}) \]
\[ f(F_{Y't}) = \frac{1}{2B} \cos \frac{1}{B} (F_{Y't} - \bar{F}_{Y't}) \]
\[ f(F_{Zt}) = \frac{1}{2C} \cos \frac{1}{C} (F_{Zt} - \bar{F}_{Zt}) \]

where \( A = 2\rho_p/\sqrt{\pi^2 - 8} \), \( B = 2\rho_p/\sqrt{\pi^2 - 8} \), and \( C = 2\rho_p/\sqrt{\pi^2 - 8} \)

and \( \bar{F}_{X't}, \bar{F}_{Y't}, \) and \( \bar{F}_{Zt} \) are given by Eq. (2). The difference between the probability distribution represented by Eq. (4) and that of Gaussian is shown in Fig. 3.

Since \( F_{X't} \) and \( F_{Zt} \) are the random variables represented by Eq. (4), the ratio of \( F_{Zt} \) to \( F_{X't} \) must also be a random variable. Interparticle sliding will occur if the following inequality is satisfied at the contact:

\[ \frac{F_{Zt}}{F_{X't}} \geq \tan \left( \frac{\pi}{2} + \phi_p - \beta \right) \]

Thus, the probability of sliding at the contact can be represented by

\[ P(\beta) = \int_{\bar{F}_{X't} - \frac{\pi}{2}A'}^{\bar{F}_{X't} + \frac{\pi}{2}A'} P(\beta) dF_{X't} \]

As \( F_{X't} \) and \( F_{Zt} \) are considered to be independent random variables with each other, \( P(\beta) \) is calculated by the following equation:

\[ P(\beta) = \int_{\bar{F}_{X't} - \frac{\pi}{2}A'}^{\bar{F}_{X't} + \frac{\pi}{2}A'} f(F_{X't}) dF_{X't} \]

where

\[ \frac{F_{X't}}{A'} = \tan \left( \frac{\pi}{2} + \phi_p - \beta \right) \frac{F_{X't}}{A} \]

It is very difficult to determine experimentally the values of standard deviations \( \rho_p, \rho_s, \) and \( \rho_p \). Since there is no reason to consider that \( F_{X't} \) (or \( F_{Zt} \)) is more variable than \( F_{Zt} \) (or \( F_{X't} \)), it is very probable to assume that the coefficients of variation \( K' \) (i.e., the ratio of the standard deviation to the mean value) are constant;

\[ \frac{\rho_p}{\bar{F}_{X't}} = \frac{\rho_s}{\bar{F}_{Zt}} = K', \quad \frac{A}{\bar{F}_{X't}} = \frac{C}{\bar{F}_{Zt}} = K \]

where

\[ K = 2K' / \sqrt{\pi^2 - 8}. \]

Because the standard deviations must increase with the decrease of fabric homogeneity, both of \( K \) and \( K' \) would also enlarge by decreasing the homogeneity of granular fabric.

By putting Eqs. (4) and (8) into Eq. (7) and by integrating Eq. (7), \( P(\beta) \) can be obtained as follows:

1) In the case of \( \frac{\sigma_{1S}S_X}{\sigma_{3S}S_Z} \leq G(\beta) \frac{1 - \frac{\pi}{2}K}{1 + \frac{\pi}{2}K} \), we get \( P(\beta) = 0 \) \( \left( 9 \right) \)

2) In the case of \( G(\beta) \frac{1 - \frac{\pi}{2}K}{1 + \frac{\pi}{2}K} \leq \frac{\sigma_{1S}S_X}{\sigma_{3S}S_Z} \leq G(\beta) \), we get
\[ P(\beta) = \frac{1}{4} \left\{ \sin \frac{1}{K} \left( R + \frac{\pi}{2} K R - 1 \right) + 1 \right\} - \frac{R^2}{4(R^2 - 1)} \]
\[ \left\{ \sin \frac{1}{K} \left( R + \frac{\pi}{2} K R - 1 \right) + \sin \frac{1}{K} \left( \frac{\pi}{2 R} K - 1 \right) \right\} \]

where

\[ G(\beta) = \tan \beta \tan \left( \frac{\pi}{2} + \phi_v - \beta \right) \]

\[ R = \frac{F_{x_1}}{F'_{x'}} = \frac{\sigma_1 \cdot S_X}{\sigma_2 \cdot S_Z} \left( \frac{G(\beta)}{G(\beta)} \right) = \frac{\tan^2 \left( \frac{\pi}{4} + \frac{1}{2} \phi_v \right)}{G(\beta)} \]

\[ 0 \leq K < 2 \frac{\pi}{\phi} \]

**Fig. 4. Probability of sliding at contacts**

**THE RELATIONS BETWEEN FABRIC INDEX, STRESS RATIO AND STRAIN RATE RATIO**

At first we consider an entirely solid path through an assembly of spherical particles as shown in Fig. 5 (also see Horne, 1965). The symbols \( G_1, G_2, \ldots, G_{m_2} \) in this figure show spherical particles having the radii of \( r_1, r_2, \ldots, r_{m_2} \) respectively. The symbols \( L_1, L_2, \ldots, L_{m_2} \) show the diametral planes through the centers of spherical particles \( G_1, G_2, \ldots, G_{m_2} \) respectively and each diametral plane is perpendicular to the reference axis \( Z \).

One contact \( (C^0_1) \) is selected as one pleases from the contacts being on the positive side of the diametral plane \( L_1 \) of particle \( G_1 \). A random choice of contact \( (C^0_2) \) is also made from the contacts being on the negative side of the same particle \( G_1 \). In the same way a random choice is made from the contacts being on the negative side of the diametral plane \( L_2 \) of particle \( G_2 \). As in this way, an entirely solid path \( (C^0_1 - C^{m_2}_{m_2+1}) \) which progresses in the general direction of the axis \( Z \) can be obtained. At this time, the distance in the direction of the axis \( Z \) between the contacts \( C^1_1 \) and \( C^{m_2+1}_{m_2+1} \) must be slightly shorter.

It is interesting to notice that the relation between \( P(\beta) \) and \( \beta \) is only dependent on the values of \( \phi_v \) and \( K \). When \( \phi_v = 22^\circ \), their relations are calculated as shown in Fig. 4. With the increase of \( K \) value, the range of angle at which the sliding occurs becomes larger and, at the same time, the magnitude of \( P(\beta) \) gradually increases.

**Fig. 5. Solid path through an assembly** (Horne, 1965)
than a unit length. The distance between the contacts \( C_t \) and \( C_{m_2} \), however, must be slightly longer than the unit length.

Let the normal direction to the tangential plane at the contact \( C_t \) be \( N_t \), and let the angle between the normal and \( Z \)-axis be \( \beta_t \). Then we get the following equation:

\[
\begin{align*}
  r_1 |\cos(\pi - \beta_t)| + \cos \beta_t + r_2 |\cos(\pi - \beta_1)| + \cos \beta_2 + \cdots \\
  + r_{m_2} |\cos(\pi - \beta_{m_2})| + \cos \beta_{m_2} \triangleq 1 \\
  \sum_{i=1}^{m_2} r_1 \cos \beta_i + \sum_{i=1}^{m_2} r_1 \cos \beta_{i-1} \triangleq 1
\end{align*}
\]  

(11)

Since the random variables \( r_t \) and \( \cos \beta_t \) are mutually independent, we get the following equations if \( m_2 \) is sufficiently large:

\[
\frac{1}{m_2} \sum_{i=1}^{m_2} r_1 \cos \beta_i = \bar{r} \cdot \cos \bar{\beta} = \bar{r} \cdot \cos \beta
\]

(12)

\[
\frac{1}{m_2} \sum_{i=1}^{m_2} r_1 \cos \beta_{i-1} = \bar{r} \cdot \cos \bar{\beta} = \bar{r} \cdot \cos \beta
\]

where \( \bar{r} \cdot \cos \beta \) and \( \cos \beta \) are mean values of \( r_t \) \( \cos \beta_t \), \( r_t \) and \( \cos \beta_t \) respectively. Putting Eq. (12) into Eq. (11), we get

\[
m_2 = \frac{1}{2 \bar{r} \cdot \cos \beta}
\]

(13)

From the same consideration, we obtain the same equations with respect to the directions \( X \) and \( Y \):

\[
m_x = \frac{1}{2 \bar{r} \cdot \cos \alpha \cdot \sin \beta}, \quad m_y = \frac{1}{2 \bar{r} \cdot \sin \alpha \cdot \sin \beta}
\]

(14)

where

\[
\cos \beta = \int_0^{\pi/2} \int_0^{2\pi} E(\alpha, \beta) \sin 2\beta \, d\alpha \, d\beta
\]

\[
\cos \alpha \cdot \sin \beta = \int_0^{\pi} \int_{-\pi/2}^{\pi/2} 2 E(\alpha, \beta) \cos \alpha \cdot \sin^2 \beta \, d\alpha \, d\beta
\]

\[
\sin \alpha \cdot \sin \beta = \int_0^{\pi} \int_0^{2\pi} E(\alpha, \beta) \sin \alpha \cdot \sin^2 \beta \, d\alpha \, d\beta
\]

In the general trend of the axis \( Z \) along the solid path as shown in Fig. 5, the proportion of all contacts whose normals lie within the solid angles \( \alpha \) to \( \alpha + d\alpha \) and \( \beta \) to \( \beta + d\beta \) is \( 2E(\alpha, \beta) \sin \beta \, d\alpha \, d\beta \). The number of contacts lying within these solid angles along the solid path from \( C_t \) to \( C_{m_2} \) is given by

\[
2m_2 E(\alpha, \beta) \sin \beta \, d\alpha \, d\beta
\]

(15)

Hence the number of sliding contacts in the contacts of distance \( 2m_2 E(\alpha, \beta) \sin \beta \, d\alpha \, d\beta \) can be estimated by

\[
2m_2 P(\beta) E(\alpha, \beta) \sin \beta \, d\alpha \, d\beta
\]

(16)

where \( P(\beta) \) is the probability of sliding calculated by Eqs. (9) and (10). Let the mean increment of sliding length at contacts be \( \mathcal{L}U \) as shown in Fig. 5. The component of relative shortening between \( C_t \) and \( C_{m_2} \) along the axis \( Z \) due to the relative sliding movements \( \mathcal{L}U \) at the contacts whose normals lie within the solid angles \( \alpha \) to \( \alpha + d\alpha \) and \( \beta \) to \( \beta + d\beta \) is therefore given by

\[
2m_2 \mathcal{L}U P(\beta) E(\alpha, \beta) \sin^2 \beta \, d\alpha \, d\beta
\]

(17)
The summation of all the components of relative shortening due to the sliding movements along the entirely solid path is equal to the increment of axial strain. In order to take account of all the components of relative shortening along the axis Z, the integration of Eq. (17) over the entirely solid angles \((0 \leq \alpha \leq 2 \pi, \ 0 \leq \beta \leq \frac{\pi}{2})\) is necessary. Then we get

\[
\Delta \varepsilon_z = 2m_x \int_0^{\pi/2} \int_0^{2\pi} \Delta U \ P(\beta) \ E(\alpha, \ \beta) \ \sin^2 \beta \ d\alpha \ d\beta
\]  

(18)

From the same consideration with respect to the directions X and Y, we get

\[
\Delta \varepsilon_x = -m_x \int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \Delta U \ P(\beta) \ E(\alpha, \ \beta) \ \cos \alpha | \sin 2\beta | \ d\alpha \ d\beta
\]

\[
\Delta \varepsilon_y = -m_y \int_0^{\pi/2} \int_0^{2\pi} \Delta U \ P(\beta) \ E(\alpha, \ \beta) \ \sin \alpha | \sin 2\beta | \ d\alpha \ d\beta
\]  

(19)

When we only consider an axial symmetrical stress state, we get

\[
\left(1 - \frac{dv}{d\varepsilon_z}\right) = -\frac{2\Delta \varepsilon_z}{S_z} = \frac{m_x}{m_x} \int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \Delta U \ P(\beta) \ E(\alpha, \ \beta) \ \cos \alpha | \sin 2\beta | \ d\alpha \ d\beta
\]

\[
= \frac{m_x}{m_x} \int_0^{\pi/2} \int_0^{2\pi} \Delta U \ P(\beta) \ E(\alpha, \ \beta) \ \sin^2 \beta \ d\alpha \ d\beta
\]

\[
= \frac{S_y}{S_x} \cdot Q
\]  

(20)

where

\[
m_x = \frac{S_y}{S_x} \quad \text{and} \quad Q = \frac{\int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \Delta U \ P(\beta) \ E(\alpha, \ \beta) \ \cos \alpha | \sin 2\beta | \ d\alpha \ d\beta}{\int_0^{\pi/2} \int_0^{2\pi} \Delta U \ P(\beta) \ E(\alpha, \ \beta) \ \sin^2 \beta \ d\alpha \ d\beta}
\]  

(21)

From Eqs. (3) and (20), we obtain the following three equations as the most fundamental relations between \(\frac{S_y}{S_x}, \sigma_1, \sigma_3\) and \(\left(1 - \frac{dv}{d\varepsilon_z}\right);\)

\[
\frac{\sigma_1}{\sigma_3} = \frac{S_y}{S_x} \tan^2 \left(\frac{\pi}{4} + \frac{1}{2} \phi \right)
\]

(3)

\[
\left(1 - \frac{dv}{d\varepsilon_z}\right) = \frac{S_y}{S_x} \cdot Q
\]

(20)

\[
\frac{\sigma_1}{\sigma_3} = \frac{1}{Q} \left(1 - \frac{dv}{d\varepsilon_z}\right) \tan^2 \left(\frac{\pi}{4} + \frac{1}{2} \phi \right)
\]

(22)

From the studies on granular model composed of regular packing of rigid spheres, Rowe (1962, 1963) proposed the following equation in regard to the behaviour of a random assembly of irregular particles on the problematic assumption of minimum energy ratio theory (Gibson and Morgenstern, 1963; Trollope and Parkin, 1963; Scott, 1964; Roscoe and Schofield, 1964);

\[
\frac{\sigma_1}{\sigma_3} = \left(1 - \frac{dv}{d\varepsilon_z}\right) \tan^2 \left(\frac{\pi}{4} + \frac{1}{2} \phi \right)
\]

(23)

It is interesting to notice that both of Eqs. (22) and (23) which were obtained on the different consideration with each other have the identical form of equation except for a new introduction of \(\frac{1}{Q}\) in Eq. (22).
DETERMINATION OF THE VALUE OF Q

When the standard deviations ($\rho_1$ and $\rho_2$) of random variables are extremely small in comparison with the mean values $\bar{F}_{x'1}$ and $\bar{F}_{x'2}$, the angle range of sliding contact is limited within $\beta_c - \frac{\Delta \beta}{2}$ to $\beta_c + \frac{\Delta \beta}{2}$, where $\beta_c = \frac{\pi}{4} + \frac{1}{2} \phi_2$ and $\Delta \beta \ll \beta_c$.

In such a case the value of $Q$ can be calculated easily by

$$Q = \frac{2 \sin 2 \beta_c}{\pi \cdot \sin^2 \beta_c} = \frac{4}{\pi \tan \left(\frac{\pi}{4} + \frac{1}{2} \phi_2\right)}$$

(24)

It must be noted that the value of $Q$ calculated by Eq. (24) is independent of the functional form of $\mathcal{A} \mathcal{U}$ and $E(\alpha, \beta)$. Putting Eq. (24) into Eqs. (20) and (22), we get

$$\left(1 - \frac{dv}{d\varepsilon_1}\right) = \frac{S_2}{S_X} \cdot \frac{4}{\pi \tan \left(\frac{\pi}{4} + \frac{1}{2} \phi_2\right)}$$

(25)

$$\sigma_1 = \frac{\pi}{4} \left(1 - \frac{dv}{d\varepsilon_1}\right) \tan \left(\frac{1}{4} \pi + \frac{1}{2} \phi_2\right)$$

(26)

In more general cases, the mean increment of sliding length $\mathcal{A} \mathcal{U}$ must be expressed to integrate Eq. (21) by a function of $\alpha$ and $\beta$. It is reasonable to consider that the sliding length at a contact $C_i$ is proportional to the ratio of the shearing force parallel to the sliding direction ($F_{r1}$) to the normal force ($F_{n1}$). Then we get

$$\Delta U = \eta \frac{F_{r1}}{F_{n1}} \tan \beta - 1 = \eta \frac{(F_{21}/F_{X'1}) \tan \beta - 1}{(F_{21}/F_{X'1}) + \tan \beta}$$

(27)

where $\eta$ is a proportional coefficient which may be determined by the stress state, the configuration relation of grains and some other factors. So, the mean value of $\Delta U$ can be represented by

$$\Delta U = \eta \frac{(F_{21}/F_{X'1}) \tan \beta - 1}{(F_{21}/F_{X'1}) + \tan \beta}$$

(28)

Fig. 6 shows the linear positive relation between

$$\frac{F_{21}}{F_{X'1}}$$

Fig. 6. Diagram showing linear relationship between

$$\frac{F_{21}}{F_{X'1}}$$

and $1/\tan \beta$

Putting Eq. (28) as follows;

$$\Delta U = \eta \frac{(a + b \tan \beta - 1) \cdot \tan \beta}{a + b \tan \beta + \tan^2 \beta}$$

(30)

Putting Eq. (28) into Eq. (21) and integrating Eq. (21), we can determine the value of $Q$. Fig. 7 shows the relations between $Q$ calculated and $S_2/S_X$ of the specimens compacted by the plunging method (Oda, 1972b). Fig. 8 also shows the same relations of the specimens compacted by the tapping method (Oda, 1972b). The data in these figures in-
Fig. 7. Diagram showing linear relationship between \((Q)\) and \(\left(\frac{S_z}{S_x}\right)^c\) for the specimens compacted by the plunging method.

Fig. 8. Diagram showing linear relationship between \((Q)\) and \(\left(\frac{S_z}{S_x}\right)^c\) for the specimens compacted by the tapping method.

Fig. 9. Theoretical relationship between dilatancy factor \(\left(1 - \frac{d\phi}{d\varepsilon_i}\right)\) and fabric index \(\frac{S_z}{S_x}\).
dicate that the relation between \( Q \) calculated and \( \frac{S_2}{S_1} \) can be represented approximately by the following linear equation:

\[
Q = c \frac{S_2}{S_1} + d
\]  

(31)

where the constants \( c \) and \( d \) are determined by the value of \( K \) as well as by the compaction method of sand specimen. With the decrease of \( K \) to zero, the constant \( c \) becomes zero and the constant \( d \) becomes

\[
\frac{A}{\pi \tan \left( \frac{\pi}{4} + \frac{1}{2} \phi_p \right)}
\]

Putting Eq. (31) into Eq. (20), we get

\[
\left(1 - \frac{dv}{d\varepsilon_1}\right) = c \left( \frac{S_2}{S_1} \right)^2 + d \left( \frac{S_2}{S_1} \right)
\]

(32)

When the standard deviations of \( F_X \), and \( F_G \) can be regarded as extremely small in comparison with their mean values, the relation between \( \frac{S_2}{S_1} \) and \( \left(1 - \frac{dv}{d\varepsilon_1}\right) \) is given by Eq. (25) which is represented by a broken line \( A \) in Fig. 9. When we use 0.5 as the constant \( K \), the theoretical lines showing Eq. (32) for the specimens compacted by the plunging and the tapping methods are represented by solid lines \( B \) and \( C \) in Fig. 9 respectively. Circles, squares, solid circles, solid squares and crosses in this figure are obtained from experimental results (Oda, 1972b; Oda, 1972c). As the three theoretical lines are so close but experimental results are so scattered, it is rather difficult to choose a single theoretical line to fit the experimental results. However, it seems that the solid lines are more suitable to fit the experimental data than the broken line. It is worthy to note that the experimental points of both sands \( A \) and \( B \) having different values of \( \phi_p \) (see Fig. 4) which fall almost in the same banded field as shown in Fig. 9.

**STRESS-DILATANCY RELATION**

The nonlinear relation between \( \frac{\sigma_1}{\sigma_3} \) and \( \left(1 - \frac{dv}{d\varepsilon_1}\right) \) can be determined by putting Eq. (32) into Eq. (22) as follows:

\[
\left(1 - \frac{dv}{d\varepsilon_1}\right) = c \left( \frac{\sigma_1/\sigma_3}{\tan \left( \frac{\pi}{4} + \frac{1}{2} \phi_p \right)} \right)^2 + d \left( \frac{\sigma_1/\sigma_3}{\tan \left( \frac{\pi}{4} + \frac{1}{2} \phi_p \right)} \right)
\]

(33)

**Fig. 10. Theoretical relationship between principal stress ratio \( \frac{\sigma_1}{\sigma_3} \) and dilatancy factor \( \left(1 - \frac{dv}{d\varepsilon_1}\right) \)**

**Fig. 11. Experimental relationship between stress ratio \( \frac{\sigma_1}{\sigma_3} \) and dilatancy factor \( \left(1 - \frac{dv}{d\varepsilon_1}\right) \) (Barden and Khayatt, 1966)**
For example, the stress-dilatancy relations of Eq. (33) for the $P$- and $T$-specimens are represented by the curved lines in the plot of $\frac{\sigma_1}{\sigma_s}$ against $\left(1 - \frac{dv}{d\epsilon_i}\right)$ as shown in Fig. 10.

The curvature of lines increases with the increase of the value of $K$. It is noted, however, that the theoretical line is plotted within a fan shaped domain enclosed by two straight lines passing through the origin (upper and lower lines in Fig. 10).

In order to obtain a reliable stress-dilatancy plot, experiments of high degree accuracy are required in the measurement of both stress and strain. Barden and Khayatt (1966) represented one of the most reliable results in consequence of diminishing the various error sources (Fig. 11). It is worthy of attention that the data are never plotted on a single straight line passing through the origin as proposed by Rowe and Horne. The plots are everywhere between the two limit lines $K_{pp}$ and $K_{pu}$. When the stress ratio nearly equal to unity, the plots are near the upper limit line $K_{pp}$. With the increase of mobilized stress ratio, the data are plotted near the lower limit line $K_{pu}$. These tendencies are well in accordance with the theoretical lines given in Fig. 10.

**MAJOR FACTOR TO DETERMINE THE MAXIMUM PRINCIPAL STRESS RATIO MOBILIZED IN GRANULAR MATERIAL**

The maximum principal stress ratio $\left(\frac{\sigma_1}{\sigma_s}\right)$ mobilized in the granular material at failure must be determined directly by the maximum value of $\frac{S_2}{S_x}$ which can be accomplished in the granular fabric in the compressional deformation. As has been described in the previous paper (Oda, 1972c), the increase of the stress ratio occurs in association with the increase of void ratio at the severely deformed zone (dilated domain). So, it is very probable to think that the mechanical stability of granular material begins to reduce and the strain hardening due to the concentration of $N_i$ toward $\sigma_1$-direction ceases when the void ratio at

![Fig. 12. Stress-strain-volumetric strain curves for two dimensional model of rods made from photoelastic material (Konishi, 1972)](image)

![Fig. 13. Relationship between stability of column and concentration of $N_i$ to $\sigma_1$](image)
Photo. 1. Photoelastic figure to show the construction of column in a granular assembly (Konishi, 1972)

the dilated domain becomes larger than a critical value.

Konishi (1972) performed the following interesting experiment: Rods made from photoelastic material, which are 19 mm in length and 6, 8 or 10 mm in diameter, were piled up two-dimensionally in a loading framework of 170 mm in width and 250 mm in height.
to simulate an assembly of granular material. The assembly of rods mixed at random was loaded in a plane stress condition up to the peak stress state (confining pressure being constant, 0.056 kg/cm²). The deformation characteristics of the assembly have remarkable resemblance to those of a sand as shown in Fig. 12. From the close observations of Photo. 1, the following important facts can be detected:

(1) Axial load is mostly transmitted along some rows of rod pile, which are henceforth called "columns".

(2) The elongation axes of the columns have a tendency to coincide with the direction of compression.

(3) Rods within the domains enclosed by the columns have not any role to transmit the axial load parallel to \( \sigma_1 \) but serve only for stabilizing the columns.

In the same way as the assembly of rod, the concentration of \( N_i \) toward \( \sigma_1 \)-direction in sand seems to be due to the construction of "column" in the granular fabric as shown in Fig. 13. Axial load parallel to \( \sigma_1 \) must be mostly transmitted along the elongation axis of the column. With the decrease of angle \( \beta \) at the contacts jointing the particles in the column, the bearing ability to axial load must gradually increase. Therefore the mobilized principal stress \( (\sigma_1) \) must be determined by the bearing ability and the number of column constructed in the granular assembly, which is directly estimated by the three dimensional distribution of \( N_i \), i.e., \( E(\alpha, \beta) \).

The stability of the column can not be maintained if there is no supporting contact around it. However, the column can withstand the external load if some numbers of the supporting contacts exist, because the stability of the column increase due to the concentration of \( N_i \) toward \( \sigma_1 \)-direction during deformation (Fig. 13). This is the reason why the mobilized stress ratio increases during the strain hardening stage even though the void ratio at the dilated domain increases. It is also conceivable that there must be a certain critical minimum number of supporting contacts per particle to stabilize the column and when the number of supporting contacts is reduced under the critical one, the column collapses to make a new stable fabric (shear zone).

In this paper, the author has dealt with a theoretical model of granular materials which is applicable to sand under the pre-peak stress state. In the following paper, the author will discuss a granular model to explain the occurrence of shear domain.

ACKNOWLEDGEMENTS

The author wishes to express his sincere gratitude to Prof. T. Onodera and Y. Seki of Saitama University for their kind advice throughout this work and also for their critical reading of this manuscript. He is greatly indebted to Prof. T. Mogami of Nihon University, Prof. K. Ishihara, Dr. F. Tatsuoka and Mr. T. Tokue of Tokyo University for their kind advice and discussion. He is also grateful to Prof. Konishi of Shinshu University for his admission to use his valuable photoelastic figure.

NOTATION

\( A, B, C \) = parameters to represent probability density function of random variables \( F_X, F_Y, F_Z \) respectively

\( E(\alpha, \beta) \) = probability density function to represent three-dimensional distribution of \( N_i \)

\( F_{X1}, F_{Y1}, F_{Z1} \) = contact forces resolved in the principal stress directions acting on a contact
\( \bar{F}_{x}, \bar{F}_{y}, \bar{F}_{z} \) = mean values of the contact forces \( F_{x}, F_{y}, F_{z} \) respectively

\( K' \) = coefficient of variation for the random variables \( F'_{x}, F'_{y}, F'_{z} \)

\( K = 2K' / \sqrt{\pi^2 - 8} \)

\( m_x, m_y, m_z \) = average number of particles traversed in proceeding unit length in the principal directions \( X, Y \) and \( Z \) respectively

\( n \) = number of contacts

\( N_t \) = direction of normal to tangential plane at a contact

\( P(\beta) \) = probability of sliding at contact

\( A_S \) = average area of contact surfaces

\( S_X, S_Y, S_Z \) = summations of projected area of each contact surface on \( YZ, ZX \)- and \( XY \)-planes respectively

\( S_Z/S_X \) = fabric index representing fabric anisotropy

\( X, Y, Z \) = orthogonal reference axes

\( \alpha, \beta \) = spherical coordinates to define the direction of \( N_t \)

\( \sigma_1 \) = axial stress

\( \sigma_3 \) = confining pressure

\( \sigma_1/\sigma_3 \) = principal stress ratio

\( \phi_p \) = interparticle friction angle

\( \left( 1 - \frac{\partial v}{\partial \varepsilon} \right) \) = dilatancy factor

\( \rho_1, \rho_2, \rho_3 \) = standard deviations of the random variables \( F'_{x}, F'_{y}, F'_{z} \) respectively

REFERENCES


(Received May 22, 1973)