STRESS–STRAIN RELATIONSHIPS OF SANDS 
BASED ON THE MOBILIZED PLANE

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ABSTRACT
Based on stress–strain relationships on the mobilized plane, general stress–strain relationships of soils under three different principal stresses were formulated by introducing a new concept of three compounded mobilized planes. The formulas of these stress–strain relationships were verified with respect to measured data of triaxial compression tests, triaxial extension tests, plane strain tests and true triaxial tests on sands and glass beads.

Key words: dilatancy, drained shear, microscopy, plane strain, sandy soil, soil structure, special shear tests, stress–strain curve, triaxial compression test

IGC: D6/D3

INTRODUCTION
In the previous papers (Matsuoka, 1972; 1974), the author studied the shear mechanism of such granular materials as sands from the microscopic point of view by chiefly using two-dimensional models composed of aluminum rods or photoelastic rods. Here, he proposes to derive stress ratio–strain relationships of soils applicable to the mobilized plane from the relationships which were confirmed through the microscopic study of the previous papers. He then proceeds to derive from these expressions general stress ratio–strain formulas under three different principal stress conditions, with emphasis focussed on the mobilized planes involved. The general stress–strain relationships will be verified on the basis of data obtained by triaxial compression tests, plane strain tests and true triaxial tests.

SPECIMENS AND APPARATUS USED IN TESTS
Toyoura sand, Sagami River sand and glass beads were investigated here. Toyoura sand had the average grain size of 0.19 mm, the uniformity coefficient of 1.6, the specific gravity of 2.65 and the maximum and the minimum void ratio of 0.95 and 0.58 respectively. Sagami River sand had the average grain size of 0.42 mm, the uniformity coefficient of 2.2, the specific gravity of 2.68 and the maximum and the minimum void ratio of 1.08 and 0.53 (by Tatsuoka and Shiba, 1972; Tatsuoka and Ishihara, 1973). The glass beads had the average grain size of 0.15 mm, the uniformity coefficient of 1.2, the specific gravity of 2.48 and the maximum and the minimum void ratio of 0.71 and 0.50.

For the present tests were used an ordinary triaxial apparatus (N.G.I. type) and a newly manufactured true triaxial apparatus (Matsuoka and Hashimoto, 1973) capable of applying three different principal stresses. The schematic diagram of the true triaxial apparatus is

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shown in Fig. 1. In the case of the triaxial apparatus, a test specimen was 3.5 cm in diameter and 7 cm in height (for Toyoura sand and glass beads). To reduce possible friction on the upper and lower end surfaces, a rubber membrane coated with silicone grease was inserted each between the specimen and the cap and between the specimen and the pedestal. In the true triaxial apparatus, a test specimen was a cube of about 7 cm in edge length and the load was applied to the specimen via the loading plates. One pair of opposite loading plates (disposed in the direction of the minor principal stress) were so designed that a built-in board was pushed out of each face in proportion to the displacement of the loading plate (disposed in the direction of the intermediate principal stress) by virtue of the resilience of springs, so as to fill up any possible gap between loading plates. To minimize possible friction between each loading plate and the specimen, a thin polyethylene membrane coated with grease was inserted. By actual measurement, the frictional coefficient existing between the loading plates and the specimen under the condition was found to be about 0.02, a value which does not seem to have any appreciable effect upon test results. It is added that all test specimens used for the triaxial test were in a saturated condition and those for the true triaxial test were in a dry condition.

STRESS-STRAIN RELATIONSHIPS ON MOBILIZED PLANE

In the previous papers (Murayama and Matsuoka, 1971; Matsuoka, 1972; Murayama and Matsuoka, 1973), the author et al. drew out the relationship between the shear-effective normal stress ratio \( \tau/\sigma_N \) and the normal–shear strain increment ratio \( (d\varepsilon_N/dT) \) on the mobilized plane (potential sliding plane).

\[
\frac{\tau}{\sigma_N} = \lambda \left( -\frac{d\varepsilon_N}{dT} \right) + \mu \tag{1}
\]

Here, \( \varepsilon_N \) presumes compression as positive, \( \mu \) is a coefficient of friction between soil particles (=tan \( \phi \)) and \( \lambda \) is a constant of an approximate value of 1.1–1.5 to be determined by the value of \( \mu \). When \( \tau/\sigma_N \) is still on the way to the peak, the normal strain on the
mobilized plane can be expressed as follows (Matsuoka, 1974).

$$
\varepsilon_N = \log_e \left| \frac{\cos \tilde{\theta}}{\cos \tilde{\theta}_0} \right|
$$

(2)

where $\tilde{\theta}$ and $-\tilde{\theta}_0$ are the average value of angles of interparticle contact and that immediately after the start of shear respectively. By developing power series of $\log_e \cos \tilde{\theta}$ ($=-\tilde{\theta}_0^2/2-\tilde{\theta}_0^4/12-\cdots$) from Eq. (2) and disregarding terms of higher order, $\varepsilon_N$ can be approximated as shown below.

$$
\varepsilon_N = \log_e \left| \frac{\cos \tilde{\theta}}{\cos \tilde{\theta}_0} \right| \approx -\frac{1}{2} \tilde{\theta}_0^2 - \left( -\frac{1}{2} \tilde{\theta}_0^2 \right)
$$

$$
= \frac{1}{2} \left( \tilde{\theta}_0^2 - \tilde{\theta}_0^2 \right) = \frac{1}{2} (\tilde{\theta}_0^2 + \tilde{\theta}) \cdot (\tilde{\theta}_0 - \tilde{\theta})
$$

(3)

Since the increment $d\tau$ in shear strain is the macroscopic change in the angle of soil element, it can be also considered to correspond to the average value $d\tilde{\theta}$ of the change in the angles of interparticle contact (cf. Matsuoka, 1974). This is verified by measured data of a simple shear test on rod mass. Based on this relation of $d\tau = d\tilde{\theta}$, one can approximate $\tau = (\tilde{\theta}_0 + \tilde{\theta})/2$ for the neighborhood where $\tilde{\theta} = \tilde{\theta}_0$ or $\varepsilon_N = 0$. Since this gives rise to $\tilde{\theta} = -\varepsilon_N / \tau$, one can derive the following equation from the expression $\tau/\sigma_N = \lambda \cdot \tilde{\theta} + \mu$ (Matsuoka, 1972).

$$
\frac{\tau}{\sigma_N} = \lambda \cdot \tilde{\theta} + \mu \equiv \lambda \left( -\frac{\varepsilon_N}{\tau} \right) + \lambda \cdot \tilde{\theta}_0 + \mu
$$

$$
= \lambda \left( -\frac{\varepsilon_N}{\tau} \right) + \mu'
$$

(4)

Here, $\mu' = \lambda \cdot \tilde{\theta}_0 + \mu$. By combining Eqs. (4) and (1) and solving the differential equation, one obtains the following equations.

$$
\frac{\tau}{\sigma_N} = (\mu' - \mu) \cdot \log_e \frac{\tau}{\tau_0} + \mu
$$

$$
= 2.3(\mu' - \mu) \cdot \log_{10} \frac{\tau}{\tau_0} + \mu
$$

(5)

$$
\varepsilon_N = \frac{\mu - \mu'}{\tau} \cdot \log_e \frac{\tau}{\tau_0} - 1
$$

$$
= \frac{\mu - \mu'}{\tau} \cdot \left( 2.3 \log_{10} \frac{\tau}{\tau_0} - 1 \right)
$$

(6)

where, $\tau_0$ represents $\tau$ at the maximum compression point of $\varepsilon_N$. Eq. (5) evaluates this point as the value inherent to the material soil by taking into account the fact that $\tau/\sigma_N$ amounts to the coefficient of interparticle friction $\mu$ where $\tau = \tau_0$. Further by dividing both members of Eq. (5) by $\mu$, one obtains the following equation which means that $\tau/\mu \cdot \sigma_N$ and $\tau/\tau_0$ are straight lines passing the point (1, 1) on the semi-logarithmic graph paper.

$$
\frac{\tau}{\mu \cdot \sigma_N} = \left( \frac{\mu'}{\mu} - 1 \right) \cdot \log_e \frac{\tau}{\tau_0} + 1
$$

$$
= 2.3 \left( \frac{\mu'}{\mu} - 1 \right) \cdot \log_{10} \frac{\tau}{\tau_0} + 1
$$

(7)

Then by assuming the condition that the direction of the principal stress and that of the principal strain increment are the same with reference to Eqs. (5) and (6), the said equations can be converted into the following equations which govern the relationships between the effective principal stress ratio and the principal strains. It should be noted that an approximation is involved in the course of integral computation. For the convenience of
expression, $X$ is taken to denote $\sqrt{\sigma_1/\sigma_3} - \sqrt{\sigma_3/\sigma_1}$.

\begin{align*}
\varepsilon_1 &= \frac{\tau_0 \cdot \exp\left( -\frac{\mu}{\mu' - \mu} \cdot \exp\left( \frac{X}{2(\mu' - \mu)} \right) \right)}{2} \cdot \left\{ \frac{X^2}{8} + \left( \frac{1}{2} - \frac{1}{\lambda} \right) \cdot \frac{\mu'}{2} \cdot X + (\mu' - \mu)^2 - (\mu' - \mu) + \frac{2\mu'}{\lambda} + 1 \right\} \\
&\equiv f\left( \frac{\sigma_1}{\sigma_3} \right) \\
(8)
\end{align*}

\begin{align*}
\varepsilon_3 &= \frac{\tau_0 \cdot \exp\left( -\frac{\mu}{\mu' - \mu} \cdot \exp\left( \frac{X}{2(\mu' - \mu)} \right) \right)}{2} \cdot \left\{ -\frac{X^2}{8} + \left( \frac{1}{2} - \frac{1}{\lambda} \right) \cdot \frac{\mu'}{2} \cdot X - (\mu' - \mu)^2 - (\mu' - \mu) + \frac{2\mu'}{\lambda} - 1 \right\} \\
&\equiv g\left( \frac{\sigma_1}{\sigma_3} \right) \\
(9)
\end{align*}

These are the expressions of the relationships between the principal stress ratio and the principal strain on one mobilized plane.

**STRESS–STRAIN RELATIONSHIPS UNDER THREE PRINCIPAL STRESSES**

The term *mobilized plane* refers to the plane of $(\tau/\sigma_3)_{\text{max}}$ which touches on Mohr's stress circle and this plane, as is evident from Fig. 2(a), forms an angle of $(45^\circ + \phi_{\text{mo}}/2)$ with the major principal stress plane. The condition in which the stress is exerted on this mobilized plane may be depicted in terms of two dimensions as shown in Fig. 2(b). Murayama (1964) theoretically derived an equation of the stress–strain relationship of soils by attaching his attention to the plane having soil particles mobilized to the highest extent. Karube et al. (1972) studied the stress–strain relation of clays by focusing his attention on the soil element running in parallel with the direction of the intermediate principal stress $\sigma_2$ and forming a certain angle with the major principal stress plane. The behavior of soil particles under three principal stresses is generally three-dimensional. So, the author introduces the new concept of three mobilized planes as illustrated in Fig. 3. This concept presumes that, under three different principal stresses, a mobilized plane touching upon

![Diagram](image-url)
Three mobilized planes on which \( \frac{\tau}{\sigma_N} \) is maximum under three different principal stresses

Three mobilized planes under three different principal stresses

Mobilized planes in a triaxial compression test

Mobilized planes in a triaxial extension test

Fig. 3. Three compounded mobilized planes for three-dimensional behavior of particles

it's corresponding Mohr's stress circle exists, as shown in Fig. 3 (a), in the stress planes governed by \( \sigma_1 \) and \( \sigma_2 \) on the one part and \( \sigma_2 \) and \( \sigma_3 \) on the other part as well as in the stress plane governed by \( \sigma_1 \) and \( \sigma_3 \). From this, it follows there generally exist three mobilized planes as shown in Fig. 3 (b). It is assumed that these mobilized planes are parallel to their corresponding third principal stress axes in the same way that the mobilized plane governed by \( \sigma_1 \) and \( \sigma_2 \) is paralled to the \( \sigma_3 \) axis. Here, the following expressions can be drawn on the basis of Fig. 3 (b) if it is assumed that Eqs. (8) and (9) hold true for the principal stress ratios \( \sigma_1/\sigma_2 \), \( \sigma_1/\sigma_3 \) and \( \sigma_3/\sigma_1 \) of the three principal stresses (\( \sigma_1 \geq \sigma_2 \geq \sigma_3 \)), and strains can be superposed.

\[
\begin{align*}
\varepsilon_1 &= f \left( \frac{\sigma_1}{\sigma_2} \right) + f \left( \frac{\sigma_1}{\sigma_3} \right) \\
\varepsilon_2 &= f \left( \frac{\sigma_2}{\sigma_3} \right) + g \left( \frac{\sigma_1}{\sigma_2} \right) \\
\varepsilon_3 &= g \left( \frac{\sigma_3}{\sigma_2} \right) + g \left( \frac{\sigma_1}{\sigma_3} \right)
\end{align*}
\]

(10)

where the following equations are assumed to hold good for \( X = \sqrt{\sigma_i/\sigma_j} - \sqrt{\sigma_j/\sigma_i} \) (\( i, j = 1, 2, 3, i \neq j \)).

\[
f \left( \frac{\sigma_1}{\sigma_j} \right) = \frac{r_0 \cdot \exp \left( -\frac{\mu'}{\mu' - \mu} \right) \cdot \exp \left( \frac{X}{2(\mu' - \mu)} \right)}{\frac{X^2}{8} + \left( \frac{1}{2} - \frac{1}{k} \right) \cdot \frac{\mu' - \mu}{2} \cdot X + (\mu' - \mu)^2 - (\mu' - \mu) + \frac{2\mu'}{k} + 1} \quad (i < j)
\]
\[ g(\frac{\sigma_i}{\sigma_j}) = \frac{\tau^2 \exp\left(-\frac{\mu}{\mu' - \mu}\right) \exp\left\{ \frac{X}{2(\mu' - \mu)} \right\}}{\left(\frac{X^2}{8} + \left(\frac{1}{2} \frac{1}{\lambda} + \frac{\mu' - \mu}{2}\right)X - (\mu' - \mu)^2 - (\mu' - \mu) + \frac{2\mu'}{\lambda} - 1 \right) (i \neq j)} \]

\[ f(\frac{\sigma_i}{\sigma_j}) = g(\frac{\sigma_i}{\sigma_j}) = 0 \quad (i = j) \]

According to Eq. (10), the principal strains (\( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \)) and the volumetric strain (\( \Delta V/V \)) for a triaxial compression test (\( \sigma_1 \geq \sigma_2 = \sigma_3 \)) can be expressed as follows:

\[ \varepsilon_1 = 2f(\frac{\sigma_1}{\sigma_2}), \quad \varepsilon_2 = \varepsilon_3 = g(\frac{\sigma_1}{\sigma_3}) \quad \text{(11)} \]

\[ \frac{\Delta V}{V} = \varepsilon_1 + 2\varepsilon_2 = 2f(\frac{\sigma_1}{\sigma_3}) + 2g(\frac{\sigma_1}{\sigma_3}) \quad \text{(12)} \]

The mobilized planes in the case of triaxial compression are illustrated in Fig. 3 (c).

In the case of the triaxial extension test (\( \sigma_1 = \sigma_2 \geq \sigma_3 \)), the principal strains and the volumetric strain are to be expressed as follows:

\[ \varepsilon_1 = \varepsilon_2 = f(\frac{\sigma_1}{\sigma_2}), \quad \varepsilon_3 = 2g(\frac{\sigma_1}{\sigma_3}) \quad \text{(13)} \]

\[ \frac{\Delta V}{V} = 2\varepsilon_1 + \varepsilon_2 = 2f(\frac{\sigma_1}{\sigma_3}) + 2g(\frac{\sigma_1}{\sigma_3}) \quad \text{(14)} \]

The mobilized planes in the case of triaxial extension are illustrated in Fig. 3 (d).

In the case of the plane strain condition (\( \varepsilon_3 = 0 \)), the principal strains are to be expressed as follows:

\[ \varepsilon_1 = f(\frac{\sigma_1}{\sigma_2}) + f(\frac{\sigma_1}{\sigma_3}) \equiv f(\frac{\sigma_1}{\sigma_2}) \quad \text{(15)} \]

\[ \varepsilon_2 = g(\frac{\sigma_1}{\sigma_2}) + g(\frac{\sigma_2}{\sigma_3}) \equiv g(\frac{\sigma_1}{\sigma_2}) \quad \text{(15)} \]

In most cases, the ratios \( \sigma_1/\sigma_2 \) and \( \sigma_2/\sigma_3 \) in Eq. (15) have values which are at most on the order of 2. Thus, it is believed that the approximate equation is true in most cases. In this connection, comparison of Eq. (11) and Eq. (13) reveals that, in the case of test specimens having the same initial granular structure, \( \varepsilon_1 \) of triaxial compression is twice \( \varepsilon_1 \) in triaxial extension and \( \varepsilon_3 \) in triaxial extension is conversely twice \( \varepsilon_3 \) in triaxial compression. In order to obtain the relationships relating to one mobilized plane from measured data, it is clear, from Eqs. (11), (13) and (15), that the measured value of \( \varepsilon_1 \) must be multiplied by 1/2 in the case of triaxial compression tests, the measured value of \( \varepsilon_3 \) multiplied by 1/2 in the case of triaxial extension tests and the measured value used in its unaltered form in the case of plane strain tests respectively prior to proceeding to the required computations. It is predicted from Eqs. (12) and (14) that the volumetric strain \( \Delta V/V \), if computed by using the principal stress ratio \( \sigma_1/\sigma_3 \) or the shear-normal stress ratio \( \tau/\sigma_N \) on the mobilized plane as a parameter, will correspond with either of triaxial compression and triaxial extension. These relationships will be verified by measured data.

VERIFICATION BY TEST DATA

Verification of Relationship between Shear-Effective Normal Stress Ratio (\( \tau/\sigma_N \)) and Normal-Shear Strain Increment Ratio (\( d\varepsilon_N/d\tau \))

Fig. 4 represents the results of a triaxial compression test on Toyoura sand as rearranged in terms of relationship between \( \tau/\sigma_N \) and \( d\varepsilon_N/d\tau \) on the mobilized plane. Computation of the values of stress and strain on one mobilized plane from the data of triaxial compression test can be accomplished by using the measured value of \( \varepsilon_1 \) multiplied by 1/2 and that of
\[ \varepsilon_\delta \] in its unaltered form, and converting the values into those on the mobilized plane, on the assumption that the principal stress and the principal strain increment coincide in direction. The test conditions were the mean effective principal stress \( \sigma_m = 1.0 \text{ kg/cm}^2 \) and the initial void ratio \( \varepsilon_i = 0.889 \). Fig. 5 shows the results of a triaxial extension test on Toyoura sand. The test conditions were \( \sigma_m = 3.0 \text{ kg/cm}^2 \) and \( \varepsilon_i = 0.641 \). Fig. 6 shows the results of a plane strain test conducted on Toyoura sand by Ichihara and Matsuzawa (1970). The test conditions were \( \sigma_\delta = 2.0 \text{ kg/cm}^2 \) and \( \varepsilon_\delta = 0.663 \). These diagrams indicate that the data depict substantially the same linear relationship as is implied by Eq. (1). The ordinate intercept 0.23-0.25, therefore, represents the interparticle friction \( \mu \) and the linear gradient 1.2 is a reasonable value for \( \lambda \) relative to this value of \( \mu \) (Matsuoka, 1974).

In the case of Toyoura sand, direct measurement of the frictional coefficient was impossible because of unavailability of its mother rock. As an alternative, glass beads whose frictional coefficient is easy to determine were used as the material for a test specimen. Fig. 7 shows the results of a triaxial compression test, (indicated by \( \circ \) marks; \( \sigma_m = 3.0 \text{ kg/cm}^2 \)), a triaxial extension test (indicated by \( \bullet \) marks; \( \sigma_m = 3.0 \text{ kg/cm}^2 \)) and a plane strain test (indicated by \( \triangle \) marks; \( \sigma_\delta = 1.6 \text{ kg/cm}^2 \)) conducted on plots glass beads. From this diagram, it is seen that the plots in the three tests give virtually the equal gradient (\( \lambda = 1.2 \)) and the equal ordinate intercept (\( \mu = 0.10 \)). It is interesting to note that the value of this ordinate intercept is in sufficient agreement with the value 0.10-0.13 of the frictional coefficient measured between glass plates or between a glass plate and glass beads placed on the glass plate. In view of the basic idea which underlies the process of deriving the
strain of Eq. (1), this strain is believed to be ascribable to dilatancy. Strictly speaking, therefore, the verification should be made by tests conducted at a constant mean effective principal stress \( \sigma_m \). It is possibly because of the relatively small compressibility of sands that the said formula practically holds good as mentioned above even if \( \sigma_m \) is not constant.

**Verification of Relationship between Effective Principal Stress Ratio (\( \sigma_1/\sigma_3 \)) and Principal Strain Increment Ratio (\( d\varepsilon_3/d\varepsilon_1 \))**

In the previous paper (Matsuoka, 1972; 1974), the relationship between the principal stress ratio (\( \sigma_1/\sigma_3 \)) and the principal strain increment ratio (\( d\varepsilon_3/d\varepsilon_1 \)) was derived by assuming that the principal stress and the principal strain increment would agree in direction.

\[
\frac{d\varepsilon_3}{d\varepsilon_1} = \frac{\sigma_1/\sigma_3 - 2\mu \cdot \sqrt{\sigma_1/\sigma_3} \cdot (\lambda - 1)}{(1 - \lambda) \cdot \sigma_1/\sigma_3 - 2\mu \cdot \sqrt{\sigma_1/\sigma_3} \cdot \lambda - 1}
\]  
(16)

Eq. (16) represents a relationship which is established in the case of a single mobilized plane. It is, therefore, considered to be such that, under the plane strain condition, measured values may approximately be applied, in their unaltered form, to this equation. It is seen from Eqs. (11) and (13) that Eq. (16) is required to be modified in the cases of a triaxial compression test and a triaxial extension test respectively, as follows:

\[
\frac{d\varepsilon_3}{d\varepsilon_1} = \frac{1}{2} \frac{\sigma_1/\sigma_3 - 2\mu \cdot \sqrt{\sigma_1/\sigma_3} \cdot (\lambda - 1)}{(1 - \lambda) \cdot \sigma_1/\sigma_3 - 2\mu \cdot \sqrt{\sigma_1/\sigma_3} \cdot \lambda - 1}
\]  
(17)

\[
\frac{d\varepsilon_3}{d\varepsilon_1} = 2 \frac{\sigma_1/\sigma_3 - 2\mu \cdot \sqrt{\sigma_1/\sigma_3} \cdot (\lambda + 1)}{(1 - \lambda) \cdot \sigma_1/\sigma_3 - 2\mu \cdot \sqrt{\sigma_1/\sigma_3} \cdot \lambda - 1}
\]  
(18)

With a view to verifying these expressions, the results of the triaxial compression test (\( \sigma_m = 1.0 \) kg/cm\(^2\) and \( \varepsilon_1 = 0.889 \)), the plane strain (by Ichihara and Matsuzawa (1970), \( \sigma_3 = 2.0 \) kg/cm\(^2\) and \( \varepsilon_1 = 0.663 \)) and the triaxial extension test (\( \sigma_m = 3.0 \) kg/cm\(^2\) and \( \varepsilon_1 = 0.641 \)) conducted on Tooyoura sand have been plotted as shown in Fig. 8. In the diagram, the solid lines represent calculated curves based on Eqs. (16) through (18). In the calculation, the constants \( \mu = 0.24 \) and \( \lambda = 1.2 \) proper to Tooyoura sand were used. It is seen from this diagram that theoretical curves and measured values obtained from the three tests show a relatively satisfactory agreement with each other.

**Verification of Relationship between Shear–Effective Normal Stress Ratio (\( \tau/\sigma_N \)) and Normal–Shear Strain Ratio (\( \varepsilon_N/\gamma \))**

Fig. 9 represents a \( \tau/\sigma_N \) vs. \( \varepsilon_N/\gamma \) relationship on a single mobilized plane obtained from the results of a triaxial compression test on Tooyoura sand (\( \sigma_m = 1.0 \) kg/cm\(^2\), \( \varepsilon_1 = 0.889 \)). It is observed that the plot approximately overlaps the straight line described by the gradient coefficient \( \lambda = 1.1 \) to 1.2 as implied by Eq. (4).

**Verification of Relationship among Shear–Effective Normal Stress Ratio (\( \tau/\sigma_N \)), Shear Strain (\( \gamma \)) and Normal Strain (\( \varepsilon_N \))**

Fig. 10 represents the results of a triaxial compression test on Tooyoura sand (\( \sigma_m = 1.0 \)
Fig. 9. Relationship between $\tau/\sigma_N$ and $\varepsilon_N/\gamma$ on one mobilized plane during a triaxial compression test on Toyoura sand

Fig. 10. Relationship among $\tau/\sigma_N$, $\gamma$ and $\varepsilon_N$ on one mobilized plane during a triaxial compression test on Toyoura sand

kg/cm² and $e_t=0.889$) which is expressed in terms of the relationship of the mobilized plane, in conjunction with the curves given by the calculation based on Eqs. (5) and (6). For the related calculations, the coefficients $\lambda=1.2$, $\mu=0.25$, $\mu'=0.44$ proper to Toyoura sand and $\gamma_0=0.45\%$ were used.

Verification of $\tau/\mu \cdot \sigma_N$ vs. $\gamma/\gamma_0$ Relationship

Fig. 11 shows the data of triaxial compression tests on Toyoura sand. Three sets of test conditions were used, involving different mean effective principal stresses ranging from 0.5 to 2.0 kg/cm² as indicated in the graph. Fig. 12 represents the plotted data of triaxial compression and triaxial extension tests conducted on Sagami River sand (by Tatsuoka and Shiba, 1972). It is observed from these graphs that Eq. (7) is established almost similarly on the mobilized plane, no matter how the confining stress may vary or how a triaxial compression test or triaxial extension test is performed. A review of Figs. 11 and 12 leads to a conclusion that the void ratio has no marked effect. From this, it may be inferred that, so far as the test specimen is fixed, the gradient of the said linear relationship under
the condition of the normal initial granular structure is practically definite to first approximation.

Verification of Relationships between Effective Principal Stress Ratio ($\sigma_1/\sigma_3$) and Principal Strains ($\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$) under Triaxial Compression and Triaxial Extension Stress Conditions

Fig. 13 represents the data of a triaxial compression test ($\sigma_m=1.0$ kg/cm$^2$ and $\varepsilon_1=0.889$) on Toyoura sand as plotted in terms of the relationship of $\sigma_1/\sigma_3 \sim \varepsilon_1 \sim \Delta V/V (=\varepsilon_1 + 2\varepsilon_3)$ and the curves of values calculated from Eq. (11). In the calculations, the coefficients $\lambda=1.2$, $\mu=0.25$, $\mu'=0.44$ proper to Toyoura sand and $\gamma_o=0.45\%$ were used similarly to the previous case. Fig. 14 represents the data (by Tatsuoka and Shibata, 1972) of a triaxial compression test ($\sigma_m=2.0$ kg/cm$^2$, loose) and a triaxial extension test ($\sigma_m=1.0$ kg/cm$^2$, loose) on Sagami River sand as plotted in terms of the $\sigma_1/\sigma_3$ vs. $\varepsilon_1$ relationship and the $\sigma_1/\sigma_3$ vs. $\varepsilon_3$ relationship and the curves of values calculated in accordance with Eqs. (11) and (13). In the calculations, the same coefficients $\lambda=1.2$, $\mu=0.28$, $\mu'=0.46$ and $\gamma_o=0.3\%$ were used for the results of both triaxial compression and triaxial extension tests. This implied that the initial granular structures in both tests were same. It is, therefore, observed that the relationship in which the value of $\varepsilon_1$ of triaxial compression is twice as large as that of $\varepsilon_1$ of triaxial extension and the value of $\varepsilon_3$ of triaxial extension is twice as large as that of $\varepsilon_3$ of triaxial compression is also satisfied.

Verification of Relationships between Effective Principal Stress Ratios ($\sigma_1/\sigma_3$, $\sigma_1/\sigma_2$, and $\sigma_3/\sigma_2$) and Principal Strains ($\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$) under Three Different Principal Stress Conditions

Fig. 15 represents the results of a plane strain test (by Ichihara and Matsuzawa (1970),
\( \sigma_3 = 2.0 \text{ kg/cm}^2 \) and \( \epsilon_t = 0.663 \) and the curves obtained from Eq. (15). In the calculations, the same coefficients \( \lambda = 1.2, \mu = 0.25, \mu' = 0.44 \) proper to Toyoura sand and \( \tau_0 = 0.2\% \) as described above were used. Fig. 16 shows the results of a plane strain test \( (\sigma_3 = 1.4 \text{ kg/cm}^2 \) and \( \epsilon_t = 0.660) \) on Toyoura sand run by means of the true triaxial apparatus in comparison with the curves obtained by Eq. (15). Since, in the case of this apparatus, shear usually starts from \( K_0 \)-compression state, the curves of the calculated values are parallelly translated to compare them with measured values as shown in the diagram. This parallel translation of the curves is supported by a concept that \( K_0 \)-compression has already caused the amount of shear corresponding to the initial principal stress ratio \( (1/K_0) \). In these calculations, the coefficients \( \lambda = 1.2, \mu = 0.25, \mu' = 0.44 \) proper to Toyoura sand and \( \tau_0 = 0.1\% \) were used. Fig. 17 shows the result of a test conducted on loose Ottawa sand by the use of the true triaxial apparatus (by Ko and Scott, 1968) and the curves of values calculated from Eq. (10). In the calculations, the coefficients \( \lambda = 1.3, \mu = 0.20, \mu' = 0.39 \) and \( \tau_0 = 0.06\% \) determined by rearranging the various relationships on the mobilized plane with respect to the result of a triaxial compression test conducted on Ottawa sand were used. Fig. 18 represents the result of a test on Toyoura sand by the use of the true triaxial apparatus \( (\sigma_3 = 1.5 \text{ kg/cm}^2 \) and \( \epsilon_t = 0.638) \), in comparison with the curves of values calcu-
lated in accordance with Eq. (10). In the calculations, the coefficients $\lambda = 1.2$, $\mu = 0.25$, $\mu' = 0.44$ proper to Toyoura sand and $\gamma_0 = 0.1\%$ were used. For the same reason as mentioned above, the curves of calculated values are parallely translated to eliminate possible effect of the initial shear.

**METHOD FOR DETERMINATION OF STRESS-STRAIN RELATIONSHIPS**

Of the four coefficients ($\lambda$, $\mu$, $\mu'$ and $\gamma_0$) which are involved in dealing with the shear phenomenon, $\mu$ is a physical constant proper to soil which is believed to constitute a frictional coefficient ($=\tan \phi$) between soil particles. Then, $\lambda$ is a constant which is determined by the value of $\mu$ of the soil. According to Eq. (1), the value of $\lambda$ can be determined from the linear gradient and the value of $\mu$ from the ordinate intercept, respectively, of the $\tau/\sigma_N$ vs. $d\varepsilon_N/d\tau$ relationship. These values ought to be approximately constant in spite of the variability of confining pressure and void ratio. And, $\mu'$ is a coefficient which, according to the defining formula $\mu' = \lambda / \theta_0 + \mu$, has something to do with the initial granular structure and the interparticle friction. According to Eq. (7), the linear gradients of Figs. 11 and 12 are found to correspond to $2.3 (\mu'/\mu - 1)$. An inspection of these diagrams leads one to infer that the values of the gradients are not appreciably affected by the confining pressure and the void ratio. In consideration of the constancy of the value of $\mu$, therefore, the value of $\mu'$ may be assumed to be roughly constant by first approximation under the normal granular structure. Finally, $\gamma_0$ means $\tau$ at the maximum compression point of the normal strain $\varepsilon_N$ on the mobilized plane and, therefore, is considered to serve as a parameter for evaluation of the granular structure of soil. Under the normal soil structure, it is considered to constitute a coefficient governed by the initial void ratio ($\varepsilon_i$) and the confining pressure ($\sigma_m$) (see Fig. 19). Because of the equation, $\tau/\sigma_N = 2.3 (\mu' - \mu) (\log_{10} \tau - \log_{10} \gamma_0) + \mu$, which is derived from Eq. (5), the value of $\mu'$ is determined from the gradient in the $\tau/\sigma_N$ vs. $\log_{10} \tau$ relationship and the value of $\gamma_0$ from the ordinate for $\tau = 1$. Since the value of $\gamma_0$ is affected by $\sigma_m$ and $\varepsilon_i$, the value at the present position is estimated by carrying out rearrangement of relevant data in the manner of
Fig. 19.

Once the coefficients ($\lambda$, $\mu$, $\mu'$ and $\tau_0$) are thus determined, it becomes possible to calculate stress-strain relationships prevailing basically under various conditions. The generality and applicability of the proposed formulas in that the stress-strain relationships obtained by triaxial compression tests, triaxial extension tests, plane strain tests and true triaxial tests on the respective test specimens can be explained by these substantially constant coefficients ($\lambda$, $\mu$, $\mu'$) deserves special attention. Accurate determination of the remaining coefficient $\tau_0$ is an extremely difficult task. In the elucidation of stress-strain relationships of soils, such parameters as involved in the evaluation of granular structure are by all means necessary. The coefficient $\tau_0$ is the very parameter that should be estimated to evaluate the soil structure.

CONCLUSIONS

The conclusions may be summarized as follows:

(1) The formulas of various stress-strain relationships on the mobilized plane induced in the previous papers from the microscopic point of view were further developed to derive relationships among shear-effective normal stress ratio ($\tau/\sigma_N$), shear strain ($\gamma$) and normal strain ($\varepsilon_N$) on the mobilized plane.

(2) On the basis of the new concept of three compounded mobilized planes, general stress-strain relationships under different principal stresses were derived from the formulas of basic stress-strain relationships on the single mobilized planes.

(3) The formulas of various relationships mentioned above were verified by various test data. First, the relationship between shear-effective normal stress ratio ($\tau/\sigma_N$) and normal-shear strain increment ratio ($d\varepsilon_N/d\gamma$) on the single mobilized plane was verified by triaxial compression tests, triaxial extension tests and plane strain tests on Toyoura sand and glass beads.

(4) The relationship between effective principal stress ratio ($\sigma_1/\sigma_3$) and principal strain increment ratio ($d\varepsilon_3/d\varepsilon_1$) derived by converting the $\tau/\sigma_N$ vs. $d\varepsilon_N/d\gamma$ relationship on the mobilized plane into those on the principal stress plane was examined by triaxial compression tests, triaxial extension tests and plane strain tests using Toyoura sand.

(5) The shear-normal stress ratio ($\tau/\sigma_N$) versus normal-shear strain ratio ($\varepsilon_N/\gamma$) relationship on the mobilized plane was examined in comparison with the result of a triaxial compression test conducted on Toyoura sand.

(6) The relationship between shear-normal stress ratio ($\tau/\sigma_N$), shear strain ($\gamma$), and normal strain ($\varepsilon_N$) was examined in comparison with the result of a triaxial compression test conducted on Toyoura sand.

(7) When the measured values corresponding to the $\tau/\mu \cdot \sigma_N$ vs. $\log_{10} (\tau/\gamma_0)$ relationship were plotted on a semi-logarithmic scale, it was found that the data for Toyoura sand and Sagami River sand would show substantially linear relationships.

(8) When the principal stress ratio vs. principal strain relationship was examined with respect to the results of triaxial compression and triaxial extension tests on Toyoura sand and Sagami River sand, there was found a satisfactory agreement.

(9) The formula of the principal stress ratio vs. principal strain relationship was similarly examined with reference to the results of plane strain tests and true triaxial tests on Toyoura sand and Ottawa sand. Consequently, a satisfactory agreement was found between the theoretical values and the measured values. It was additionally ascertained that the stress-strain characteristics found by the shear test conducted subsequent to $K_0$-consolidation could similarly be explained.

(10) A method for determining the values of the coefficients ($\lambda$, $\mu$, $\mu'$ and $\tau_0$) was
discussed in detail.

(11) From the foregoing discussion, it is judged proper even from the standpoint of the microscopic consideration of shear mechanism to conclude that the mechanical properties of soils are governed by the value of $\tau/\sigma_N$ on the mobilized plane (or principal stress ratio). In other words, it is concluded that the substance soil is a material which basically satisfies the law of friction (Coulomb's) under normal density.

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NOTATION

\[ \frac{\Delta V}{V} = \text{volumetric strain} \]
\[ \tau = \text{shear strain on the potential sliding plane} \quad (\text{mobilized plane}) \]
\[ \tau_0 = \text{peak at the maximum compression point of} \varepsilon_N \]
\[ \varepsilon_N = \text{normal strain on the potential sliding plane} \quad (\text{mobilized plane}) \]
\[ \varepsilon_1, \varepsilon_2, \varepsilon_3 = \text{major, intermediate and minor principal strain} \]
\[ \bar{\theta}, -\bar{\theta} = \text{average value of angles of interparticle contact and that immediately after the start of shear} \]
\[ \lambda = \text{a constant decided by the value of} \mu \text{ and taking an approximate value} (1.1 \text{ to} 1.5) \]
\[ \mu = \text{coefficient of interparticle friction} \quad (= \tan \phi_p) \]
\[ \mu' = \text{one of the coefficients of soil} \quad (\equiv \lambda' \bar{\theta} + \mu) \]
\[ \sigma_N = \text{effective normal stress on the mobilized plane} \]
\[ \sigma_1, \sigma_2, \sigma_3 = \text{major, intermediate and minor effective principal stress} \]
\[ \sigma_m = \text{mean effective principal stress} \]
\[ \tau = \text{shear stress on the mobilized plane} \]
\[ (\tau/\sigma_N) = \text{shear-effective normal stress ratio on the mobilized plane} \]
\[ (d\tau/d\sigma) = \text{normal-shear strain increment ratio on the mobilized plane} \]
\[ \phi_{m} = \text{mobilized angle of internal friction} \quad (= \arctan (\tau/\sigma_N)) \]

REFERENCES


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