ROTATION OF PRINCIPAL STRESSES IN GRANULAR MATERIAL DURING SIMPLE SHEAR

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ABSTRACT
On the basis of the experimental evidences for the simple shear test of two-dimensional granular model and for the direct shear test of sand, some fundamental problems such as (1) the orientation of the principal stress axes and (2) the relation between the principal axes of stress and those of strain increment are discussed to make the deformation and strength behaviours of granular material clear from a theoretical point of view. The distribution law of contact force, which makes the determination of mean contact force at each contact possible, is successfully applied to calculate the theoretical value with respect to the mobilized stress ratio and the inclination angle of the maximum principal stress axis in the simple shear test. The rotation of the principal stress axes during simple shear occurs in the process of increasing stress ratio, and the principal axes of stress and of strain increment do not generally coincide with each other, at least up to the peak stress ratio. The equations obtained theoretically accord well with the experimental results performed by the present authors and Roscoe, et al. The same principle in regard to the mechanism of strain hardening is also applicable to the direct shear test of sand.

Key words: direct shear test, progressive failure, sand, soil structure
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INTRODUCTION
Oda and Konishi (1974) has shown that the principal stress axes gradually rotate during the progressive shearing of two-dimensional granular model, in agreement with the gradual rotation of the preferred direction of $N_t$ ($N_t$ is the direction normal to the tangential plane at the contact). Roscoe, Bassett and Cole (1967) also observed the rotation of the maximum principal stress axis during the simple shear of a sand. According to their experimental results, the rotation of the principal axis occurs during the time when the stress ratio $\frac{\tau}{\sigma_N}$ increases and the inclination angle, $\phi$, of the maximum principal stress axis to the vertical direction tends to approach a certain asymptotic value after the peak stress ratio.

At present there are two alternative interpretations to define the orientation of the principal stress axes in a granular assembly sheared under the application of shear stress (Gibson, 1953; Hansen, 1961; Morgenstern and Tchalenko, 1967) (Fig. 1). The commonest interpretation is obtained by assuming that the direction of applied shear stress (parallel to the horizontal plane of shear apparatus) coincides with the shear plane on which the Mohr's failure criterion is satisfied. This is equivalent to the assumption that the

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maximum principal stress axis is inclined at 
\( \left( \frac{\pi}{4} - \frac{1}{2} \phi \right) \) to the horizontal (Fig. 1). Another interpretation is based on the assumption that the shear stress parallel to the horizontal is equal to the maximum shear stress. That is, the conjugate shear planes on which the Mohr's failure criterion must be satisfied are inclined at \( \phi/2 \) and \( \left( \frac{\pi}{2} - \frac{1}{2} \phi \right) \) to the horizontal (Fig. 1).

Both of these two explanations are based on the basic assumption that the Mohr's concept, which takes account of static stability on the shear planes, is applicable to understand the failure mechanism of granular material in a microscopic scale. Rowe (1962), however, has stated that the shear planes are the delayed result of failure and not the cause. Oda (1972) has also supported the Rowe's idea by the detailed microscopic observation on the granular fabrics of deformed sand. Therefore, the Mohr's concept is not sufficient to interpret the failure mechanism of granular material in microscopic scale.

Rowe (1969) obtained the following equation to determine the inclination angle \( \phi \) of the maximum principal stress axis to the vertical direction during shear:

\[
\cos 2\phi = \frac{dv}{d\varepsilon_1 - dv} \tag{1}
\]

Eq. (1) was derived from the problematic assumption that the principal axis of stress coincides with the principal axis of strain increment. However, as elucidated by Hill (1950), both of these two principal axes do not, in general, coincide in an anisotropic material. The experiments performed by Roscoe, et al. (1967) clearly indicate that these two principal axes do not always coincide during the deformation while these two axes coincide with each other when sand is sheared after the stress ratio at the minimum volume is over.

Therefore, it is necessary to find (1) the orientation of principal stress axes and (2) the relation between the principal axes of stress and of strain increment in order to explain the mechanism of the deformation and hardening behaviour of granular material during the tests by simple shear and direct shear. In this paper, the present authors will discuss these problems on the basis of some experimental evidences obtained from the simple shear test of two-dimensional granular model and the direct shear test of a sand (Oda and Konishi, 1974).

THEORETICAL CONSIDERATION ON TWO-DIMENSIONAL GRANULAR MODEL

Let us consider a square having a dimension 1×1 in a sheared zone (abcd in Fig. 2). The centroids of the grains \( G_1, G_2, \ldots \) and \( G_{n_x} \) exist within the square. These grains contact at \( C_1, C_2, \ldots \) and \( C_{n_x} \) with the grains \( G_1', G_2', \ldots \) and \( G_{n_x}' \) whose centroids exist outside of the square. The curved lines a-b, b-c, c-d and d-a in Fig. 2 are determined so as to satisfy the following two conditions: (1) The number of the selected contacts \( C_1, C_2, \ldots \) and \( C_m \) is independent of the general directions of curved lines a-b, b-c, c-d and d-a.
(2) The distribution of $N_i$ at these contacts is sufficiently represented by the function $E(\alpha, \beta)$ which has been introduced in the previous paper (Oda, 1972) to show the probability density of $N_i$ in a granular assembly.

Contact force transmitted through the contact from the grain $G_i'$ to the grain $G_i$ is resolved in the principal directions $Y$ and $Z$. The resolved values of the contact force may be considered to be random variables. However, we can consider the mean values of the random variables denoted by $\bar{F}_{Yi}$ and $\bar{F}_{Zi}$ which are positive in the negative directions of the reference axes $Y$ and $Z$. The summation of forces transmitted through all the contacts $C_1, C_2, \ldots$ and $C_{m_2}$ across the curved line (a-b) in Fig. 2 can be represented by $\sum_{i=1}^{m_2} \bar{F}_{Yi}$ and $\sum_{i=1}^{m_2} \bar{F}_{Zi}$. If the number of contacts is sufficiently large, the normal and shear stresses acting on the curved line (a-b) can be represented as follows:

$$\sigma_N=\sigma_Z=\sum_{i=1}^{m_2} \bar{F}_{Zi} \quad \text{and} \quad \tau=\tau_{YZ}=-\sum_{i=1}^{m_2} \bar{F}_{Yi} \quad (2)$$

where the first subscript of stresses refers to the direction of the normal to the plane (a-b) and the second refers to the direction of the stress component itself. The compressional stress is taken as positive and the shear stress acting on the plane containing the reference axis is taken as positive toward the negative direction of the axis.

The following condition must be satisfied to keep the stresses exerted on the unit square in equilibrium:

$$\tau_{YZ}=\tau_{YZ}=\tau \quad (3)$$

Now, we consider that it is always possible to find the two mutually orthogonal principal planes with zero shear stress component. The directions of normals to these two planes are called principal stress axes and are denoted by $Y'$ and $Z'$. The inclination angle of $Z'$ to $Z$ is represented by $\phi(0 \leq \phi \leq \pi/2)$. The contact force acting on a contact $C_i$ are resolved in the principal axes $Y'$ and $Z'$. The resolved forces $\bar{F}_{Y'i}$ and $\bar{F}_{Z'i}$ (Fig. 3) are taken as positive when acting in the negative directions of the principal stress axes $Y'$ and $Z'$, and the inclination angle of $N_i$ to the axis $Z'$ is denoted by $\beta'$.

Then, the following relations must be satisfied:

$$\beta=\beta'+\phi$$

$$\bar{F}_{Y'i}=\bar{F}_{Y'i} \cos \phi+\bar{F}_{Z'i} \sin \phi$$

$$\bar{F}_{Z'i}=-\bar{F}_{Y'i} \sin \phi+\bar{F}_{Z'i} \cos \phi \quad (4)$$

Let us consider that the deformation in a microscopic scale is due to the relative sliding between rigid particles without rolling (Horne, 1965). Because the ratio of the tangential force to the normal force at the contact cannot exceed the value of $\tan \phi_n$, the interparticle sliding must occur when the following equation is satisfied:

$$\frac{\bar{F}_{Z'i}}{\bar{F}_{Y'i}}=\tan\left(\frac{\pi}{2}+\phi_n-\beta'\right) \quad (5)$$

Fig. 2. Unit square in granular assembly

Fig. 3. Contact forces resolved in principal stress axes $Y'$ and $Z'$
when $0 \leq \beta' \leq \frac{\pi}{2}$ or $\pi \leq \beta' \leq \frac{3}{2} \pi$, and

$$\frac{\bar{F}_{x'}}{\bar{F}_{Y'}} = \tan \left( \frac{\pi}{2} - \phi_s - \beta' \right)$$

(6)

when $\frac{\pi}{2} < \beta' < \pi$ or $\frac{3}{2} \pi < \beta' < 2\pi$.

According to the distribution law of contact forces by Oda (1972 and 1974), the contact forces resolved in the principal stress axes $Y'$ and $Z'$ can be represented by the following equations:

$$\begin{align*}
\bar{F}_{Y'} &= k_{Y'} \bar{\Delta S} \sigma_{Y'} \sin \beta' \\
\bar{F}_{Z'} &= k_{Z'} \bar{\Delta S} \sigma_{Z'} \cos \beta'
\end{align*}$$

(7)

where $\sigma_{Y'}$ and $\sigma_{Z'}$ are the principal stresses in the direction of $Y'$ and $Z'$ respectively, $\bar{\Delta S}$ is the average area of contact surfaces and the parameters $k_{Y'}$ and $k_{Z'}$ are determined only by the fabric character of granular assembly. Eq. (7) was derived from the assumption that the forces resolved in the directions of principal stresses ($Y'$ and $Z'$) should be proportional to the areas of the contact surfaces projected on the principal planes and also to the principal stresses $\sigma_{Y'}$ and $\sigma_{Z'}$ (Oda, 1972). The experimental evidences concerning Eq. (7) were discussed in the previous papers (Oda, 1972 and 1974). In the previous study by Oda and Konishi (1974), they used the cylinder assembly made of the photoelastic material as constituent particles. Therefore, the contact forces can be easily determined as far as the fringe order at the contacts and the radii of these particles are known. The distribution law of contact force is also supported by the photoelastic analysis which will be described in detail in the appendix of this paper.

Putting Eq. (7) into Eqs. (5) and (6), we get

$$\frac{k_{Y'} \sigma_{Y'}}{k_{Y'} \sigma_{Y'}} = \tan \beta' \cdot \tan \left( \frac{\pi}{2} - \phi_s - \beta' \right)$$

(8)

$$\frac{k_{Z'} \sigma_{Z'}}{k_{Y'} \sigma_{Y'}} = \tan \beta' \cdot \tan \left( \frac{\pi}{2} - \phi_s - \beta' \right)$$

(9)

Oda and Konishi (1974) showed that slidings in a microscopic scale do not occur in the majority of contacts in the granular assembly and are confined to the preferred contacts. This fact means that when there are some critical contacts at which the resolved forces $\bar{F}_{Y'}$ and $\bar{F}_{Z'}$ satisfy Eqs. (5) and (6), the static stability of sand is entirely lost and the interparticle slidings occur at these contacts, and the granular mass begins to be hardened by the reconstruction of granular fabric. The contacts minimizing the right side of Eqs. (8) and (9) with respect to $\beta'$ seem to be the critical sliding contacts which determine the static stability of granular fabric, because these contacts can slide easily under a minimum principal stress ratio for a given granular fabric. Since the right sides of Eqs. (8) and (9) have the same minimum value when $\beta' = \left( \frac{\pi}{4} + \frac{1}{2} \phi_s \right)$ and $\beta' = -\left( \frac{\pi}{4} + \frac{1}{2} \phi_s \right)$ respectively, we get the following equation by inserting these values into Eqs. (8) and (9);

$$\frac{k_{Z'} \sigma_{Z'}}{k_{Y'} \sigma_{Y'}} = \tan^2 \left( \frac{\pi}{4} + \frac{1}{2} \phi_s \right)$$

(10)

In the above discussion, the contact forces were defined by using the reference axes $Y'$ and $Z'$ which corresponded to the principal stress axes. Now, when inserting Eq. (7) into Eq. (4) and using the relation $\beta' = \beta - \phi$, we get the forces $\bar{F}_{Y1}$ and $\bar{F}_{Z1}$ resolved in the axes $Y$ and $Z$ as follows;
\[ F_{Y_1} = \bar{F}S \left( (k_y \sigma_y \cdot \cos^2 \phi + k_z \sigma_z \cdot \sin^2 \phi) \sin \beta - \frac{\sin 2 \phi}{2} (k_y \sigma_y - k_z \sigma_z \cdot \cos \beta) \right) \]
\[ F_{Z_1} = \bar{F}S \left( (k_y \sigma_y \cdot \sin^2 \phi + k_z \sigma_z \cdot \cos^2 \phi) \cos \beta - \frac{\sin 2 \phi}{2} (k_y \sigma_y - k_z \sigma_z \cdot \sin \beta) \right) \]  
(11)

According to Eq. (2), \( \sigma_N = \sum_{i=1}^{m_1} F_{Z_1} \) and \( \tau = \sum_{i=1}^{m_2} F_{Y_1} \). Within the all contacts \( C_t, C_{t+1}, \ldots \)
and \( C_{m_2} \) (Fig. 2), the number of contacts whose normals lie within the angle \( \beta \) to \( \beta + d\beta \) is given by \( 2m_2 E(\beta) d\beta \) where \( E(\beta) \) represents a probability density of \( N_t \) in regard to two-dimension and each of these contacts transmits the contact force given by Eq. (11). Therefore, we get

\[ \sigma_N = \sum_{i=1}^{m_2} F_{Z_1} = \sum_{-\pi/2 \leq \beta \leq \pi/2} 2m_2 E(\beta) F_{Z_1} d\beta \]

\[ = m_2 S \left[ (k_y \sigma_y \cdot \sin^2 \phi + k_z \sigma_z \cdot \cos^2 \phi) \int_{-\pi/2}^{\pi/2} 2E(\beta) \cos \beta d\beta \right. \]

\[ + \left. \frac{(k_z \sigma_z - k_y \sigma_y)}{2} \sin 2\phi \int_{-\pi/2}^{\pi/2} 2E(\beta) \sin \beta d\beta \right] \]  
(12)

\[ \tau = \tau_{YZ} = \sum_{i=1}^{m_Y} F_{Y_1} = \sum_{-\pi/2 \leq \beta \leq \pi/2} 2m_Y E(\beta) F_{Y_1} d\beta \]

\[ = m_Y S \left[ (k_y \sigma_y \cdot \cos^2 \phi + k_z \sigma_z \cdot \sin^2 \phi) \int_{-\pi/2}^{\pi/2} 2E(\beta) \sin \beta d\beta \right. \]

\[ + \left. \frac{(k_z \sigma_z - k_y \sigma_y)}{2} \sin 2\phi \int_{-\pi/2}^{\pi/2} 2E(\beta) \cos \beta d\beta \right] \]  
(13)

From the same consideration on the line (b-c) in Fig. 2, the shear stress \( \tau_{YZ} \) can be calculated by:

\[ \tau = \tau_{YZ} = \sum_{i=1}^{m_Y} F_{Z_1} = \sum_{0 \leq \beta \leq \pi} 2m_Y E(\beta) F_{Z_1} d\beta \]

\[ = m_Y S \left[ (k_y \sigma_y \cdot \sin^2 \phi + k_z \sigma_z \cdot \cos^2 \phi) \int_{0}^{\pi} 2E(\beta) \cos \beta d\beta \right. \]

\[ + \left. \frac{(k_z \sigma_z - k_y \sigma_y)}{2} \sin 2\phi \int_{0}^{\pi} 2E(\beta) \sin \beta d\beta \right] \]  
(14)

where \( m_Y \) is the number of contacts which transmit the contact forces across the line (b-c). Since the number of contacts is independent of the general direction of line, \( m_Y \) is equal to \( m_2 \).

**ROTATION OF PRINCIPAL STRESS AXES**

Because the stresses exerted on the unit square must be in equilibrium without rotational component, \( \tau_{YZ} \) is equal to \( \tau_{YZ} \). Then, we get the following equation from Eqs. (13) and (14) and the relation given by Eq. (10);

\[ \left\{ J_1 \tan^2 \left( \frac{\pi}{4} + \frac{1}{2} \psi \right) - J_2 \right\} \tan^2 \psi + (J_3 - 1) \left\{ \tan^2 \left( \frac{\pi}{4} + \frac{1}{2} \psi \right) - 1 \right\} \tan \psi \]

\[ + \left\{ J_1 - J_2 \tan^2 \left( \frac{\pi}{4} + \frac{1}{2} \psi \right) \right\} = 0 \]  
(15)

where

\[ J_1 = \frac{\int_{-\pi/2}^{\pi/2} 2E(\beta) \sin \beta d\beta}{\int_{0}^{\pi/2} 2E(\beta) \sin \beta d\beta}, \quad J_2 = \frac{\int_{-\pi/2}^{\pi/2} 2E(\beta) \cos \beta d\beta}{\int_{0}^{\pi/2} 2E(\beta) \sin \beta d\beta} \]
and

\[ J_b = \frac{\int_{-\pi/2}^{\pi/2} 2E(\beta)\cos \beta d\beta}{\int_0^{\pi/2} 2E(\beta)\sin \beta d\beta} \]

Since the probability density function of \( N_i \) (i.e., \( E(\beta) \)) was determined as shown in Figs. 3 and 4 in the previous paper (Oda and Konishi, 1974), the quadratic Eq. (15) with respect to \( \tan \phi \) gives us a solution to satisfy the condition of \( \tan \phi \geq 0 \). Therefore, we can determine the direction of the principal stress axes.

In Figs. 4 and 5, the angle \( \phi \) thus determined are plotted against the shear distortion \( \gamma \). It must be noted that the relation between \( \gamma \) and \( \phi \) is almost the same as the relation between \( \gamma \) and \( \frac{\tau}{\sigma_N} \). This means that the rotation of the principal stress axes mostly occurs during the process of increasing the stress ratio and that the value of \( \phi \) becomes a certain asymptotic value. These facts accord well with the experimental evidences obtained by Roscoe, et al. (1967).

Fig. 6.a shows the relation between the maximum principal stress direction thus determined and the preferred direction of \( N_i \) (Oda and Konishi, 1974). This indicates that the maximum principal stress direction is nearly parallel to the preferred direction of \( N_i \).

The inclination angle \( \xi \) of major principal strain increment axis to the vertical axis \( Z \) can be easily determined when the Mohr’s circle of strain increment is derived from the observed displacements. The change of \( \xi \) during the shear strain is also represented in

Fig. 4. Relations between shear force (\( S \)), shear distortion (\( \gamma \)), rate of thickness change (\( \Delta H/H \)), inclination angle of maximum principal stress axis (\( \phi \)) and inclination angle of maximum principal strain increment axis (\( \xi \)) for experimental series of dense model.
Fig. 5. Similar diagram to Fig. 4 for experimental series of loose model

Fig. 6. Direction of maximum principal stress axis and its relation to preferred direction of $N_i$ (Rosette diagram and Schmidt's equal area projection show two and three dimensional distribution of $N_i$ (Oda and Konishi, 1973))
Figs. 4 and 5 by broken lines. It must be noted that there is a remarkable discordance between the principal axes of stress and of strain increment, at least up to the peak stress ratio.

MOBILIZED STRESS RATIO IN SIMPLE SHEAR TEST

By using Eqs. (10) and (12), the mobilized stress ratio, \( \frac{\tau}{\sigma_N} \), in the simple shear test is given by

\[
\frac{\tau}{\sigma_N} = \frac{\left(1 + \tan^2 \left(\frac{\pi}{4} + \frac{1}{2} \phi_p\right)\right) K + \left[\tan^2 \left(\frac{\pi}{4} + \frac{1}{2} \phi_p\right) - 1\right] \tan \phi} {\tan^2 \phi + \tan^2 \left(\frac{\pi}{4} + \frac{1}{2} \phi_p\right) + \left[\tan^2 \left(\frac{\pi}{4} + \frac{1}{2} \phi_p\right) - 1\right] \cdot K \cdot \tan \phi}
\]

where

\[
K = \frac{\int_{-\pi/2}^{\pi/2} 2E(\beta) \sin \beta d\beta}{\int_{-\pi/2}^{\pi/2} 2E(\beta) \cos \beta d\beta}
\]

The calculated values of \( \frac{\tau}{\sigma_N} \) are shown in Fig. 7 by plotting them against the experimental values. Since the points in this figure are plotted near (slightly above) the solid line showing the relation \( \frac{\tau}{\sigma_N} \) (by experiment) = \( \frac{\tau}{\sigma_N} \) (by theory), we are sure that Eq. (16) as well as Eq. (7) which represents the distribution law of contact forces were supported by our experimental evidences.

THEORETICAL CONSIDERATION OF DIRECT SHEAR TEST OF SAND

It is assumed that there is also a narrow shear zone with a homogeneous stress distribution in a direct shear test as shown in Fig. 8. Consider a cube having a dimension 1\( \times \)1 \( \times \)1 in the homogeneously sheared zone. From the previous consideration on the interparticle forces transmitted through the planes of the unit cube, the mobilized stress ratio \( \frac{\tau}{\sigma_N} \) and the inclination angle of maximum principal stress

![Fig. 7. Relations between experimentally determined \( \tau/\sigma_N \) values and calculated ones of two-dimensional granular model deformed under simple shear](image)

![Fig. 8. Unit cube in granular assembly homogeneously sheared in direct shear apparatus and reference axes X, Y and Z](image)
axis to the vertical are given by Eqs. (16) and (15) respectively, with some modifications in regard to the parameters $K$, $J_1$, $J_2$ and $J_3$ as follows:

\[
K = \frac{\int_0^{\pi/2} \int_0^{2\pi} 2E(\alpha, \beta) \sin \alpha \sin^2 \beta \, d\alpha \, d\beta}{\int_0^{\pi/2} \int_0^{2\pi} 2E(\alpha, \beta) \sin \beta \cos \beta \, d\alpha \, d\beta}, \quad J_1 = \frac{\int_0^{\pi/2} \int_0^{\pi} 2E(\alpha, \beta) \sin \alpha \sin^2 \beta \, d\alpha \, d\beta}{\int_0^{\pi/2} \int_0^{\pi} 2E(\alpha, \beta) \sin \beta \cos \beta \, d\alpha \, d\beta},
\]

\[
J_2 = \frac{\int_0^{\pi/2} \int_0^{\pi} 2E(\alpha, \beta) \sin \beta \cos \beta \, d\alpha \, d\beta}{\int_0^{\pi/2} \int_0^{\pi} 2E(\alpha, \beta) \sin \alpha \sin^2 \beta \, d\alpha \, d\beta}, \quad J_3 = \frac{\int_0^{\pi/2} \int_0^{\pi} 2E(\alpha, \beta) \sin \alpha \sin^3 \beta \, d\alpha \, d\beta}{\int_0^{\pi/2} \int_0^{\pi} 2E(\alpha, \beta) \sin \beta \cos \beta \, d\alpha \, d\beta}
\]

where $E(\alpha, \beta)$ is a probability density function of $N_i$ in a three-dimensional case and can be calculated from the fabric diagrams as shown by Figs. 11 and 12 in the previous paper (Oda and Konishi, 1974).

Theoretical values of $\tan \phi$ and $\frac{\tau}{\sigma_N}$ can be calculated on the basis of the interparticle friction angle $26^\circ$ of sand. The maximum principal stress direction thus determined is shown by $\sigma_1$ in Fig. 6.b. It must be noted that the normal $N_i$ has a tendency to concentrate in the narrow domains enclosed by a dotted lines in these figures and the centers of the domains accord with the maximum principal stress direction.

Putting the values of $\tan \phi$ and $K$ into Eq. (16), the value of $\frac{\tau}{\sigma_N}$ is also calculated (Fig. 9). Some calculated values of $\frac{\tau}{\sigma_N}$ are less than the experimentally determined ones, especially in the range of 0 to 0.5. This is probably due to the fact that we cannot exactly determine a true homogeneous shear zone in the deformed granular mass which suffers shear displacement less than 1 mm. This ambiguity results in incorrect estimation of $N_i$-distribution in the deformed granular mass. Nevertheless, it is reasonable to conclude from the result shown in Fig. 9 that the theoretical Eq. (16) was supported by the experimental results throughout the deformation to the residual strength state, irrespective of the difference in the experimental conditions such as the initial void ratio, the initial fabric and the normal stress.

**Fig. 9.** Relations between experimentally determined $\tau/\sigma_N$ values and calculated ones of sand deformed under direct shear

**RELATIONSHIP BETWEEN MOBILIZED STRESS RATIO $\frac{\tau}{\sigma_N}$ AND DIRECTION OF PRINCIPAL STRESS AXES**

It has been mentioned in this paper that the inclination angle $\phi$ becomes large with the increase of the mobilized stress ratio $\frac{\tau}{\sigma_N}$. This fact can be theoretically explained as follows:
Fig. 10. Relation between $\tan \psi$ and $K$. Solid lines show hypothetical ones of Eq. (17) with $\kappa=0.6$, 0.8 and 1.0.

Since both of $\tan \psi$ and $K$ are determined only by the functional form of $E(\beta)$ for a given value of $\phi_p$, it seems to be natural to expect that there is an interrelationship between them. In Fig. 10, the values of $\tan \psi$ are plotted against the values of $K$. The curved lines in the figure show the following hypothetical relation between them:

$$K = \frac{\kappa \tan^3 \phi + \left( (\kappa-1) \tan^2 \left( \frac{\pi}{4} + \frac{1}{2} \phi_p \right) + 1 \right) \tan \phi}{\left( \tan^2 \left( \frac{\pi}{4} + \frac{1}{2} \phi_p \right) + 1 \right) - \kappa \tan^2 \left( \frac{\pi}{4} + \frac{1}{2} \phi_p \right) - 1} \tan^2 \phi$$  \hspace{1cm} (17)

where $\kappa$ represents a parameter determined only by the interparticle friction angle. Since the measured values of $\tan \psi$ and $K$ seem to satisfy approximately the hypothetical relation between them, Eq. (18) can be obtained by inserting Eq. (17) into Eq. (16):

$$\frac{\tau}{\sigma_N} = \kappa \cdot \tan \phi$$  \hspace{1cm} (18)

Roscoe, Bassett and Cole (1967) showed the stress-strain curves and the inclination angle $\phi$ in the same figures (Figs. 9 and 10 in their paper). Experimental data are represented in Fig. 11 as the relation between $\frac{\tau}{\sigma_N}$ and $\tan \psi$. The points in this figure are close to a unique linear line passing through the origin. This fact accords well with Eq. (18).

In the case of triaxial compression test, Rowe (1962 and 1971) and Barden and Khayatt (1966) pointed out that the stress-dilatancy equation can be represented by $R=DK$, where $R=\text{principal stress ratio}$, $\frac{\sigma_1}{\sigma_2}$, $D=\text{dilatancy defined by granular fabric}$ and $K=\text{component of internal friction}$. Oda (1972) obtained the following equation almost identical to the stress-dilatancy equation:

$$\left( \frac{\sigma_1}{\sigma_2} \right) = \frac{S_x}{S_D} \tan^2 \left( \frac{\pi}{4} + \frac{1}{2} \phi_p \right)$$
Fig. 11. Close relation of inclination angle of maximum principal stress axis to mobilized stress ratio

where the value of \( \frac{S}{N} \) depends only on the granular fabric of sand. It is interesting to note that the mobilized stress ratio \( \frac{\tau}{\sigma_N} \) in the simple shear test is also separated into two fundamental components, \( \tan \phi \) representing granular fabric and \( \kappa \) representing frictional component.

CONCLUSIONS

The conclusions are summarized as follows:

1. The distribution law of contact force which is valid to predict the mean value of contact force is successfully applied to obtain the relations between the granular fabric, the orientation of principal stress axes and the stress ratio, \( \frac{\tau}{\sigma_N} \), in the granular mass sheared in the simple shear apparatus.

2. The direction of the maximum principal stress axis in the simple shear tests on granular mass is determined by the quadratic Eq. (15) so as to satisfy the condition of \( \tan \phi \geq 0 \). The principal direction thus determined is nearly parallel to the principal axis of anisotropy with respect to the three-dimensional distribution of \( N_i \) (i.e., the preferred direction of \( N_i \)).

3. The rotation of the principal stress axes during the simple shear test occurs mostly in the process of increasing the stress ratio, \( \frac{\tau}{\sigma_N} \). The inclination angle \( \phi \) of the maximum principal stress axis to the vertical direction is related to the stress ratio by the following equation:

\[
\frac{\tau}{\sigma_N} = \kappa \cdot \tan \phi
\]

where the constant \( \kappa \) is determined only by the interparticle friction angle \( \phi_f \) of granular material.

4. The principal axes of stress and of strain increment do not generally coincide with each other, at least up to the peak stress ratio.

5. The mobilized stress ratio \( \frac{\tau}{\sigma_N} \) in the simple shear test or direct shear test is sepa-
rated into the two fundamental components, one component representing resistance due to the granular fabric and another component representing the frictional resistance.

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NOTATION

\( E(\beta) \) or \( E(\alpha, \beta) \) = probability density of \( N_i \) in two- or three-dimensional
\( \bar{F}_{X1}, \bar{F}_{Y1} \) and \( \bar{F}_{Z1} \) = interparticle contact forces resolved in reference axes \( X, Y \) and \( Z \), respectively
\( \bar{F}_{X'1}, \bar{F}_{Y'1} \) and \( \bar{F}_{Z'1} \) = interparticle contact forces resolved in principal stress axes \( X', Y' \) and \( Z' \), respectively
\( \bar{J}_1, \bar{J}_2, \bar{J}_3 \) or \( \bar{K} \) = parameter defined by distribution character of \( N_i \) in two- or three-dimensional
\( k_{X'}, k_{Y'} \) or \( k_{Z'} \) = coefficient which determines the relation between contact force and inclination angle of \( N_i \)
\( m \) = number of contacts transmitting contact forces across unit line
\( \bar{n}_i \) = mean value of corrected fringe order which is proportional to mean contact force
\( \overline{dS} \) = mean area of contact surface
\( dv \) = volumetric strain increment
\( X, Y \) or \( Z \) = reference axis
\( X', Y' \) or \( Z' \) = principal stress axis
\( \alpha \) and \( \beta \) = solid angles to define direction of \( N_i \) with respect to reference axes \( X, Y \) and \( Z \)
\( \alpha' \) and \( \beta' \) = solid angles to define direction of \( N_i \) with respect to principal stress axes \( X', Y' \) and \( Z' \)
\( \tau \) = shear strain
\( e \) = longitudinal strain
\( \xi \) = inclination angle of maximum principal strain increment axis to axis \( Z \) (= vertical)
\( \sigma_N \) = normal stress applied in simple shear test
\( \sigma_{X'}, \sigma_{Y'} \) or \( \sigma_{Z'} \) = principal stress
\( \tau \) = shear stress applied in simple shear test
\( \phi \) = internal friction angle
\( \phi_f \) = interparticle friction angle
\( \phi \) = inclination angle of maximum principal stress axis to axis \( Z \) (= vertical)

REFERENCES

2) Gibson, R. E. (1953): “Experimental determination of the true cohesion and true

APPENDIX

Distribution Law of Contact Force
Since the mean forces resolved in the principal stress axes $Y'$ and $Z'$ are given by the distribution law of contact force (Eq. (7)), the mean resultant force $\bar{F}_t$ at the contact can be given by

$$\bar{F}_t = \sqrt{(\bar{F}_{Y'})^2 + (\bar{F}_{Z'})^2}$$
$$= k_{Z'} A \bar{\sigma}_{Z'} \sqrt{\left(\frac{k_{Y'} \bar{\sigma}_{Y'}}{k_{Z'} \bar{\sigma}_{Z'}}\right)^2 \sin^2 \beta' + \cos^2 \beta'}$$

(19)

Inserting the relations given by Eq. (10) and $\beta' = \beta - \phi$ into Eq. (19), we get

$$\bar{F}_t = k_{Z'} A \bar{\sigma}_{Z'} \sqrt{\tan^{-1}\left(\frac{\pi}{4} + \frac{1}{2} \phi_P\right) \cdot \sin^2(\beta - \phi) + \cos^2(\beta - \phi)}$$

(20)

where the value of $\phi$ is given by Eq. (15).

The mean interparticle force ($\bar{F}_n$) at the contact whose normal coincides with the maximum principal stress axis (that is, $\beta = \phi$) is given by

$$F_n = k_{Z'} A \bar{\sigma}_{Z'}$$

Then, the resultant force at any contact can be represented by

$$\bar{F}_t = \bar{F}_n \sqrt{\tan^{-1}\left(\frac{\pi}{4} + \frac{1}{2} \phi_P\right) \cdot \sin^2(\beta - \phi) + \cos^2(\beta - \phi)}$$

(21)

The maximum fringe order ($n_t$) which is observed in the photoelastic isochromatic at a contact between two particles is related to both of the resultant force $F_t$ and the dimension of cylinder such as the radius $r_t$ and the length $l_t$. That is,
A proportional coefficient in Eq. (22) depends only on the character of photoelastic material (according to the experiment, other neighbouring contacts scarcely disturb this relation).

When the radius of cylinder is 0.4 cm, the fringe order is recorded directly at each contact. When the radius of cylinder is 0.3 or 0.5 cm, however, the fringe order is corrected by Eq. (22) to the equivalent fringe order in the case of cylinder of 0.4 cm in radius. The fringe order thus corrected is only proportional to the resultant contact force $F_t$. If we use the mean value of the corrected fringe order in stead of $F_t$ and $F_0$, Eq. (21) can be rewritten as follows;

$$\bar{n}_t = \bar{n}_0 \sqrt{\tan^{-1} \left( \frac{\pi}{4} + \frac{1}{2} \phi_r \right) \cdot \sin^2 (\beta - \psi) + \cos^2 (\beta - \psi)}$$

where $\bar{n}_t =$ the mean value of corrected fringe order at the contacts whose normals incline to the vertical direction at $\beta$ and $\bar{n}_0 =$ the mean value of corrected fringe order at the contacts whose normals accord with the maximum principal stress direction.

The two solid lines in Fig. 12 show the experimentally observed relations between $\beta$ and

![Fig. 12. Effect of inclination angle $\beta$ on intensity of contact force](image-url)
\( \bar{n}_1 \) for the densely and loosely packed granular assemblies sheared up to \( \gamma = 9\% \) and \( = 15\% \), respectively. In these figures, the theoretical lines calculated by Eq. (23) are also represented by the broken lines. The accordance of the experimental results with the calculated ones as shown in Fig. 12 indicates that the distribution law of contact forces from which Eq. (22) were derived must be a basic formula which expresses the mechanical behaviour of granular material.

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