THE BEHAVIOUR OF UNIFORMLY LOADED PILED STRIP FOOTINGS

P. T. Brown* and T. J. Wiesner**

ABSTRACT

Graphical results are presented for load taken by piles, maximum displacement and differential displacement, and maximum positive and negative bending moments, due to uniformly distributed loading applied to a smooth strip footing which is supported by piles and a very deep homogeneous isotropic elastic foundation.

The results presented are for piles whose length/diameter ratio is 50, footings whose width is 5 pile diameters and an incompressible foundation. Corrections for other pile length/diameter ratios, other footing widths and soil Poisson's ratio of zero, are discussed.

Key words: elasticity, footing, pile, soil mechanics
IGC: E2

INTRODUCTION

Analysis of a strip footing under a loadbearing wall may indicate that its behaviour is unsatisfactory because of excessive total or differential settlement. Although the total settlement may be reduced by broadening the footing, and the differential settlement may be reduced by stiffening the footing, in some cases it may prove more economical to reduce total or differential settlement by installation of piles beneath the footing.

The traditional approach to piled footing analysis, which involves the assumption that all the load is taken by the piles, will clearly not lead to maximum economy, and a more exact method of analysis seems desirable. Poulos (1968), Butterfield and Banerjee (1971) and Davis and Poulos (1972) have considered problems of rigid footings supported by piles and Hongladaromph, Chen and Lee (1973) have described a method of analysis for a piled raft of any flexibility. The present paper is an attempt to provide a parametric study of the problem, including the effects of footing and pile stiffness, for use as a basis for design. Despite the idealisation of the problem, it is believed that the relative effect of addition of piles will be indicated with satisfactory accuracy.

The analysis presented is based on idealisation of the proposed problem as a uniformly loaded strip footing supported by a number of identical, equally spaced piles located on the centre-line of the footing, and by an isotropic homogeneous elastic half-space (see Fig. 1). In every case considered, the distance between the end of the footing and the nearest pile is half the pile spacing. The results given are for an incompressible soil (Poisson's ratio $\nu_s=0.5$) but the effect of lower values of Poisson's ratio is discussed briefly.

* Senior Lecturer, Dept. of Civil Engineering, University of Sydney, Australia.
** Research Student, Dept. of Civil Engineering, University of Sydney, Australia.
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METHOD OF ANALYSIS

The pressure between the soil and the footing is assumed to consist of a number of zones of uniform pressure extending over the full width of the footing. The force between a pile and the footing is treated as a concentrated upward force at the centre of a reaction zone. The interaction stresses on the exterior surface of a pile are assumed to consist of zones of uniform vertical stress, composed of one zone of normal stress at the base of the pile and a number of zones of uniform shear stress along the shaft of the pile.

The adopted forms of pressure distribution cannot exactly represent those which would occur if the soil were a perfect linearly elastic continuum. However footing settlements are calculated on the longitudinal footing centreline, where settlements are primarily controlled by pile settlement, not by the form of pressure distribution beneath the footing. Some errors in pile settlement will arise due to inadequate representation of the shear stress singularities which would occur in an ideal soil, but these are only about 3% for a common value of pile stiffness factor \( K_p = 1000 \). When \( K_p = 100 \), this error may rise to 12%, but is likely to be exceeded by the effects of uncertainties in the value of soil modulus \( E_s \).

Real clays do not behave as perfect linearly elastic materials in that they cannot withstand unlimited stress and their stress-strain relationship is non-linear. For the former

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![Diagram of footing and pile layout](attachment:image1.png)

**Fig. 1(a)** The type of problem considered

**Fig. 1(b)** Integration of Mindlin's equation

**Fig. 1(c)** Internal forces and stresses on footing and piles

**Fig. 1(d)** Stresses acting on soil boundaries
reason, some local bearing failure is to be expected at the ends of a footing, but results
given by Brown (1968) suggest that this will cause only a minor redistribution of reaction
and have very little effect on the results. Mattes and Poulos (1969) have shown that
unless a pile is extremely compressible, slip failure on the shaft of a pile will only affect
displacements by a few percent at normal working loads.

The non-linearity of the stress-strain relationship for clay cannot be fully allowed for in
the type of analysis carried out, but as Poulos has shown in Lee (1974), pile settlement pre-
dictions of very satisfactory accuracy can be made on the basis of an elastic modulus back-
figured from a pile test. Where the piles are comparatively stiff (\(K_p > 1000\)), modulus is
probably best backfigured from a pile test since most of the load is taken by the piles.
Where the piles are compressible and most of the load is taken by the footing, it is prob-
ably more satisfactory to use a modulus backfigured from a plate-bearing test.

By application of the integration technique described by Poulos and Davis (1968) to the
expression given by Mindlin (1936) for displacement at any point due to an interior vertical
point load in an elastic half-space, it is possible to determine numerical relationships
between the interaction stress associated with every zone of the footing and piles, and the
vertical soil displacement at the centre of every zone of the footing and piles, caused by
that stress. This technique may be illustrated by considering the determination of the
displacement of point \(i\) (see Fig. 1(b)) due to the effect of a uniform pressure \(p_b\) applied
to the soil by the base of the pile. In this case the vertical displacement due to pressure
\(p_b\) is

\[
\rho_{ib} = \int_0^{2\pi} \int_0^R p_b \text{Idr} \ d\theta
\]

where

\[
I = \frac{1+\nu}{8\pi(1-\nu)E_p} \left[ \frac{Z^2}{R_1} + \frac{3-4\nu + 5-12\nu + 8\nu^2}{R_1} \frac{R}{R_1} \right] + \frac{5-12\nu + 8\nu^2}{R_1} \frac{R}{R_1} \left[ \frac{3-4\nu + 2\nu^2}{R^2} + \frac{6\nu^2}{R^2} \right]
\]

\[
R^2 = \frac{Z^2}{R_1} + a^2 - 2ra \cos \theta
\]

\[
R_1^2 = Z^2 + a^2 + r^2 - 2ra \cos \theta
\]

The integration with respect to \(r\) can be done analytically but the integration with respect
to \(\theta\) is evaluated most readily numerically.

Relations between the stresses and relative displacements of zone centres in the piles and
footing can be determined by considering the piles as simple compression members (Mattes
and Poulos, 1969) and by use of simple bending theory for the footing (Barden, 1962).
Vertical soil and pile or footing displacements are equated at the centre of each zone, and
the resulting equations plus the equation of vertical equilibrium enable the interaction stresses
to be determined. Bending moments and settlements of the footing are then determined
from these values of stress. The values of bending moment obtained are average values at
the cross-section, but values of settlement should strictly apply to the longitudinal centrel-
ine of the footing.

DISCUSSION OF RESULTS

The results presented are for a pile length to diameter ratio \(L_p/d\) of 50, and for
Poisson’s ratio of the soil \(\nu_s\) of 0.5. However the effect of a change in \(\nu_s\), and a
method for extension of the results for other values of \(L_p/d\) are discussed in the next
section. The results are also based on the assumption that the piles are solid and have a
Young’s modulus \(E_p\) equal to the Young’s modulus of the footing \(E\). Hollow piles
are considered for by reducing \(E_p\) so as to maintain the required compression stiffness,
and a difference between \(E_p\) and \(E\) can be allowed for by alteration of the second moment
of area \(I\) of the footing cross-section so as to maintain the required value of the product
EI.

The parameters of the problem used for nondimensionalisation or discussion of the results include the footing length $L$, the footing width $B$, the loading pressure $q$, and the pile spacing $s$, as well as the two relative stiffnesses $K_p = E_p/E_s$, as used by Mattes and Poulos (1969) and

$$K_{st} = \frac{16EI(1-\nu^2)}{\pi E_s L^4},$$

as used by Brown (1973),

When $K_p = 100$ the pile is comparatively compressible and when $K_p = 10000$ the pile is regarded as being stiff. Values of $K_{st} < 0.001$ correspond to flexible footings, and values $> 0.1$ to stiff footings.

(a) Percentage of Total Load Taken by Piles

The percentage of the total load on the footing which is supported by the piles ($F$), is shown in Fig. 2 for a wide range of pile spacings and for $K_p = 100$, 1000 and 10000 and $K_{st} = 0.1$ and 0.001. The results given are for the case of 4 piles and $B/d = 5$. However the plotted curves are almost independent of the number of piles, and for $L/d = 50$ and 9 piles the analysis gives a value of $F$ which is only 2%, higher than that given by Fig. 2. The effects of changes in $B/d$ are discussed in the section of this paper entitled 'Extension of Results'.

![Fig. 2. The percentage load taken by piles](image)

It is interesting to note that all piles take approximately equal loads. The distribution of load between piles is least uniform for stiff footings and larger numbers of piles. When $K_{st} = 0.1$ and $L/d = 50$, with 4 piles the outer piles take 9% more than the average pile load, however with 8 piles, the outermost piles take 28% more than the average and all other piles take from 86% to 97% of the average.

(b) Maximum Footing Displacement

Maximum footing displacement is plotted against $K_{st}$ in Fig. 3 for $K_p = 100$, 1000 and 10000 and various numbers of piles for $B/d = 5$ and $L/d = 50$. The variation of displacement
with footing length can be corrected for by multiplying displacement values for \( L/d=50 \) by a correction factor \( f_L \). The following equations have been found to give \( f_L \) adequately for the range 15<\( L/d<150 \) where \( \alpha=L/50d. \)

\[
\begin{align*}
f_L &= 0.063\alpha^3 - 0.443\alpha^2 + 1.158\alpha + 0.222, & K_p &= 100 \\
f_L &= 0.035\alpha^3 - 0.305\alpha^2 + 1.120\alpha + 0.150, & K_p &= 1000 \\
f_L &= 0.034\alpha^3 - 0.294\alpha^2 + 1.134\alpha + 0.126, & K_p &= 10000
\end{align*}
\]

The comparatively large spacing between corresponding curves for \( K_p=100 \) and 1000 indicates that stiffening (or lengthening) of piles may be more effective in reducing settlement than increasing the number of piles, as was pointed out by Davis and Poulos (1972).

The results shown in Fig. 3 have been replotted in Fig. 4 to show the effect on maximum displacement of a change in number of piles for a given footing. The curves shown in Fig. 4 are for \( L/d=50, B/d=5, K_{st}=0.001 \) and \( K_p=100, 1000, \text{and} \ 10000. \) However the trends are similar for other values of \( K_{st} \), with additional piles being less effective as \( K_{st} \) increases.

![Fig. 3. Maximum footing displacement](image)

![Fig. 4. Relative maximum displacement(Piled footing/Non-piled footing)](image)

(c) Maximum Differential Displacement

For the purpose of this paper, maximum differential displacement is defined as the maximum difference between the displacement of the top of a pile and the displacement of the footing at half the pile spacing on either side of that pile. Maximum differential displacement (\( v \) in the dimensionless form \( vE_I/dq_B \)) is plotted against number of piles in Fig. 5 for \( K_{st}=0.001, 0.01 \text{ and } 0.1 \text{ and } K_p=1, 100, 1000 \text{ and } 10000. \) \( K_p=1 \) corresponds to the case of no piles and in this case the differential displacement is measured between the end of the footing and a point \( L/2n \) from the end, i.e. at what would have been the top of the outer pile. The fact that the curves are reasonably constant in ordinate demonstrates that \( v \) is approximately proportional to \( s \ (s=L/n). \)
Fig. 5. Differential displacement for a footing with $L/d=50$, $B/d=5$

The curves plotted are for $L/d=50$ and $B/d=5$. However results for other values of $L/d$ can be approximately determined by multiplying by the correction factor $f_L$ which is shown in Fig. 6. The factor $f_L$ is almost independent of pile stiffness $K_p$. Correction for other values of $B/d$ is discussed later.

(d) Maximum Positive Moment

Maximum positive (sagging) moment in the footing is plotted against pile spacing in Fig. 7, for a wide range of footing stiffnesses and $K_p=100$, 1000 and 10000. These curves are based on results for 4 piles and $B/d=5$, but will provide approximate results for larger numbers of piles if it is observed that the effect of an increase in number of piles reverses when $K_{st}$ is about 0.005. For example at $s/d=10$ and $K_{st}=0.1$, 0.01 with 8 piles, the bending moment is 17% and 6% higher respectively than for 4 piles and the same value of $s/d$, but is still appreciably less than for a footing without piles (Brown, 1972). When $K_{st}=0.001$, the bending moment drops below the value for 4 piles and $s/d=10$, by 24% for 5 piles and 48% for 8 piles and the same value of $s/d$. When the footing is flexible ($K_{st}<0.001$), bending moments in a piled footing, although small, are larger than in a footing without piles.

(e) Maximum Negative Moment

Maximum negative (hogging) moment in the footing is plotted
against pile spacing in Fig. 8 for \( K_p = 100, 1000 \) and 10000, for the case of 4 piles. When \( K_p = 100 \), negative bending moments rarely arise for \( K_{st} > 0.001 \), and are shown for \( K_{st} = 10^{-4} \) and \( 10^{-4} \). When \( K_p = 1000 \) or 10000, negative bending moments occur in almost all cases and usually have their maximum value for \( K_{st} = 0.001 \). The curves given for these larger values of \( K_p \) are envelopes of maximum values and in extreme cases for long stiff footings or short flexible footings, may overestimate the actual moments by up to 40%.

An increase in the number of piles (for a given \( s/d \)) causes a marked decrease in negative moment. For example increasing the number of piles from 4 to 8 causes the negative moments to decrease by a factor of about 3.

**EXTENSION OF RESULTS**

(a) *Pile Length*

When the piles being considered have an \( L_p/d \) other than 50, Fig. 9 can be used to determine the ‘equivalent’ pile stiffness \( (K_p = E_p/E_t) \) for a pile whose \( L_p/d = 50 \). These curves were derived from analyses of a footing supported by four piles, and as a means of indicating bounds for errors arising from the use of these curves, were applied to a case with two piles. When the actual \( L_p/d = 25 \) the value of \( F \) is 4% (of total load) low and the negative moment over the pile is 40% low. When the actual \( L_p/d = 100 \), the value of \( F \) is 7% high and the negative moment over the pile is 20% high. In both cases the accuracy of displacements is less than 4% low and positive moments are less than

**Fig. 9. Relationship between pile length and stiffness**
3% low.

(b) Footing Width

While the results presented are for $B/d=5$, corrections for other $B/d$ can be based on the following information. Values of $F$ for $B/d=5$ can be adjusted for other $B/d$ by multiplying by the factor $f_b$, which is given in the inset of Fig. 2. These correction factors are virtually independent of $K_{st}$ and $L$.

Maximum displacement/$B$ decreases almost linearly with increasing width and when $K_p = 100$, 1000 decreases by 13% and 7% respectively when $B/d$ increases to 10. Maximum differential displacement/$B$ increases with $B/d$ except when $K_{st}<0.0005$. When $K_{st}=0.01$, 0.001 and $B/d$ increases to 10, differential displacement increases by 23% and 8% respectively.

Maximum positive moment/$B$ varies almost linearly with $B/d$ and an increase of $B/d$ to 10 causes the positive moment to change by +22%, +6%, -12% and -24% for $K_p = 10^{-1}$, $10^{-2}$, $10^{-3}$ and $10^{-4}$ respectively. Maximum negative moment/$B$ decreases with increased footing width and an increase of $B/d$ to 10 causes negative moment to decrease by 65%, 42%, 30% and 22% for $K_{st}=10^{-1}$, $10^{-2}$, $10^{-3}$ and $10^{-4}$ respectively for the case of $K_p = 1000$.

(c) Poisson's Ratio

When Poisson's ratio of the soil is reduced from 0.5 to zero the percentage of load taken by the piles is increased by 20% of the total load, but displacement and differential displacement are not perceptibly changed. The value of maximum positive moment increased by 10% for $K_{st}=0.001$ and decreased by 15% for $K_{st}=0.1$, while for both values of footing stiffness the maximum negative moment was increased by 30%.

CONCLUSIONS

Installation of piles beneath a uniformly loaded strip footing is seen to enable

(i) considerable transfer of load to the piles,
(ii) significant reduction of displacement and differential displacement.
(iii) small reductions in positive moment when the footing is moderately or very stiff.

However addition of piles gives rise to negative bending moments which would not otherwise have arisen, and when the footing is flexible ($K_{st}<0.001$) the positive moments are larger than would have occurred without piles.

Long and/or stiff piles are seen to be more effective than short and/or compressible piles.

Due to the large number of parameters required to specify this type of problem, the results presented cannot be regarded as complete, but they should prove sufficient for purposes of preliminary design.

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NOTATION

$B$ = width of strip footing
PILE FOOTINGS

\[ d = \text{diameter of pile} \]
\[ E = \text{Young's modulus of strip footing} \]
\[ E_p = \text{Young's modulus of pile} \]
\[ E_s = \text{Young's modulus of soil} \]
\[ F = \text{Percentage of total load on footing which is supported by the piles} \]
\[ f_b = \text{correction factor for a change in strip width} \]
\[ f_L = \text{correction factor for a change in strip length} \]
\[ I = \text{moment of inertia of strip footing cross-section} \]
\[ K_p = \text{relative pile stiffness} = E_p/E_s \]
\[ K_{st} = \text{relative strip stiffness} = 16EI(1-\nu_s^2)/\pi E_sL^4 \]
\[ L = \text{length of strip footing} \]
\[ L_p = \text{length of pile} \]
\[ n = \text{number of piles} \]
\[ q = \text{loading pressure on footing} \]
\[ s = \text{pile spacing (centre to centre)} \]
\[ \nu = \text{maximum differential displacement} \]
\[ \alpha = L/50d \]
\[ \nu_s = \text{Poisson's ratio of the soil} \]

REFERENCES


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