SHEAR MODULI OF SANDS UNDER CYCLIC TORSIONAL SHEAR LOADING

Toshio Iwasaki*; Fumio Tatsuoka** and Yoshikazu Takagi***

ABSTRACT

To evaluate degree of reduction in shear moduli of sands with an increase in shear strain amplitude, dynamic soil tests were performed with use of hollow cylindrical samples applying torsional shearing forces. Two types of soil testing equipments, a resonant-column apparatus and a torsional shear apparatus, were employed. It is pointed out that shear moduli at shear strain amplitude of $10^{-4}$ obtained from these two different equipments agree satisfactorily. Furthermore, extents of reduction in shear moduli with an increase in shear strain amplitude are shown for various sands. These data are compared with those presented by other investigators. On the basis of test results, a simplified procedure for predicting reduction in shear modulus with an increase in shear strain amplitude is proposed.

Key words: deformation, dynamic, sand, special shear test, torsion, vibration

IGC: D6/D7

INTRODUCTION

It is essential to properly estimate strain–dependent shear moduli and damping properties of soils in analysing soil–structure interactions and earthquake responses of grounds and soils structures. Shear moduli and damping properties of soils at a shear strain level to be induced in grounds during strong earthquakes may be evaluated as follows. First, shear moduli at shear strain $\gamma$ of around $10^{-4}$ are obtained from in situ seismic surveys. Then, shear strains which are induced in grounds during strong earthquakes are estimated by computation and found to be around $10^{-4}$ or around $10^{-2}$ at the largest, which are much larger than $10^{-4}$. Reduction in shear moduli with increasing shear strain can be obtained by laboratory soil testings. Internal dampings of soils can also be obtained by laboratory testings. To determine a reliable relationship between shear moduli and shear strains, it firstly becomes necessary to know shear moduli at shear strain of around $10^{-4}$ by laboratory soil testings. For this purpose, the resonant–column method is one of the best laboratory soil testing procedures. For alluvial-origin sands, Stokoe and Richart (1973), Cunyn and Fry (1973), Iwasaki and Tatsuoka (1977) and Richart, Anderson and Stokoe (1977) have found good agreements between resonant–column results and in situ seismic survey values. This is due to the fact that the secondary time effect is negligible for sands. At the shear strain level larger than $10^{-4}$, sands behave as non-elastic materials and have rather large

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dampings. For this range of strain, the resonant-column method is not a proper one in evaluating soil properties. To obtain shear moduli and dampings of soils for the shear strain level larger than $10^{-4}$, several devices have been developed. Typical examples are dynamic triaxial test devices (Park and Silver, 1975), box type simple shear devices (Kovacs, Seed and Chan, 1971; Sugimoto, Suzuki, Terada and Watanabe, 1974), NGI type simple shear devices (Theis and Seed, 1968; Silver and Seed, 1971; Hara and Kiyota, 1976), hollow cylinder static torsional shear devices (Hardin and Drnevich, 1972a, 1972b; Sheriff and Ishibashi, 1976) and a ring torsion shear apparatus (Yoshimi and Oh-Oka, 1973).

In the present study, two devices, a resonant-column apparatus applicable for the strain of $\gamma = 10^{-6}$ to $10^{-4}$ and a torsional shear device for the strain of $\gamma = 10^{-4}$ to $10^{-2}$, were employed. For the two types of testings, hollow cylindrical samples with identical cross-sections were used, because nearly uniform shear strain are induced throughout hollow cylindrical specimens. Furthermore, the identical consolidation stress condition was adopted for the both tests.

Presented herein are shear moduli of various kinds of sands for shear strain of $10^{-6}$ to $10^{-2}$ which were obtained by utilizing a Drnevich-type resonant-column apparatus and a torsional shear apparatus. Sands tested involved clean ones and other natural sands which include fine soils to some extent. Consecutive values of shear moduli for the strain range of $\gamma = 10^{-6}$ to $10^{-2}$ were obtained from the two testing procedures. Test results are compared with those by other investigators. A simplified relationship between shear moduli and shear strains is proposed to provide a design engineer with a convenient tool to estimate shear moduli of sands subjected to large strains.

**APPARATUS**

*Resonant-Column Apparatus*

A hollow cylindrical specimen in resonant-column test is 25 cm in height, 10 cm in outside diameter and 6 cm in inside diameter. A sample is fixed at the bottom and torsional forces are applied at the top. The total rotational inertia of the top mass is 0.1926 kg-cm $^2$. The spring constant is 1,000 kg-cm/radian at the amplitude of 0.00043 radians and the spring constant varies slightly with the amplitude. Isotropical confining stress condition is adopted in this study. The consolidation time for each confining pressure is taken about 2 hours. Loosely packed samples were made by different two methods; the spooning method and the raining method. In the spooning method, air-dry sands were poured with a spoon into a mold and a loosely packed saturated sample was made by pouring de-aired saturated sands with a spoon into a mold which was fulfilled with de-aired water in advance. In the raining method, air-dry sands were poured into a mold through a small hole with the diameter of 3 mm from a constant height of 10 cm above the sand surface. In the spooning and tapping method for dense samples, densification of samples made by the spooning method was achieved by tapping the mold with a wooden hammer. In each test, fresh sands were used.

By using the resonant-column apparatus, shear moduli and dampings of sands for the shear strain ranging from about $10^{-6}$ to about $10^{-4}$ were measured. Details of the apparatus and the testing procedures are described in the previous papers (Kuribayashi, Iwasaki and Tatsuoka, 1975; and also Iwasaki and Tatsuoka, 1977).

*Torsional Shear Device*

For obtaining soil properties for strain range of $\gamma = 10^{-4}$ to $10^{-2}$, a torsional shear device was developed. In the device, torsional loads and torsional displacements can be measured directly. Fig. 1 shows the schematic drawing of the apparatus. Photo. 1 shows a close
view of the triaxial cell of the torsional shear device without a lucid cylinder. Soil samples with dimensions of 6 cm in inside diameter, 10 cm in outside diameter and 10 cm in height are tested. The samples are fixed at bottom and torsional loads are applied at the top of the samples. Equal lateral confining pressures are applied to both the outside and the inside of the hollow samples. Axial loads exerted by air pressures are applied through a loading shaft as shown in Fig. 1. The initial stresses in a sample prior to application of dynamic shear stress are under isotropical consolidating conditions where no initial torsional stresses exist. As described in the above, for both resonant-column tests and torsional shear tests, shapes and cross-sectional dimensions of samples, confining stress conditions and the procedures for applying torsional loads are identical. Heights of samples and strain rates in cyclic loadings, however, are different. Accordingly, it is seen that deformation properties for the shear strain of $7 = 10^{-6}$ to $10^{-2}$ evaluated by the two types of devices in this study can be compared under the identical conditions of sample shapes, confining stresses and loading manners. Torsional loads are measured by a torque pickup which is located in the loading shaft just above a sample to eliminate errors due to the friction. Cyclic torsional forces were applied by controlling air-pressure in a cylinder which is arranged above the triaxial cell. The change in air-pressures in the cylinder is rendered into the torsional forces in the loading shaft through gears. The air-pressures are controlled by handoperations using pressure-regulators. Pseudo-sinusoidal torsional forces with the period of about 10 seconds are produced. Torsional displacements at the top of a sample were monitored by a potentiometer which is set up just above the sample as shown in Fig. 1 and Photo. 1. With use of a wheel, cyclic torsional displacements of the sample are first amplified mechanically five times. The rotation of this wheel is transformed into the change in electric current by the potentiometer. Then, the current is amplified and the amplified current is recorded. In this arrangement, the change in the electric current is linearly proportional to the rotational displacement of the wheel, from infinitely small to infinitely large displacements. This arrangement permits a measurement of shear moduli and dampings for a wide range of shear strains. However, from the limitation of mechanical systems, the smallest shear to be measured reliably will be about $\gamma = 5 \times 10^{-4}$. 
The relationship between cyclic torsional forces and cyclic torsional displacements can be recorded in two forms, time histories and hysteresis loops.

TEST PROCEDURES IN TORSIONAL SHEAR TESTS

In torsional shear tests, air-dry and saturated samples were prepared by the procedures identical to those in resonant-column tests. All tests were performed under the drained condition. In each test, fresh sands were used. After isotropically consolidated, a specimen was sheared with a constant shear stress amplitude. Fig. 4 shows one of the typical test results. In this test, the confining lateral pressure $\sigma_{c}$ was 2.0 kg/cm$^2$. On the other hand, the measured initial static axial stress $\sigma_a$ was 2.078 kg/cm$^2$ while the axial load was arranged so that $\sigma_a = 2.0$ kg/cm$^2$. A small difference between $\sigma_a$ and $\sigma_c$ is indispensable because $\sigma_a$

Fig. 2. Definition of shear modulus and damping ratio

Fig. 3. Stress-strain records at 10th cycle of I to VII stages (Shear moduli and damping ratios of this test are shown in Fig. 4)

Fig. 4. Typical test result

and $\sigma_c$ are controlled independently in this apparatus. The differences were small in all the tests conducted. The mean principal stress $\sigma = (\sigma_a + 2\sigma_c)/3$ was adopted to represent a confining stress. In the test shown in Fig. 4, $\sigma$ becomes $(2.078 + 2 \times 2.0)/3 = 2.026$ (kg/cm$^2$). First, ten times of shear tests were conducted for a sample at shear strain of about $7 \times 10^{-4}$. This first stage is represented by the Greek letter I in Fig. 4. The hysteresis loop at the number of cyclic loading $N=10$ at the Stage I is presented at the extremely left in Fig. 3. Then with a larger cyclic shear stress amplitude, the second stage (Stage II) with ten times of shear tests was done. In Fig. 4, seven stages of tests were illustrated. It is seen from Fig. 4 that shear strain amplitudes decrease with the increase in cycles of loadings for stages with larger shear strains. It is also seen from Fig. 4 that as the numbers of
Table 1. Test materials and test programs

<table>
<thead>
<tr>
<th>Material</th>
<th>$G_s$</th>
<th>$D_{10}$</th>
<th>$D_{50}$</th>
<th>$U_c$</th>
<th>$e_{max}$</th>
<th>$e_{min}$</th>
<th>F.C. (%)</th>
<th>Air–Dry or Saturated Preparation</th>
<th>Density</th>
<th>Confining Pressures $p$ (kg/cm$^2$)</th>
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<tbody>
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<td>Toyoura Sand</td>
<td>2.640</td>
<td>12</td>
<td>0.162</td>
<td>1.460</td>
<td>0.960</td>
<td>0.640</td>
<td>A–D</td>
<td>RN Loose</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>S</td>
<td>SP Med-Loose</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A–D, S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>SP+TP Dense, Med-Dense</td>
<td>0.25, 0.5, 1.0, 2.0</td>
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</tr>
<tr>
<td>Model-Test Sand***</td>
<td>2.880</td>
<td>310.0</td>
<td>481.520</td>
<td>930.660</td>
<td>0.5</td>
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<td>Air-Dry</td>
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<td>Ban-nosu Sand</td>
<td>2.670</td>
<td>168</td>
<td>0.251</td>
<td>70.170</td>
<td>821.1</td>
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<td>Spoonsing and Tapping</td>
<td>Dense</td>
<td>Medium-Dense</td>
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<tr>
<td></td>
<td>A</td>
<td>B</td>
<td>2.660</td>
<td>310.923</td>
<td>351.010</td>
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<td>C</td>
<td>2.680</td>
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<td>Ogishima Sand</td>
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<td>250.720</td>
<td>490.36</td>
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<td>090.323</td>
<td>891.420</td>
<td>928.8</td>
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<td>Monterey No. 0 Sand</td>
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<td>320.441</td>
<td>850.560</td>
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</table>

*) F.C.: Content of Fine Soils less than 0.074 mm.
***) SP: Spoonsing Method, SP+TP: Spoonsing and Tapping Method, RN: Raining Method
****) After oven-dried several times.

cyclic loadings increase, shear moduli increase and dampings decrease. Shear moduli at the tenth cyclic loading are taken as the representative values in the following. Variations of shear moduli with respect to the number of cyclic loadings will also be discussed.

Material used and test programs are listed in Table 1. A wide variation of testings were performed on Toyoura Sand. For the other sands, only one case of testing, for air-dry samples prepared by the spoonsing and tapping method and with the confining pressure of 1.0 kg/cm$^2$, was conducted.

DEFINITION OF SHEAR MODULUS AND SHEAR STRAIN IN THE TORSIONAL SHEAR DEVICE

In the torsional shear tests, it is necessary to define shear stress and shear strain, because the distributions of torsional shear stresses and shear strains on a cross section of a hollow cylindrical sample are not uniform when torsionally sheared (Ishihara and Li, 1972). Therefore, in this study, the torsional shear stress $\tau$ was defined as

$$\tau = \tau_{av} = \frac{S}{A} \quad (1)$$

where $\tau_{av}$ means the average shear stress on a cross section of a sample. $S$ in Eq. (1) is the total magnitude of shear stress defined as

$$S = \int_{r_1}^{r_2} \tau_r \cdot 2\pi r \cdot dr \quad (2)$$

in which $\tau_r$ is the shear stress at the radius $r$ from the axis of the sample and $r_1$ and $r_2$ denote the inside radius and the outside radius of the sample, respectively. Furthermore, $A$ in Eq. (1) is the net area of the cross section of the sample, given by

$$A = \pi (r_2^2 - r_1^2) \quad (3)$$
On the other hand, the torque $T$ which is applied to the axis given by

$$ T = \int_{r_1}^{r_2} \tau r^2 \, 2\pi r \, dr $$

(4)

is measured. There could be two extreme states in respect of deformation properties of soils, one is the perfectly linear elastic state and the other is the perfectly plastic state. In the perfectly linear elastic state, shear stress $\tau_r$ is proportional to the radius $r$. In this case, from Eqs. (2) and (4) the value of $S$; $S_r$ is given by

$$ S = S_r = \frac{4}{3} \frac{r_2^3 - r_1^3}{r_2^3 - r_1^3} T $$

(5)

From Eqs. (1), (3) and (5), the average shear stress at the perfectly linear elastic state is obtained as

$$ \tau_{av, r} = \frac{S}{A} = \frac{4}{3\pi} \frac{r_2^3 - r_1^3}{(r_2^3 - r_1^3)(r_2^3 - r_1^3)} T $$

(6)

In the perfectly plastic case, shear stress is identical over the cross section of the sample and given by

$$ \tau_r = \tau_{av, p} $$

(7)

in which $\tau_{av, p}$ represents the shear stress in the perfectly plastic case. From Eqs. (4), (5) and (7), the value of $S$ is obtained as

$$ S = \frac{3}{2} \frac{r_2^3 - r_1^3}{r_2^3 - r_1^3} T $$

(8)

Eventually, the average shear stress in the perfectly plastic state is obtained as

$$ \tau_{av, p} = \frac{S}{A} = \frac{3}{2\pi} \frac{1}{r_2^3 - r_1^3} T $$

(9)

In this study, the approximate values of $r_1$ and $r_2$ are 3.0 cm and 5.0 cm, respectively. With these values, $\tau_{av, r}$ and $\tau_{av, p}$ are given by

$$ \begin{align*}
\tau_{av, r} &= 0.00478 T \\
\tau_{av, p} &= 0.00487 T
\end{align*} $$

(10)

Since the difference between these two values is small, the mean value of the two values was defined as shear stress for the entire range of shear strain, i.e. $\tau = 10^{-4} \sim 10^{-2}$ as

$$ \tau = \frac{\tau_{av, r} + \tau_{av, p}}{2} $$

(11)

The shear strain may be proportional to the distance from the center in a given plane section perpendicular to the axis. Therefore, in this study, the shear strain $\gamma$ can be defined as

$$ \gamma = \frac{\tau_{av, r} + \tau_{av, p}}{2} $$

(12)

In equation (12), $\gamma_{av}$ is the average shear strain, $\theta$ is the an angle of distortion in radian and $H$ is the height of the sample.

In determining values of $\tau$ and $\gamma$ from Eqs. (11) and (12), values of $T$ and $\theta$ were read off from their recorded time histories. Secant shear moduli $G$ were defined as

$$ G = \frac{|\tau|}{|\gamma|} $$

(13)

in which $|\tau|$ and $|\gamma|$ means the amplitudes of shear stress $\tau$ and shear strain $\gamma$ in each loading cycle, respectively.

Dampings were obtained from hysteresis loops of shear stress and shear strain as shown in Fig. 2. Fig. 3 shows typical recorded hysteresis loops. In this figure, $N$ denotes the
number of cyclic loadings for a given cyclic shear stress amplitude.

**SHEAR MODULUS OF TOYOURA SAND**

$G/G^* \sim \gamma$ Relationship

From resonant-column tests on fifteen clean sands, Iwasaki and Tatsuoka (1977) found that shear moduli of clean sands at shear strains of $\gamma = 10^{-6}$, $10^{-4}$ and $10^{-4}$ can be approximately represented by

![Graph of $G/G^*$ versus \( \gamma \) relationship of Toyoura sand (\( p = 0.25 \text{kg/cm}^2 \))](image1)

![Graph of $G/G^*$ versus \( \gamma \) relationship of Toyoura sand (\( p = 0.5 \text{kg/cm}^2 \))](image2)

Fig. 5. $G/G^*$ versus $\gamma$ relationship of Toyoura sand ($p = 0.25 \text{kg/cm}^2$)

Fig. 6. $G/G^*$ versus $\gamma$ relationship of Toyoura sand ($p = 0.5 \text{kg/cm}^2$)
\[ G = 900 \frac{(2.17 - e)^2}{1 + e} \rho^{0.10} \text{ (at } \gamma = 10^{-6}) \] (14)

\[ G = 850 \frac{(2.17 - e)^2}{1 + e} \rho^{0.44} \text{ (at } \gamma = 10^{-3}) \] (15)

and

\[ G = 700 \frac{(2.17 - e)^2}{1 + e} \rho^{0.50} \text{ (at } \gamma = 10^{-4}) \] (16)

where \( G \) is the shear modulus in kg/cm², \( e \) is the void ratio and \( \rho \) is the mean principal stress in kg/cm². Eq. (16) was originally established by Hardin and Richart (1963) for round-grained Ottawa sands \((e < 0.80)\) and for shear strains of \(10^{-4}\) or less. It seems important to compare shear moduli by resonant-column tests with those by torsional shear.

![Fig. 7. \( G/G^* \) versus \( \gamma \) relationship of Toyoura sand (\( p=1.0 \text{ kg/cm}^2 \))](image1)

![Fig. 8. \( G/G^* \) versus \( \gamma \) relationship of Toyoura sand (\( p=2.0 \text{ kg/cm}^2 \))](image2)
tests under the identical conditions of void ratios, confining pressures and shear strains. As both of resonant-column tests and torsional shear tests can give shear moduli at shear strain of $10^{-4}$, both results from these two tests were compared with the shear moduli estimated by Eq. (16). Data from the two tests were plotted in Figs. 5 through 8 where $G/G^*$ versus $\tau$ relationships are presented. In the figures, $G$ is the measured shear modulus and $G^*$ is obtained by substituting void ratio $e$ and confining pressure $p$ into Eq. (16). The values of $e$ and $p$ were measured when shear modulus $G$ was measured for each cyclic loading at each stage. By this procedure, effects of changes in void ratios and confining pressures on shear moduli can be eliminated in each of Figs. 5 through 8. It is seen from Figs. 5 through 8 that at shear strains of around $10^{-4}$, shear moduli by resonant-column tests agree well with those by torsional shear tests evaluated at the 10th cyclic loading for confining pressures ranging 0.25 to 2.0 kg/cm$^2$. Furthermore, it is seen from these figures that the methods of sample preparation adopted in this investigation have little effects on shear modulus of Toyoura sand for both resonant-column tests and torsional shear tests. However, additional studies are necessary to obtain more general conclusions in this respect. It is also seen that consecutive values of shear moduli between $\tau=10^{-6}$ and $10^{-2}$ can be obtained using both of the resonant-column apparatus and the torsional shear device.

It may be induced from the test results shown in Figs. 5 through 8 that for shear strain of $10^{-4}$ or less the effects of strain rate and repeated loading on shear modulus of clean sands are rather small, because both of strain rate and number of repetitions of loading in resonant-column tests are about 500 to 1,000 times as many as those in torsional shear tests. However, with increasing shear strain amplitude the effects of repeated loadings on shear modulus get larger as discussed later. It is also seen from these figures that the $G/G^*$-values of saturated samples are slightly less than those of air-dry samples in the case of resonant-column tests. The difference is rather small.

Shear Modulus and Void Ratio

It is noteworthy that in Figs. 5 through 8 the effect of void ratio on $G/G^* \sim \tau$ relationship is not significant. Fig. 9 shows the relationship between shear modulus for $p=1.0$ kg/cm$^2$ and void ratio at different shear strain levels. In obtaining shear moduli for $p=1.0$ kg/cm$^2$ from the data of tests where confining pressure $p$ was slightly different from 1.0 kg/cm$^2$, measured shear moduli were divided by $p^{-3}$. This correction was 4% at the largest. Solid curves in Fig. 9 represent shear moduli which are proportional to a function $(2.17-e)/(1+e)$. It is seen from Fig. 9 that the change in shear moduli due to the change in void ratio can be evaluated by the function $(2.17-e)/(1+e)$ for a wide range of shear strain ($\tau=10^{-6}$ to $3 \times 10^{-3}$).

Shear Modulus and Mean Principal Stress

To evaluate the effect of confining pressures on shear moduli, shear moduli $G$ divided by $(2.17-e)/(1+e)$ were plotted as functions of $p$ as shown in Figs. 10 and 11. This is to eliminate the effects of void ratios on shear moduli. The values of $G$ in these figures were read off from $G \sim \tau$ curves as shown in Fig. 4. It may be seen in Figs. 10 and 11 that for a wide range of shear strain, there are linear relationships between log $G$ and log $p$ for both of air-dry and saturated Toyoura Sands.

![Fig. 9. $G$ versus $e$ relationship of Toyoura sand](image)
### Table 2. List of other investigations

<table>
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<tr>
<th>No.</th>
<th>Sand</th>
<th>Void Ratio</th>
<th>Device</th>
<th>Sample</th>
<th>$N^{*}$</th>
<th>$\sigma_{o}/\sigma_{r}$</th>
<th>References</th>
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<td>Dry Ottawa No. 20-30</td>
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<td>Resonant-Column (RC)</td>
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<td>Hardin and Richart (1963)</td>
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<tr>
<td>9</td>
<td>Dry Reid-Bedford-Model Sand</td>
<td>0.65</td>
<td>RC(Hall)</td>
<td>Solid Cylinder</td>
<td>—</td>
<td>1.0</td>
<td>Skoglund, Marcuson and Cunny (1976)</td>
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<tr>
<td>10</td>
<td></td>
<td>0.65</td>
<td>RC(Shannon and Wilson)</td>
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<td>11</td>
<td></td>
<td>0.55</td>
<td>RC(CRREL)</td>
<td></td>
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<td>12</td>
<td></td>
<td>0.65</td>
<td>RC(WES)</td>
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<tr>
<td>13</td>
<td>Clean Dry Sand</td>
<td>0.57</td>
<td>Torsional Shear</td>
<td>Hollow Cylinder</td>
<td>10th</td>
<td>1.0</td>
<td>Hardin and Drnevish(1972a)</td>
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<td>14</td>
<td>Dry Ottawa</td>
<td>0.55</td>
<td>&quot;</td>
<td>&quot;</td>
<td>2nd</td>
<td>1.0</td>
<td>Sheriff and Ishibashi (1976)</td>
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<td>15</td>
<td>Dry Seward-Park</td>
<td>0.66, 77</td>
<td>&quot;</td>
<td>&quot;</td>
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<td>16</td>
<td>Dry Niigata</td>
<td>0.88</td>
<td>Ring Torsion</td>
<td>Hollow Cylinder</td>
<td>100th</td>
<td>PS</td>
<td>Oh-Oka (1974)</td>
</tr>
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<td>17</td>
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<td>0.76</td>
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<td></td>
<td>PS</td>
<td>Oh-Oka and Yoshimi (1971)</td>
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<td>0.71</td>
<td>&quot;</td>
<td>&quot;</td>
<td>10th</td>
<td>PS</td>
<td>Yoshimi et al. (1976)</td>
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<td>NGI-Simple Shear</td>
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<td>PS</td>
<td>Silver and Seed (1971)</td>
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<td>0.56</td>
<td>Simple Shear</td>
<td>Box</td>
<td>—</td>
<td>PS</td>
<td>Sugimoto et al. (1974)</td>
</tr>
<tr>
<td>23</td>
<td>Dry Toyoura</td>
<td>0.71</td>
<td>Simple Shear</td>
<td>Circular</td>
<td>10th</td>
<td>PS</td>
<td>Hara et al. (1976)</td>
</tr>
<tr>
<td>24</td>
<td>15 Clean Sands</td>
<td>0.56</td>
<td>RC</td>
<td>Hollow Cylinder</td>
<td>1.0</td>
<td></td>
<td>Iwasaki and Tatsuoka (1977)</td>
</tr>
</tbody>
</table>

* Number of Cyclic Loading (− ; unknown)

** PS ; Plane Strain Condition

This means that shear moduli for $\gamma$ and $\rho$ of interest can be represented by

$$[G]_{\gamma,\rho} = K(\gamma) \left(\frac{(2.17 - e)^2}{1 + e} \right)^{m(\gamma)}$$

(17)

in which $K(\gamma)$ is a function of shear strain and $m(\gamma)$ is the exponent of $\rho$ which is also a function of shear strain. Exponents $m(\gamma)$ of Toyoura Sands derived from Figs. 10 and
Fig. 10. $G/(2.17 - e)^2$ versus $\rho$ relationship of air-dry Toyoura sand

Fig. 11. $G/(2.17 - e)^2$ versus $\rho$ relationship of saturated Toyoura sand

Fig. 12. $m(\gamma)$ versus $\gamma$ relationship of clean sands

11 are shown in Fig. 12 as a function of shear strains. A similar representation was firstly indicated by Silver and Seed (1971). The values of exponents $m(\gamma)$ obtained by other investigators are also indicated by hollow circles in Fig. 12. Numbers beside hollow circles correspond to those in Table 2 in which shear tests on clean sands utilizing torsional resonant-column devices, torsional shear devices, ring torsion devices, and simple shear devices are listed. It is seen from Fig. 12 that with increasing shear strain amplitude the exponent $m(\gamma)$ increases from about 0.4 at the shear strain of around $10^{-6}$ or less to 0.9 at the shear strain of around $5 \times 10^{-4}$.

A SIMPLIFIED REPRESENTATION OF STRAIN-DEPENDENCY IN SHEAR MODULI OF SANDS

$G/[G]_{\gamma=10^{-6}} \sim \gamma$ Relationship for Toyoura Sand

From Eq. (17), shear modulus for Toyoura Sand at the shear strain of $10^{-6}$ is represented by

$$[G]_{\gamma=10^{-6}} = K(\gamma = 10^{-6}) \frac{(2.17 - e)^2}{1 + e} \rho^{m(\gamma = 10^{-6})}$$

(18)
From Eqs. (17) and (18), the ratio of shear modulus at an arbitrary shear strain $\gamma$ to the shear modulus at $\gamma=10^{-4}$ is derived as

$$\left[ \frac{G}{\{G\}_{\gamma=10^{-4}}} \right]_p = \frac{K(\gamma)}{K(\gamma=10^{-4})} \cdot \rho^{m(\gamma)-m(\gamma=10^{-4})}$$

(19)

For $p=1.0$ kg/cm$^2$, Eq. (19) becomes

$$\left[ \frac{G}{\{G\}_{\gamma=10^{-4}}} \right]_{p=1.0}$

(20)

From Eqs. (19) and (20), a simple equation is derived between $\rho=0.25$ kg/cm$^2$ and 2.0 kg/cm$^2$ as

$$\left[ \frac{G}{\{G\}_{\gamma=10^{-4}}} \right]_{p=1.0} = \left[ \frac{G}{\{G\}_{\gamma=10^{-4}}} \right]_{p=1.0} \cdot \rho^{m(\gamma)-m(\gamma=10^{-4})}$$

(21)

The value of $[G/\{G\}_{\gamma=10^{-4}}]_{p=1.0}$ kg/cm$^2$ at a shear strain of interest was obtained from Fig. 7 by dividing $G/G^*$-values at the shear strain by $G/G^*$-value at the shear strain of $10^{-4}$. Thick solid curve in Fig. 13 is the measured relationship between $G/\{G\}_{\gamma=10^{-4}}$ for $p=1.0$ kg/cm$^2$ and shear strain amplitude $\gamma$. Dotted curves in Fig. 13 represent $G/\{G\}_{\gamma=10^{-4}} \sim \gamma$ relationships for $p=0.25, 0.5$ and 2.0 kg/cm$^2$ which were obtained from solid curves shown in Figs. 5, 6 and 8 by dividing $G/G^*$-values at arbitrary shear strains $\gamma$ by $G/G^*$-values at $\gamma=10^{-4}$. Thin solid curves in Fig. 13 represent $G/\{G\}_{\gamma=10^{-4}} \sim \gamma$ relationships which were obtained from Eq. (21) by substituting measured exponents $m(\gamma)$ of Toyoura Sand.
and the measured value of \( G/[G]_{T=10^{-4}} \) \( \rho=1.0 \text{ kg/cm}^2 \) represented by a thick solid curve in Fig. 13 into Eq. (21). As understood from Fig. 13, when \( G/[G]_{T=10^{-4}} \sim \tau \) relationship for \( \rho=1.0 \text{ kg/cm}^2 \) and \( (m(\tau)-m(\tau=10^{-4})) \sim \tau \) relationship are established, Eq. (21) can be a convenient equation to evaluate the \( G/[G]_{T=10^{-4}} \sim \tau \) relationship for a confining pressure \( \rho \) other than 1.0 kg/cm². Fig. 14 shows the effects of the number of cyclic loadings on \( G/[G]_{T=10^{-4}} \sim \tau \) relationships of Toyoura Sand in the case of \( \rho=1.0 \text{ kg/cm}^2 \). The curves for the 2nd and the 5th cyclic loadings in Fig. 14 were obtained from \( G/G^* \sim \tau \) plottings of measured shear moduli as shown in Figs. 5 through 8. Using these curves and Eq. (21), the value of \( G/[G]_{T=10^{-4}} \) for values of \( \rho \) and \( \tau \) of interest may be evaluated for an arbitrary number of cyclic loading between two and ten.

**G/[G]_{T=10^{-4}} \sim \tau \) Relationship for Various Sands**

Before establishing \( G/[G]_{T=10^{-4}} \sim \tau \) relationship for various sands, measured strain-dependency in clean sands were summarized. In Fig. 15 shown are \( G/G^* \sim \tau \) relationships for \( \rho \) of around 1.0 kg/cm² which were obtained by various investigators. The reason why measured shear moduli were divided by \( G^*=rac{700(2.17-e)^2}{1+e} \rho^{0.8} \) in Fig. 15 is that in this way shear moduli by different methods can be compared for the identical values of void ratio and confining pressure. In Fig. 15 also presented is the \( G/G^* \sim \tau \) relationship for \( \rho=1.0 \text{ kg/cm}^2 \) for Bannosu Sand A which was obtained by the procedures as described in Fig. 7. A result for Bannosu Sand A which includes a small amount of fine soils is shown in Fig. 15 for a comparison. It may be seen From Fig. 15 that when limited to the data by torsional resonant–column tests and torsional shear tests (No.3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18 and 24), the scatterings among the data might be considered to be rather small in viewing differences of materials, strain rate, number of cyclic loadings, sample shapes and the other unknown factors. On the other hand, the difference between these data and those by simple shear tests (No. 21, 22 and 23) may be due to the differences in the fashion of shear deformations in samples and consolidation stress conditions. Accordingly, it may be concluded that resonant–column tests and torsional shear tests can give consistent results.

In Fig. 16 shown are measured relationships between \( G/[G]_{T=10^{-4}} \) and \( \tau \) for \( \rho=1.0 \text{ kg/cm}^2 \) of medium dense Bannosu Sands. These curves were plotted by the procedures similar to

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**Fig. 15. G/G* versus \( \tau \) relationship of various clean sands**
those shown in Figs. 7 and 13. As indicated in Table 1, these sands have nonuniform gradings and include fine soils to some extent. The $G/[G]_{\gamma=10^{-4}}$ versus $\gamma$ relationships of the whole sands tested are shown in Fig. 17. As seen in Fig. 17, no significant differences were found among $G/[G]_{\gamma=10^{-4}}$ relationships of these sands. It should be noted, however, that shear moduli of these sands are not identical for the identical values of shear strain amplitude $\gamma$, void ratio $e$ and confining pressure $p$. Iwasaki and Tatsuoka (1977) found that shear moduli of disturbed sands for the identical values of $\gamma$, $e$ and $p$, in general, decrease with the increases in uniformity coefficients and contents of fine soils. Furthermore, the index $B$ was defined to represent the relative largeness of shear moduli for the identical values of $\gamma$, $e$ and $p$. Using this index, shear moduli of sands can be represented by

$$G = A(\gamma) \cdot B \cdot \frac{(2.17-e)^2}{1+e} p^{m(\gamma)}$$

(22)

where $A(\gamma)$ and $m(\gamma)$ are functions only of shear strain and $B$ is not a function of shear strain but is a characteristic value for each sand. $A$ way to obtain the value of $B$ for each sand is reported in the previous paper (Iwasaki and Tatsuoka, 1977). Fig. 18 shows the relationship between the values of $G/[G]_{\gamma=10^{-4}}$ at $\gamma=10^{-4}$, $5 \times 10^{-4}$, $10^{-3}$ and $3 \times 10^{-3}$ and the index $B$ for sands tested for $p=1.0 \text{ kg/cm}^2$. The values of $G/[G]_{\gamma=10^{-4}}$ were obtained from the measured relationship between $G/[G]_{\gamma=10^{-4}}$ and $\gamma$ for each sand. Fig. 18 suggests that the value of $G/[G]_{\gamma=10^{-4}}$ for each value of $\gamma$ is almost constant for a wide range of $B$.  

Fig. 16. $G/[G]_{\gamma=10^{-4}}$ versus $\gamma$ relationship of Ban-nosu sands

Fig. 17. Average $G/[G]_{\gamma=10^{-4}}$ versus $\gamma$ relationship of sands tested
It may be derived from the result shown in Fig. 18 that an average curve of $G/\{G\}_{T=10^{-4}} \sim \tau$ relationship shown in Fig. 17 can be established for a wide variety of sands for practical engineering purposes.

Seed and Idriss (1970) established an average $G/\{G\}_{T=10^{-4}} \sim \tau$ relationship for sand on the basis of various data presented by other investigators. Furthermore, it was shown by Hardin and Drennich (1972a, 1972b) that the shape of the strain-dependency of shear modulus for a particular soil can be adequately represented by a hyperbolic curve. This is defined by the equation

$$\frac{G}{G_{\text{max}}} = \frac{1}{1+T/\tau_{\text{r}}}$$  \hspace{1cm} (23)$$

in which $G$ is shear modulus at any shear strain $\tau$, $G_{\text{max}}$ is the value of $G$ at $\tau=0$, and $\tau_{\text{r}}$ is the reference shearing strain ($\tau_{\text{r}} = \tau_{\text{max}}/G_{\text{max}}$). For clean sands, $\tau_{\text{max}}$ is given by

$$\tau_{\text{max}} = \left[ \left( \frac{1+K_{0}}{2} \right) \cdot \sin \phi \right]^{1/2} \cdot \tau^{1/2}$$

in which $K_{0}$ is the coefficient of earth pressure at rest and $\phi$ is the effective vertical stress. $G_{\text{max}}$ can be obtained from in situ seismic surveys or laboratory soil tests. For round-grained clean sands $G_{\text{max}}$ is given by Eq. (16). In the following Eq. (16) is used to obtain $G_{\text{max}}$ in Eq. (23). Shibata and Soelarno (1975) proposed a more simplified equation than Eq. (23) as

$$\frac{G}{G_{\text{max}}} = \frac{1}{1+1000 \cdot (\tau/p^{0.5})}$$  \hspace{1cm} (25)$$

Furthermore, Sherif and Ishibashi (1976) proposed a different equation on the basis of results by their own torsional shear tests with hollow cylindrical samples as

$$G = 2.8 \cdot \left( \frac{p}{0.070397} \right)^{1.467 + 0.8} \cdot (0.205)^{1/2} \quad (\text{for } \tau < 3 \times 10^{-4})$$

$$G = 2.8 \cdot \left( \frac{p}{0.070397} \right)^{0.85} \cdot (100\tau)^{0.4} \quad (\text{for } \tau \geq 3 \times 10^{-4})$$

$$\frac{p}{1.0 \text{ kg/cm}^2}$$

**Fig. 18.** $G/\{G\}_{T=10^{-4}} \sim \tau$ versus $B$ relationship of sands tested

**Fig. 19.** Comparison among $G/\{G\}_{T=10^{-4}} \sim \tau$ relationships
In Eq. (26) \( p \) is represented in \( \text{kg/cm}^2 \) and \( \gamma \) is represented in rad./rad. Note that the notations used in Eq. (26) are different from those in the original paper. Fig. 19 shows a comparison of strain-dependency of shear modulus for \( p = 1.0 \text{ kg/cm}^2 \) among the average curve in Fig. 17, the average curve by Seed and Idriss and the other proposed relationships, Eqs. (23), (25) and (26). In Eq. (23), the values of \( e \), \( \phi \) and \( K_s \) should be determined to obtain a particular relationship between \( G/G_{\text{max}} \) and \( \gamma \) for \( p = 1.0 \text{ kg/cm}^2 \). To depict the curve in Fig. 19 from Eq. (23), \( e = 0.65 \), \( \phi = 30^\circ \) and \( K_s = 0.6 \) were assumed as standard values. It may be seen that the differences among four curves is rather small except the curve by Shibata and Soelanno.

**Exponent \( m(\gamma) \sim \gamma \) Relationship**

In evaluating the reduction in \( G \) with increasing shear strain amplitude with use of Eq. (21), another coefficient \( m(\gamma) \) is necessitated. In addition to the data shown in Fig. 12, the values of \( m(\gamma) \) at \( \gamma = 10^{-6} \), \( 10^{-4} \) and \( 10^{-2} \) which were obtained for fifteen clean sands are also plotted in Fig. 20. Properties of these clean sands are reported elsewhere (Iwasaki and Tatsuoka, 1977). A solid curve in Fig. 20 represents the average curve of \( m(\gamma) \) for the whole data shown in Fig. 20. Furthermore, the values of \( m(\gamma) - m(10^{-6}) \) are shown in Fig. 21. Note that in deriving the \( G/G_{\text{max}} \sim 10^{-4} \), the values of \( m(\gamma) - m(10^{-6}) \) are needed. The value of \( m(\gamma) - m(10^{-6}) \) which is obtained from the average curve in Fig. 20 was compared with those which are derived from Eqs. (23), (25) and (26). Two black circles in the figure represent the average values of \( m(\gamma) - m(10^{-6}) \) at \( \gamma = 10^{-4} \) and \( \gamma = 10^{-2} \) which were obtained by resonant-column tests on 33 non-clean sands. Properties of these sands were also presented in the previous paper (Iwasaki and Tatsuoka, 1977). Fig. 21 indicates that the difference among black circles and these curves is rather small and that differences among other curves except the curve by Sherif and Ishibashi is also rather small.

![Fig. 20. \( m(\gamma) \sim \gamma \) relationship](image1)

![Fig. 21. Comparison among \( m(\gamma) - m(\gamma = 10^{-6}) \sim \gamma \) relationships](image2)
When earthquake response analyses are performed for a particular ground, it is desirable to conduct laboratory tests to evaluate $G/[G]_{\tau=10^{-4}}$ or $G/G_{\text{max}} \sim \tau$ relationship for each soil layer. However, this procedure is, in general, rather laborious and sometimes difficult. When such laboratory tests cannot be performed, Eq. (21) may be adopted for sand layers for preliminary studies, with the average curve of $G/[G]_{\tau=10^{-4}}$ for $p=1.0 \text{ kg/cm}^2$ shown in Fig. 17 and the average curve of $m(\tau)$ shown in Fig. 20 being utilized.

CONCLUSIONS

The test results and the analyses presented show that by combining resonant-column tests and torsional shear tests using hollow cylindrical samples, consecutive values of shear moduli of sands for shear strain amplitude ranging $\tau=10^{-4}$ to $10^{-8}$ can be evaluated. On the basis of the results by resonant-column tests and torsional shear tests on various sands, a simplified equation (Eq. (21)) was proposed to represent reduction in $G$ with increasing shear strain amplitude for sands. This equation with the average curves shown in Figs. 17 and 20 may be available in conducting earthquake response analyses.

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NOTATION

\[
e = \text{void ratio}
\]

\[
m(\tau) = \text{exponent of mean principal stress in relation with shear modulus}
\]

\[
p = \frac{1}{3}(\sigma_a + 2\sigma_r); \text{ mean principal stress}
\]

\[
A(\tau) = \text{a function of shear strain in relation with shear modulus}
\]

\[
B = \text{a parameter representing relative largeness of shear modulus}
\]

\[
G = \text{shear modulus}
\]

\[
[G]_{\tau=10^{-4}} = \text{shear modulus at } \tau = 10^{-4}
\]

\[
G_{\text{max}} = \text{shear modulus at } \tau = 0
\]

\[
G^* = 700 \frac{(2.17-e)^2}{1+e} \rho^{0.3}; \text{ the average shear modulus of clean sands at } \tau = 10^{-4}
\]

\[
N = \text{number of cyclic loadings}
\]

\[
\tau = \text{single amplitude shear strain}
\]

\[
\tau = \text{shear stress}
\]

\[
\eta = \text{damping ratio} = \frac{1}{2\pi} \frac{dW}{W}
\]

\[
\sigma_a, \sigma_r = \text{axial and radial stresses}
\]

REFERENCES


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