NONLINEAR ANALYSIS OF A SOIL WITH ARBITRARY DILATANCY BY FINITE ELEMENT METHOD

Takaji Kokusyo

ABSTRACT

Analytical techniques available for numerically solving nonlinear stress-deformation problems of a ground are not applicable to those cases where the highly dilatant materials are involved, as Poisson’s ratio $\nu$ in these analyses exceeds 0.5, which consequently violates positive definite condition of the stiffness matrix used in these analyses.

In a new analytical technique proposed herein volume change of a soil is conceptually divided into two components: consolidation and dilatancy, which are computed independently. Through this procedure the new analysis within basically the same framework as the conventional analytical method can readily be applied to any soil materials with arbitrary dilatant behaviors. Analyses carried out for some typical geotechnical problems employing this new technique reveals the effectiveness of the proposed method.

Key words: stress-strain curve, static, dilatancy, computer application

IGC: E 0/E 2

INTRODUCTION

Nonlinear stress-deformation analysis, in which soil properties are assumed nonlinear elastic and experimentally obtained stress-strain curves are directly employed, has increasingly drawn attention because of its effectiveness and simplicity. The earliest work of the nonlinear analysis was performed by Clough and Woodward (1967) for solving stresses and deformations of an embankment. After demonstrating the significance of taking account of the banking sequence of an embankment through the linear elastic analysis, the authors showed a good agreement between the measured performance of the embankment and the calculated results by the nonlinear analysis. The stress-strain relationship for the fill material used in the analysis were obtained by the triaxial compression tests with different values of lateral pressure. The modulus of deformations, $E$ evaluated as a tangent of the nonlinear axial stress vs. axial strain curve was employed along with Poisson’s ratio, $\nu$, calculated from $E$ by assuming a constant value of the bulk modulus, $K$. On the other hand, Girijavallabhan and Reese (1968), in addition to a settlement analysis of a footing resting on a clay, made the analysis of a model retaining wall translating into a mass of sand. An excellent agreement was demonstrated between the model wall performance and the analysis in which $\nu$ was taken as 0.49 for dense sand and 0.40 for loose sand on the ground that Poisson’s ratio could be assumed to be a constant for small strains. Employing almost the same method Radhakrishnan and Reese (1969) analysed a strip footing on a clay mass, and Desai and Reese (1970) analysed a circular footing on a layered clay.

* Research Engineer of Central Research Institute of Electric Power Industry, 1646 Abiko, Abikoshi, Chiba.

Written discussions on this paper should be submitted before January 1, 1979.
Duncan and Chang (1970) made an extensive study to approximate nonlinear stress-strain relationships of soils using the hyperbolic formulation proposed by Kondner and his coworkers (1963a, b). Variable modulus of deformation, \( E \), was calculated from the hyperbolic function as dependent on the principal stresses, whereas Poisson's ratio, \( \nu \), was assigned a fixed value. For the analysis of a circular footing on a clay, \( \nu \) was taken as 0.48, while in the case of a strip footing embedded in a loose sand \( \nu \) was taken as 0.35 despite the finding that during the triaxial test \( \nu \) changed from 0.1 to 0.65. Duncan and Clough (1971), Clough and Duncan (1971), and Clough (1972) made nonlinear analyses for "soil structure interaction problem" paying special attention to the interface between soil and structures, in which nonlinear soil properties were formulated in an analogous way to Duncan and Chang (1970). In particular the Poisson's ratio not constant but variable as a hyperbolic function of the principal stresses was used in the analysis by Clough and Duncan (1971).

Thus the nonlinear analysis, representative examples of which have been reviewed above, has a great advantage over the linear analysis because of its ability of taking direct account of nonlinear soil properties. However, it should be noted that the nonlinear analysis still stays within the framework of the theory of continuum elasticity, hence has some limitations to be applied to granular soil materials. Smith (1971), after reviewing previous works on stress-deformation analyses, concluded that although some of the analyses coincidentally yielded a remarkable agreement with measured performances, their results might not be reliable in quantitative manner because of the lack of their theoretical bases. It is therefore of much importance to improve the conventional nonlinear analysis so that more realistic soil response to load can readily be simulated. A new nonlinear analysis is proposed here in which any arbitrary dilatant behavior of soil can be reproduced and the stress-path dependency of soil deformation properties can be considered. In Fig. 1 the conventional analysis is compared with the new analytical method with respect to their representations of soil properties.

**THEORY OF THE ANALYSIS**

In the new analysis the volumetric strain, \( \nu \), is assumed to consist of two components corresponding to consolidation, \( \nu^c \), and dilatancy, \( \nu^d \), hence

\[
\nu = \nu^c + \nu^d
\]  

Furthermore, stress, strain, and a constitutive relations between them are divided into volumetric and deviatoric components. The two components of stress can be taken as the mean normal stress, \( \sigma_m' \), and the shear stress, \( \tau \), while corresponding components of strain are the volumetric strain, \( \nu^c \), and the shear strain, \( \gamma \). The bulk modulus, \( K \), obtained from the isotropic consolidation test and the shear modulus, \( G \), obtained from the triaxial compression test with a constant mean normal stress are employed instead of \( E \) and \( \nu \). Accordingly two constitutive equations for soil are written in the incremental form as

\[
\Delta \sigma_m' = K \Delta \nu^c
\]

\[
\Delta \tau = G \Delta \gamma
\]
where $K$ and $G$ denote tangential moduli for stress-strain curves. It should be noted that in Eq. (2) not the total volumetric strain, $\nu$, but a part of it, $\nu^d$, associated with the isotropic consolidation is related with $\sigma_m'$, since $\nu$ also involves the volume strain due to dilatancy, $\nu^d$, as assumed earlier.

Tatsuoka (1972), after making a close study of the test results obtained by the series of elaborate triaxial tests for river sands, introduced the concept of "equi $\tau$ lines". They can be defined on the $\sigma_m' - 2\tau$ stress plane by connecting many points corresponding to different stress path tests yet to the same shear strain. Experimentally these lines form a group of curves, the slopes of which decrease with increasing $\sigma_m'$ as shown in Fig. 2. The uppermost of the equi $\tau$ lines is identical with the failure line which, on the stress plane, has a slope of $M$ defined by the following equation,

$$M = \frac{\frac{2\tau}{\sigma_m'}}{\text{failure}} = \frac{6\sin\phi'}{3 - \sin\phi'}$$

where $\phi'$ stands for the angle of internal friction with regard to the effective stress. The same author, who made no conclusive remarks as to whether equi $\tau$ lines be defined uniquely irrespective of stress paths of triaxial specimen showed some experimental evidence that the stress-strain relationship $2\tau/\sigma_m'$ vs. $\tau$ does not greatly vary for different stress path tests. On the basis of this experimental result, the equi $\tau$ lines are considered independent of stress paths in this research. Also assumed is a coincidence of the equi $\tau$ lines with the yield surface concerned with shear deformation, which will be discussed later.

The stress path corresponding to the isotropic consolidation is expressed by the vector $A$ on $\sigma_m' - 2\tau$ plane in Fig. 2, along which the bulk modulus, $K$, evaluated from the consolidation curve is to be defined. In this analysis the same bulk modulus $K$ is also assigned to the stress path $A'$ which is in parallel with $A$, therefore the equi $\phi$ lines are defined as a group of straight lines perpendicular to $\sigma_m'$ axis. On the other hand, the stress path for the triaxial compression test with a constant mean normal stress, $\sigma_m'$, is represented by the vector $B$ on the stress space in Fig. 2 along which the shear modulus, $G$, is defined. Such definitions of the two material constants, $K$ and $G$, as described above, provide an easier physical interpretation of the manner in which soil material deforms than the conventional way of defining the two constants $E$ and $\nu$. Namely the vector $B$ causes shear strain, $\tau$, and the volumetric strain due to dilatancy, $\nu^d$, whereas the vector $A$ brings about the volumetric strain due to consolidation, $\nu^c$, only.

In the proposed analysis the identity of the equi-$\tau$ lines with the yield loci associated with the shear deformation is assumed. Accordingly, for stress incremental vectors going upward from a current yield locus, both plastic and elastic strains occur along with the simultaneous upward expansion of the yield locus, whereas for downward vectors only elastic rebound occurs. Similarly it may be relevant to assume that yield loci for consolidation are identical with the equi-strain lines for consolidation (Equi-$\nu^d$ lines): namely vertical lines perpendicular to $\sigma_m'$ axis on the stress plane. Thus two different yield conditions are employed for the shear deformation and consolidation. Of great significance in
this analysis is the assumption based on the experimental study by Tatsuoka (1972) that the two yield conditions are mutually independent. These yield conditions are not used here as plastic potentials which lead to the evaluations of incremental plastic strains using the flow rule and the normality in the theory of plasticity. Instead, the proposed analytical method directly takes advantage of the empirical stress-strain curves to calculate strain increments by step by step method.

In the light of the above discussion on the yield conditions, it is possible to consider how material constants be evaluated for the stress incremental vector, $C$ or $C'$, in Fig. 3a heading for an arbitrary direction. A current yield locus, $Y$, which corresponds to equi-$\gamma$ line, $\gamma = \gamma_1$, will expand due to the vector $C$ up to a new yield locus, $Y'$, corresponding to $\gamma = \gamma_1 + \Delta \gamma$ where $\Delta \gamma$ is an increase of the shear strain between the two equi-$\gamma$ lines. Denote $C_A$ and $C_B$ as the two components of the vector $C$ with regard to the change of the mean normal stress, $\Delta \sigma_n'$, and that of the shear stress, $\Delta q (= 2\Delta \gamma)$, respectively. Obviously the virgin consolidation occurs along the vector $C_A$ due to the increase of the mean normal stress, $\Delta \sigma_n'$, if $X$ is the current yield locus for consolidation. The component vector $C_B$ with its magnitude $\Delta q$ can be divided into two parts; $\Delta q_1$ and $\Delta q_2$, as shown in Fig. 3a. The shear stress change $\Delta q_1$ causes elastic shear strain, $\Delta \gamma_1'$, only, because it is located under the current yield locus $X$, while $\Delta q_2$ brings about both the shear strain, $\Delta \gamma_2$, and the volumetric strain due to dilatancy, $\Delta \sigma_d$. From Eq. (3) the shear modulus, $G$, defined along the stress path $B$ in Fig. 2, can be written in the following form.
Similarly shear modulus, $G^*$, assigned to the stress incremental vector with the direction $C$ is given as follows.

$$G^* = \frac{\Delta q}{2(\Delta r + \Delta r')}$$

(6)

The elastic shear strain, $\Delta r_1^e$, is given in terms of the slope of the yield locus, $s$, and the elastic shear modulus, $G^e$, as

$$\Delta r_1^e = \frac{\Delta q_1}{2G^e} = \frac{s \Delta \sigma_m'}{2G^e}$$

(7)

Eq. (5) is changed as

$$\Delta r_2 = \Delta r = \frac{\Delta q_2}{2G} = \frac{\Delta q - \Delta q_1}{2G} = \frac{\Delta q - s \Delta \sigma_m'}{2G}$$

(8)

which is substituted into Eq. (6) to finally yield the following expression for $G^*$,

$$G^* = \frac{s_i s}{(s_i - 1) + G/G^e} G$$

(9)

where $s_i$ stands for the slope of the vector.

$$s_i = \frac{\Delta q}{\Delta \sigma_m'}$$

(10)

When the vector $C$ goes up perpendicular to $\sigma_m'$ axis, $s_i$ is equal to infinity, hence $G^* = G$ by Eq. (9). If the vector $C$ is a tangent of a current yield locus $Y$, $s_i$ is equal to $s$, hence $G^* = G^e$. It is quite obvious that for $s_i$ less than $s$ the elastic modulus, $G^e$, should be assigned to $G^*$. The equation (9) is also applicable to the vector $C'$ in Fig. 3a to evaluate the modified shear modulus, $G^*$. In this case, unlike the vector $C$, unloading for consolidation occurs along with loading for shear stress. Obviously the value of $G^*$ decreases from $G^e$ to zero with counter clockwise rotation of the stress incremental vector $C$ from $C_1$ to $C_3$ as shown in Fig. 3b. For the vector $C$ existing in the interval between $C_3$ and $C_4$, this analysis based on the theory of elasticity can not be applicable, since the decrease of the shear stress yields the increase of the shear strain. It is evident from the above discussion that, except for the direction between $C_3$ and $C_4$, the influence of the direction of stress increment vector on shear modulus, $G^*$, can be taken account in this analysis.

In order to find out a scheme how to make an independent calculation for the volumetric strain of dilatancy, $\nu^d$, consider basic equations on which the finite element analysis is based. By introducing a matrix $[B]$ which relates the strain and the nodal displacements as

$$[\varepsilon] = [B][\delta]$$

(11)

the following equilibrium equation relating the stress vector $[\sigma]$ of an element to the nodal force vector $[F]$ can be drawn if body force and initial stress do not exist.

$$[F] = \int_{\Omega} [B]^T [\sigma] d\nu$$

(12)

The stress vector $[\sigma]$ and the strain vector $[\varepsilon]$ can be divided into the volumetric components $[\sigma_v]$, $[\varepsilon_v]$ and the deviatoric components $[\sigma_d]$, $[\varepsilon_d]$.

$$[\sigma] = [\sigma_v] + [\sigma_d]$$

(13)

$$[\varepsilon] = [\varepsilon_v] + [\varepsilon_d]$$

(14)

Under the plane strain condition these components are written in the following form.

$$[\sigma]^T = \{\sigma_x, \sigma_y, \tau_{xy}\}$$

(15)
\[ \{ \varepsilon_s \}^T = \{ \varepsilon_{m}', \varepsilon_{m}', 0 \} \]  
(16)

\[ \{ \varepsilon_s \}^T = \{ \varepsilon_x - \varepsilon_{m}', \varepsilon_y - \varepsilon_{m}', \tau_{xy} \} \]  
(17)

\[ \{ \varepsilon \}^T = \{ \varepsilon_x, \varepsilon_y, \tau_{xy} \} \]  
(18)

\[ \{ \varepsilon_\varepsilon \}^T = \left\{ \frac{\nu}{2}, \frac{\nu}{2}, 0 \right\} \]  
(19)

\[ \{ \varepsilon_\varepsilon \}^T = \left\{ \varepsilon_x - \frac{\nu}{2}, \varepsilon_y - \frac{\nu}{2}, \tau_{xy} \right\} \]  
(20)

With the material matrix \([D]\) for the plane strain condition,

\[
[D] = \begin{bmatrix}
\frac{4G^* + 3K}{3} & \frac{3K - 2G^*}{3} & 0 \\
\frac{3K - 2G^*}{3} & \frac{4G^* + 3K}{3} & 0 \\
\text{symmetric} & & G
\end{bmatrix}
\]  
(21)

incremental stress-strain equations for the volumetric and the deviatoric components are given by

\[
\{ \Delta \sigma \} = [D] \{ \Delta \varepsilon_\varepsilon \} \]

\[ \{ \Delta \sigma \} = [D] \{ \Delta \varepsilon_\varepsilon \} \]  
(22)

Eq. (23) may need some further discussions. For soil material the volume change during shear deformation has a significant effect on shearing resistance, thus the fundamental stress-strain relation should be given by

\[
\Delta \tau = G' \Delta \tau + \frac{\sigma_{m}' \Delta \varepsilon^d}{\Delta \tau} \]  
(3')

rather than Eq. (3). Here \(G'\) is the shear modulus for the same soil having the critical void ratio which causes no dilatancy. On the other hand, usual definition of shear stress–shear strain relationship is like Eq. (3) where no term of dilatancy is explicitly involved despite that dilatancy in fact occurs along with shear deformation. Eq. (3) compared with Eq. (3') implies that \(G\) is expressed as

\[ G = G' + \frac{\sigma_{m}' \Delta \varepsilon^d}{(\Delta \tau)^2} \]  
(24)

and already involves the influence of dilatancy. Consequently Eq. (3) can be considered valid as the stress-strain relationship with regard to the shear deformation, hence the analogous relationship Eq. (23) under the plane strain condition. It should be noted that in this analysis the dilatancy volume change is considered to occur as the result of the shear deformation and does not appear explicitly in the stress-strain relationship. Its contribution is included only implicitly in the magnitude of the shear modulus, \(G\), in the material matrix.

Adding Eq. (22) to Eq. (23) gives

\[
\{ \Delta \sigma \} = [D] \{ \Delta \varepsilon_\varepsilon + \Delta \varepsilon_\varepsilon \} = [D] \{ \Delta \varepsilon \} - \{ \Delta \varepsilon_\varepsilon \} \]

\[ \{ \Delta \sigma \} = [D] \{ \Delta \varepsilon_\varepsilon + \Delta \varepsilon_\varepsilon \} = [D] \{ \Delta \varepsilon \} - \{ \Delta \varepsilon_\varepsilon \} \]  
(25)

in which \(\Delta \varepsilon_\varepsilon\) and \(\Delta \varepsilon^d\) represent the strain vectors corresponding to consolidation and dilatancy respectively.

\[
\{ \Delta \varepsilon_\varepsilon \}^T = \left\{ \frac{\Delta \varepsilon^d}{2}, \frac{\Delta \varepsilon^d}{2}, 0 \right\} 
\]  
(26)

\[
\{ \Delta \varepsilon^d \}^T = \left\{ \frac{\Delta \varepsilon^d}{2}, \frac{\Delta \varepsilon^d}{2}, 0 \right\} 
\]  
(27)

Eq. (25) implies that not the total strain but the remainder subtracted by the dilatancy strain, \(\{ \Delta \varepsilon_\varepsilon \}\), are related with the corresponding stress. Substituting Eq. (25) into Eq.
Fig. 4. Schematic representation of dilatancy under constant mean stresses

(12), one obtains the following expression,

$$\{\Delta F\} + [\Delta F^d] = \int_v \left[ B \right]^T [D] \{\Delta e\} dv$$

where $\Delta F^d$ represents a fictitious nodal force vector attributed to the dilatancy given by Eq. (29).

$$\{\Delta F^d\} = \int_v \left[ B \right]^T [D] \{\Delta e,^d\} dv$$

If the dilatancy volume change, $\Delta v^d$, is a known quantity, so is the fictitious force vector, $\{\Delta F^d\}$, hence Eq. (28) becomes a solvable equation.

Dilatancy can in principle be picked up separately from the volume change for consolidation by performing the constant $\sigma_m'$ shear test under the drained condition. Fig. 4 schematically illustrates the volume change during the constant $\sigma_m'$ triaxial test for sand on the $\tau$ vs. $v^d$ plane. Through the direct use of experimental curves relating $\tau$ and $v^d$ as shown in Fig. 4, it becomes possible to solve Eq. (28) by resorting to the following iterative scheme.

(i) At the first step of a certain load increment, set $\{\Delta F^d\} = 0$, and solve the simultaneous Eq. (28) for all elements.

(ii) Calculate the increment of the dilatancy volume change, $\Delta v^d$, from the value of $\Delta \tau$ and $\Delta \sigma_m'$ by the method schematically illustrated in Fig. 4.

---

Table 1. Material constants of the dense & loose sand

<table>
<thead>
<tr>
<th></th>
<th>Dense Sand</th>
<th>Loose Sand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Void Ratio</td>
<td>0.51~0.54</td>
<td>0.72~0.74</td>
</tr>
<tr>
<td>$G_{ts}(kg/cm^2)$</td>
<td>290</td>
<td>130</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.51</td>
<td>0.62</td>
</tr>
<tr>
<td>$R_f$</td>
<td>0.95</td>
<td>0.93</td>
</tr>
<tr>
<td>$M$</td>
<td>1.91</td>
<td>1.53</td>
</tr>
<tr>
<td>$\phi_f$</td>
<td>48°</td>
<td>38°</td>
</tr>
<tr>
<td>$C_z$</td>
<td>0.023</td>
<td>0.042</td>
</tr>
<tr>
<td>$C_c$</td>
<td>0.007</td>
<td>0.014</td>
</tr>
</tbody>
</table>

---

Fig. 5. The computer flow chart
(iii) Calculate the fictitious nodal force vector, \( \{ \Delta F^d \} \), by using Eqs. (27) and (29).

(iv) Return to the step (ii) and obtain a new value of \( \Delta \theta^d \). If the relative difference between the old and the new value of \( \Delta \theta^d \) is within a certain allowance, stop the iteration, otherwise repeat the steps (ii) to (iv).

The computer program whose flow chart is described in Fig. 5 employs essentially the same numerical technique as the conventional method except for the iterative procedure for dilatancy component. The execution time of the computer program is mostly consumed by the matrix inversion routine which is carried out once in each load increment, and the iterative procedure for dilatancy claims only several percent of it.

Stress–strain relationships used in this analysis are based on the data obtained by the Tatsuoka's tests (1972) under the constant \( \sigma_m' \) condition. Two kinds of soil density were chosen from the data for this analysis: a “dense” sand with its initial void ratio \( e_i = 0.51 \) to 0.54 and a “loose” sand with 0.72 to 0.74. The original shear stress–shear strain curves can be modified using the hyperbolic formulation proposed by Kondner (1963a, b) in the form of the following equation.

\[
\tau = \frac{\gamma}{1 + G_0 (\sigma_m')^n + 2 R_f \tau / M \sigma_m'}
\]  

(30)

Differentiating this by \( \tau \), one gets the tangential shear modulus \( G \) which, after some calculations, is written as

\[
G = G_0 (\sigma_m')^n \left( 1 - \frac{R_f \tau}{M \sigma_m'} \right)^2
\]

(31)

where \( R_f \) is a constant of proportion between the shear stress at failure, \( \tau_f \), and the asymptotic shear stress \( \tau_{ult} \).

\[
R_f = \frac{\tau_f}{\tau_{ult}}
\]

(32)

By approximating the virgin consolidation curve by

\[
e - e_0 = - C_v \log \left( \frac{\sigma_m'}{\sigma_m'} \right)
\]

(33)
in which \( e_0 \) and \( \sigma_m' \) represent the void ratio and the mean stress at the start of the analysis, the bulk modulus \( K \) can be calculated by the following equation.

\[
K = \frac{2.303 (1 + e_0)}{C_v} \sigma_m'
\]

(34)
or for rebound curves

\[
K = \frac{2.303 (1 + e_0)}{C_v} \sigma_m'
\]

(35)

where \( C_v \) and \( C_v \) are compression indices. Material constants for the dense sand and the loose sand in the above equations are tabulated in Table 1.

**PRIMARY ANALYSIS**

In order to demonstrate the fundamental effectiveness of this analytical method, a four-elements model with uniform loading and simple boundary conditions was analysed using the material properties obtained for the dense sand and the loose sand mentioned in the previous chapter. Fig. 6 represents a quarter of the test specimen of the sand under the plane strain condition idealized by the finite elements model. In Fig. 6 (i) the specimen initially consolidated with the mean normal stress, \( \sigma_m' \), equal to 2 kg/cm\(^2\) is sheared under the constant \( \sigma_m' \) condition with the increase of vertical stress and the simultaneous decrease of lateral stress. Analysed \( \tau \) vs. \( \tau \) relationships for the sands of the two different densities
are shown in Figs. 8 (a) and (b) along with the theoretical curves. The $\tau$ vs. $\varphi$ relationships obtained from the same analysis are shown in Figs. 9 (a) and (b) by light circles. In Fig. 6 (ii) the specimen with the initial mean normal stress, $\sigma_m'$, equal to 1 kg/cm$^2$ is uniaxially loaded with vertical stress. The analysed $\sigma_m'$ vs. $e$ relationships for the dense sand and the loose sand are shown in Figs. 7 (a) and (b), the $\tau$ vs. $\varphi$ relationships in Figs. 8 (a) and (b), and the $\tau$ vs. $\varphi$ relationships in Figs. 9 (a) and (b) by dark circles. Analytical results of case (i), in which the increment of shear stress in each step was set.
0.1 kg/cm², show a striking agreement with the experimental curves. In case (ii) the increment of the shear stress in each step was taken as 0.2 kg/cm², and the mean normal stress was increased at the same time by 0.2 kg/cm². The dark circles pointed out by the arrow marks in Figs. 8 (a) and (b) corresponding to the point where the mean normal stress, σ_m', takes 2 kg/cm² and 3 kg/cm² well agree with the experimental curves along which σ_m' has the constant magnitude of 2 and 3 kg/cm². In the light of the previously discussed assumption that the shear strain is uniquely determined by the equi-γ lines on the stress plane, and independent of the stress paths, this agreement is thought to prove the accuracy of this analysis.

ANALYSES OF PRACTICAL PROBLEMS

In order to prove the applicability of this analytical method to more practical problems in the geotechnical engineering field, two typical boundary value problems were considered herein, namely the problem of a strip footing in the ground, and the embankment problem. In all the problems the ground was assumed to consist of uniform sand, the material properties of which have been described in the previous section.

The analysis of a Strip Footing in the Ground

A strip footing embedded in the uniform sand layer loaded with a vertical load was considered. The profile of the ground was idealized by the finite element meshes illustrated
Fig. 10. Finite elements idealization of a uniform sand layer loaded by a strip footing

Fig. 11. Analyzed load vs settlement curves for two nodal points

Fig. 12. Shear strain vs volumetric strain relationships for three elements

Contours of strains and Stress Ratio

Fig. 13. Contours of strains (%) and stress ratio
in Fig. 10 taking the advantage of the symmetry and the plane strain condition. The side boundary of the ground and the interface between the footing and the ground were assumed to be smooth according to an analogous analysis on a model footing by Duncan and Chang (1970). Furthermore the bottom of the footing was supposed to give a uniform load to the ground. Linearly increasing hydrostatic stress distributions $\sigma_m'$, as shown in Fig. 10, was supposed to exist before the application of the footing load. The sand has the initial void ratio determined from the virgin consolidation curve corresponding to $\sigma_m'$ of each depth of the ground.

The increment of the load intensity on the bottom of the footing was taken 0.5 kg/cm² at each step and totally forty incremental calculations were made to give the final footing load of 20 kg/cm². Fig. 11 shows the analysed result of the settlement vs. load intensity relationship of the two typical mesh points under the footing indicated in Fig. 10. Fig. 11 (a) is the result corresponding to the ground consisting of the dense sand, and Fig. 11 (b) of the loose sand. Both relationships are remarkably linear except for the initial part of the loading. The distributions of the maximum shear strain for the maximum footing load of 20 kg/cm² are shown by equi-strain contours in Fig. 13 (a) for the dense sand and (b) for the loose sand. The concentration of the shear strain in the vicinity of the corner of the footing is noteworthy. Fig. 12 presents the development of the volumetric strain with the increase of the shear strain for three representative elements shown in Fig. 10 in the case of the dense sand. The chain dotted line in the figure represents the volumetric strain due to consolidation, $\nu'$, while the dashed line represents the dilatancy volumetric strain, $\nu''$. They are summed up to make the total volumetric strain, $\nu$, illustrated by the solid line. It can be seen in the figure that the dilatancy component of the volume change always makes soil to expand whereas consolidation causes volume decrease in the elements No. 49 and 50. Consequently, as indicated by the solid line, the total volumetric strain $\nu$ experiences the initial contraction and the following dilatation with the increase of the shear strain. It is evident that the volume changes after taking the negative value (expansion) can not be simulated by the conventional analysis in which soil must contract with increasing mean normal stress so long as $\nu$ is less than 0.5. On the other hand, in the element No. 55 the unloading for consolidation takes place due to the decrease of the mean normal stress, hence $\nu'$ takes a negative value. Since the volumetric strain due to dilatancy, $\nu''$, will take a negative value, the volume change in this element will show considerable expansion. The distribution of $\nu''$ at the ultimate footing load of 20 kg/cm² are shown in Fig. 13 (b) and analogous distributions of the total volumetric strain, $\nu$, are also presented in Fig. 13 (c) for the two sands. A remarkable soil expansion occurs near the corner of the footing, and the extension of the expansion area is far larger for the dense sand than for the loose sand. On the contrary, soil contraction due to consolidation occurs beneath the footing load.

![Fig. 14. Stress paths for five elements](image-url)
Fig. 14 shows the analysed stress paths corresponding to the dense sand illustrated on the stress plane, \( \sigma_m' \) vs. \( 2\tau \), for the five representative elements indicated in Fig. 10. Also shown with thin lines are the equi-\( \tau \)-lines. It is evident from the figure that for the soil just beneath the footing load the slope of the stress path is relatively small, hence consolidation is more dominant than the shear deformation. For the soil near the corner of the footing the path becomes steeper and the shear deformation prevails, and so does dilatancy. A soil element just above the footing corner follows a quite different path, and reaches the failure line at an earlier stage of the loading, since it experiences the decrease of the mean normal stress along with the simultaneous increase of the shear stress. The distributions of the stress ratio, \( 2\tau/\sigma_m' \), for the ultimate loading stage are presented in Fig. 13 (d) for the dense sand and the loose sand. Obviously the failure starts near the corner of the footing and a stabler area exists just beneath the footing.

It is concluded that in the footing problem, stress conditions in the ground as well as the corresponding volume change behaviors including dilatancy and consolidation are very variable from one element to another. The conventional nonlinear analysis could not fully reproduce such complicated situations. Furthermore a soil with a high density could not exactly be simulated by the conventional nonlinear analysis where Poisson's ratio should be less than 0.5.

The Analysis of an Embankment

A number of analytical studies have been reported up to date, most of which emphasize that the multi-lift analysis where the banking procedure is simulated by changing step by step the configuration of the cross-sectional area is essential to obtain reliable prediction of deformations. On the other hand Poulos et al. (1972) showed that one-lift linear analysis can give a good estimate of deflections of an embankment provided that the definition of the embankment settlement is properly chosen. Lee et al. (1975) further proved that the one-lift analysis is accurate enough and much more economical than the multi-lift analysis, if only the stress in the embankment is concerned.

The analysis performed here is the one-lift analysis and the effect of the variable cross-section was excluded. Thus it was supposed that the embankment has the complete cross-section from the start of the analysis, and the weight of the fill material is gradually loaded on each mesh point. The profile of the analysed embankment, 18 meter in height, and with the slope of 1 to 2 was idealized by finite element meshes as shown in Fig. 15. The plane strain condition as well as the fixed boundary along the base of the embankment was also assumed. The fill material made of the dense sand mentioned earlier was assumed to have the initial mean normal stress of 0.05 kg/cm² due to the capillary force and

Fig. 15. Finite elements representation of an earth fill
experience the virgin consolidation from that low stress state. The unit bulk weight of the fill material was taken 2.0 ton/m³, and forty step by step analyses were carried out with the nodal forces corresponding to one fortieth of the weight applied at each step.

The distributions of the maximum shear strain, \( \gamma \), at the ultimate stage of the analysis are presented in Fig. 17(a). A larger shear deformation occurs in the internal part of the embankment. Fig. 16 shows the variation of the volumetric strain, \( \nu \), along with the increase of the shear strain for three representative elements indicated in Fig. 15. In all three elements \( \nu^d \) takes the negative value, hence the dilatancy effect of the dense sand makes the soil to expand. However, the volume contraction due to consolidation is considerably larger than dilatancy, and thus the soil eventually yields large contraction. The distributions of \( \nu^d \) and \( \nu \) at the completion of the analysis are shown for the dense sand in Figs. 17 (b) and (c). Volume contraction prevails everywhere because of the large amount of the consolidation although the dilatancy effect causes soil expansion.

Fig. 18 shows the stress paths in five representative elements of the embankment made of the dense sand plotted on the \( \sigma_m' \) vs. \( 2\tau \) stress plane. Both larger mean normal stress and larger shear stress are exerted in the internal than near the surface of the fill. Nonetheless the ratios between the increments of the two stresses are almost the same irrespective of the locations of the individual elements and the stress paths approximately coincide with each other. There exists a remarkable difference between this analysis and the previous one for the footing problem such that in the embankment problem the stress path is rather simple and does not greatly differ from one place to another. This fact, along with another fact that the soil contracts everywhere in the fill even if the fill material is a dense sand, implies that the conventional nonlinear analysis may well be applicable to the embankment.

Fig. 17. Contours of strains (\%) and stress ratio
problem. Namely the analysis using the modulus of deformation, $E$, obtained by the conventional triaxial test (more preferably by the constant stress ratio test) and Poisson's ratio, $\nu$, smaller than 0.5 may give a reasonable result to this problem. It should be noted, however, that the situation may be quite different for modern fill dams constructed using heavy compaction machineries which can attain such a low void ratio of the fill material as 0.2 to 0.3. Such a material possibly show smaller consolidation and significantly larger dilatation than the dense sand with the void ratio equal to 0.53. If that is the case, volume expansion may occur throughout the fill, hence this new analytical technique will be necessary instead of the conventional analysis.

SUMMARY AND CONCLUSIONS

A new nonlinear analytical technique proposed herein can be applicable to soils with arbitrary dilative or contractive behavior, whereas the applicability of the conventional analysis is limited to those soils, Poisson's ratio of which is smaller than 0.5. Another advantage of the new analysis is that it can modify the value of the shear modulus, $G$, according to the direction of the stress incremental vector. These new features do not greatly deteriorate the computational efficiency such as the computation time or core memory requirement.

The fundamental applicability of this analytical scheme to general geotechnical problems have been confirmed by the two simplified yet practical analyses.

ACKNOWLEDGEMENT

The author wishes to acknowledge his deep sense of gratitude to Prof. F. Tatsuoka of Tokyo University for various kinds of encouragement and help on the process of this research as well as for the good quality triaxial test data used in this research.

NOTATION

- $\sigma_m'$ = mean normal stress
- $\tau$ = maximum shear stress
- $\{\sigma\}$ = stress vector for the plane strain condition
- $\{\sigma_0\}$ = hydrostatic stress vector for the plane strain condition
- $\{\sigma_v\}$ = deviatoric stress vector for the plane strain condition
- $\nu$ = volumetric strain
- $\gamma$ = maximum shear strain
$\varepsilon^c$ = volumetric strain due to consolidation
$\varepsilon^d$ = volumetric strain due to dilatancy
$\{\varepsilon\}$ = strain vector for the plane strain condition
$\{\varepsilon_c\}$ = volumetric strain vector for the plane strain condition
$\{\varepsilon_d\}$ = deviatoric strain vector for the plane strain condition
$\{\varepsilon^c\}$ = volumetric strain vector due to consolidation
$\{\varepsilon^d\}$ = volumetric strain vector due to dilatancy
$E$ = modulus of deformation
$\nu$ = Poisson's ratio
$G$ = shear modulus
$K$ = bulk modulus
$M = (2\tau/\sigma_m)$ at failure
$\phi'$ = internal friction angle
$G'$ = shear modulus of a soil with critical void ratio
$G^*$ = modified shear modulus
$G_x$ = elastic shear modulus
$C_x$ = compression index for initial consolidation
$C_y$ = compression index for rebound and reconsolidation
$s$ = gradient of yield locus
$s_i$ = gradient of stress incremental vector
$e$ = void ratio
$e_o$ = initial void ratio
$\{F\}$ = nodal force vector
$\{\delta\}$ = nodal displacement vector
$\{B\}$ = matrix relating nodal displacements to strain
$\{D\}$ = material matrix

REFERENCE


(Received July 23, 1977)