TENSILE STRENGTH, TENSION CRACKS, AND STABILITY OF SLOPES

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ABSTRACT

The paper describes a procedure for the evaluation of the effect of tensile strength on stability of slopes. The procedure is based on variational approach to limiting equilibrium problems. Based on this approach it was found possible to estimate the depth of tension cracks, and their effect on slope stability in a consistent manner. Formulas and charts which enable the determination of the depth of tension cracks and stability number for every combination of slope angle, mobilized friction, and tensile strength are given.

Based on this analysis the following conclusions may be presented:

a) The maximum depth of tension cracks is 25% of the slope height, and this value occurs for vertical slopes.

b) The increase in stability number (decrease in critical height), which corresponds to a maximum crack is of the order of 20%. This is valid for cracks which are not filled with water.

Key words: stability analysis, tensile strength

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INTRODUCTION

Cracks are often found at the crest of soil and rock slopes. The existence of such cracks indicates that in a certain zone the tensile stress exceeds the tensile strength of the medium. Tension cracks affect stability of slopes in a number of ways:

a) By shortening the length of the slip surface, thus reducing the resistance to failure.

b) The water pressure acting on the crack face constitutes an additional driving force contributing to failure.

c) The water in the crack tends to soften the soil, degrading its strength properties.

It is therefore desirable to be able to predict the depth to which cracks will penetrate, and estimate their effect on stability. Particularly, it is of interest to determine the conditions under which no cracks are expected to develop, thus avoiding altogether the softening effect, which is difficult to analyze quantitatively. Moreover, when the depth of cracks is known, it is not difficult to make a satisfactory allowance for their effect in the stability analysis; it is the determination of the crack depth that remains the weak link in slope stability computations. In fact Taylor reviewing this problem in

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1948 wrote:

"The questionable action and the questionable depth of the tension zone have considerable bearing on the limited dependability of many stability analyses, and the action within the tension zone is a subject that is worthy of much study" (Taylor, 1948, p. 450).

Unfortunately this state of affairs has not changed much since 1948, and at present (Chowdhury, 1978) it is customary to estimate the depth of the tension crack z_c using Eq. (1) below:

$$z_c = \frac{2c_m}{\gamma} \tan\left(45 + \frac{\phi_m}{2}\right) \quad (1)$$

where $c_m = c/F$ is the mobilized cohesion, $\phi_m = \arctan(\tan\phi/F)$ is the mobilized friction angle, F —factor of safety, and γ —unit weight. Eq. (1), which is based on Rankine's theory of earth pressure in an infinite half space, predicts that z_c is independent of the geometry of the slope (height H , and inclination i), and thus can not be expected to be applicable to finite slopes.

There exist almost no means other than Eq. (1) for estimation of the depth of tension cracks; an exception is a procedure suggested by Spencer (1968) which used his method of parallel interslice forces (Spencer, 1967) to determine the line of thrust of the interslice forces, assuming $z_c = 0$. If the position of the line of thrust is not reasonable, a tension crack is introduced, and a different line of thrust is determined. The process is repeated, increasing z_c until a reasonable line of thrust is obtained.

While Spencer's procedure overcame the main objection to the use of Eq. (1), since the results are not independent of the problem's geometry, it has a number of shortcomings:

- a) The requirement of a "reasonable" line of thrust is not (and can not) be defined.
- b) The depth of the tension cracks is not related to the tensile strength. In fact (a) and (b) are probably related.
- c) Spencer's procedure requires extensive trial and error calculations, since the depth of the tension crack has to be determined for every slip surface tested, not only the critical one.

Recently a generalized variational procedure to extreme-value problem's of limiting equilibrium has been presented by Garber and Baker (1979). This approach has been applied to the slope stability problem by Baker and Graber (1977 a, b, 1978). In the framework of this approach, it is possible to consider the question of tension cracks in a consistent manner which is free from the limitations mentioned above. The present work is an extension of the approach presented by Baker and Garber (1977 b) to include finite tensile strength and its effects on cracking and stability of slopes. Only straight, homogeneous slopes without pore pressure or external loading are considered here, but the approach may be easily extended to include homogeneous pore pressure distributions, and cracks filled with water.

TENSILE STRENGTH OF SOIL

The Coulomb failure criterion, in combination with the definition of the factor of safety can be written as

$$\tau = \frac{c + \sigma\phi}{F} \quad (2)$$

$$\sigma \geq -\frac{c}{\phi} \quad (3)$$

where $\tau = \tau(x)$ and $\sigma = \sigma(x)$ are the distributions of the shear and normal stresses along a potential slip surface $y(x)$, and $\phi = \tan\phi$.

A graphical representation of Eqs. (2) and (3) are shown in Figs. 1(a). From this figure it is seen that the quantity (c/ϕ) may be interpreted as a tensile strength (t) limiting the possible values of the normal stress in tension. Notice that the limiting tensile stress t is independent of F , which means that the usual definition of the factor of safety represents factor of safety with respect to shear strength only.

While Eqs. (2) and (3) are a consistent representation of the Coulomb criterion, it is a poor representation of actual soil behaviour, since in the zone of negative (tensile) stress the yield condition is highly non linear as shown by the dashed line in Fig. 1(b) (Brace, 1960; Frydman, 1967).

The presently available experimental information is not extensive enough to allow a definition of the non-linear part of the failure criterion. It is reasonable therefore to define the tensile strength t as an additional independent variable characterizing the material, considering a Coulomb relation with a tension cut-off.

Fig. 1. Graphical representation of failure criteria

$$\tau = \frac{c + \sigma\phi}{F} \quad (4)$$

$$\sigma \geq -t \quad (5)$$

Eqs. (4) and (5) are represented by the solid line in Fig. 1(b). The dependence of the yield criterion on the factor of safety is shown in Fig. 1(c). From Fig. 1(b) and 1(c) one sees that

$$c/\phi \geq t \geq 0$$

In order to incorporate the inequality (5) into the analysis it is convenient to introduce a reduced stress Σ^2 defined as:

$$\Sigma^2 \equiv \sigma + t \quad (6)$$

or

$$\sigma = \Sigma^2 - t \quad (7)$$

Eq. (7) indicates that regardless of the value of Σ the inequality (5) is satisfied. In terms of the variable Σ the yield condition becomes

$$\tau = \frac{\tilde{c} + \Sigma^2\phi}{F} \quad (8)$$

where $\tilde{c}$ is given by $\tilde{c} = c - t\phi$

$$(9)$$

The convenience of the definition (6) is evident from the fact that the two relations (4) and (5) are condensed into the single Eq. (8). Fig. 1(c) shows that by expressing the yield condition in the form of Eqs. (4) and (5), the meaning of F as a factor of safety with respect to shear strength only is retained since the tensile strength is independent of F . An alternative definition in which F scales down the tensile strength as well as the shear

strength is equally possible (Fig. 1(d)). It is felt however that the definition shown in Fig. 1(c) is superior for the following reasons:

a) It enables the use of different factors of safety with respect to shear and tensile strength.

b) It leads to a somewhat simpler formulation. Consequently Eqs. (4) and (5) or equivalently Eq. (8) has been adopted as the yield criterion for the present analysis.

VARIATIONAL ANALYSIS

The analysis of slope stability as a problem in the calculus of variations was presented by Baker and Garber (1977 a, b, 1978). In those papers, the yield condition was taken as Eq. (4) only without considering the restriction (5). However, the formal similarity between Eq. (4) and the complete relation (8) makes it possible to use part of the analysis of Baker and Garber (1977 b) merely replacing c by $\tilde{c}$ and σ by Σ^2 . It is convenient to introduce the following non dimensional quantities:

$$N = \frac{c}{\gamma H}; \quad S = \frac{\sigma}{\gamma H}; \quad T = \frac{t}{\gamma H}; \quad \tilde{C} = \frac{\tilde{c}}{\gamma H}; \quad \bar{\Sigma}^2 = \frac{\Sigma^2}{\gamma H}; \quad Y = y/H; \quad \bar{Y} = \bar{y}/H;$$

$$X = x/H \text{ and } \zeta = \frac{z_c}{H} \quad (10)$$

where $\bar{y} = \bar{y}(x)$ represents the equation of the slope face. In terms of the non-dimensional quantities, the variational problem may be stated as follows (Baker and Garber, 1977 b).

Find a pair of functions $Y(X)$ and $\bar{Y}(X)$ which minimize the auxiliary functional G defined below and satisfy the three equations of equilibrium for the sliding mass.

$$G = \int_{x_1}^{x_2} g dx = \int_{x_1}^{x_2} [\bar{\Sigma}^2 f_1 + f_2] dx \quad (11)$$

where f_1 and f_2 are given by

$$f_1 = \phi_m [(1/\lambda_2 + Y) + Y'(\lambda_1/\lambda_2 - X)] + [(\lambda_1/\lambda_2 - X) - Y'(1/\lambda_2 + Y)] \quad (12)$$

$$f_2 = \tilde{C}_m [(1/\lambda_2 + Y) + Y'(\lambda_1/\lambda_2 - X)] - (\bar{Y} - Y)(\lambda_1/\lambda_2 - X) \quad (13)$$

and λ_1, λ_2 are two Lagrange undetermined multipliers, $Y' = dY/dX$, $\phi_m = \tan \phi / F_s'$, $\tilde{C}_m = \tilde{C} / F_s$, and F_s is the minimum factor of safety.

The function $Y(X)$ and $\bar{Y}(X)$ which minimize the functional G must satisfy the following two Euler differential equations for the function g .

$$\frac{d}{dx} \left(\frac{\partial g}{\partial \bar{\Sigma}'} \right) - \frac{\partial g}{\partial \bar{\Sigma}} = 0 \quad (14)$$

$$\frac{d}{dx} \left(\frac{\partial g}{\partial Y'} \right) - \frac{\partial g}{\partial Y} = 0 \quad (15)$$

where $\bar{\Sigma}' = d\bar{\Sigma}/dx$

Applying the first Euler equation to the function g yields

$$2\bar{\Sigma}f_1 = 0 \quad (16)$$

This equation has two solutions

$$\text{either } f_1 = 0 \quad (17)$$

$$\text{or } \bar{\Sigma} = 0 \quad (18)$$

The relation $f_1 = 0$ is a first order differential equation for the function $Y(X)$. The solution of this equation may be written as:

$$r = A \exp(-\phi_m \beta) \quad (19)$$

where A is an integration constant and r, β are polar variables, that are related to X, Y

by the coordinate transformation:

$$X = \frac{\lambda_1}{\lambda_2} + r \sin \beta = X_c + r \sin \beta \quad (20)$$

$$Y = -\frac{1}{\lambda_2} - r \cos \beta = Y_c - r \cos \beta \quad (21)$$

The quantities X_c, Y_c , which are defined by Eqs. (20) and (21), represents the coordinates of the focus of the log-spiral given in Eq. (19) (Fig. 2).

Notice that the coordinate transformation (20) and (21) are different from the transformation used by Baker and Garber (1977 b), resulting in a different geometrical meaning for the polar angle β (see Fig. 2).

The second solution; $\bar{S}=0$; implies $\sigma=-t$ or $S=-T$. It is seen, therefore, that as a result of incorporating the inequality (5) into the analysis it is found that the slip surface may consist of a part which is a log-spiral (Eq. (19)) and a second part along which the tensile strength is fully mobilized. On physical grounds it is expected that at any point where $\sigma \rightarrow -t$ a tension crack tends to appear. Assuming, as usual, that the tension cracks are vertical, one has to conclude that the tension crack will extend to the depth at which the normal stress distribution along the slip surface approaches the tensile strength (see Fig. 2). Assuming that the distribution of S along the slip surface $S=S[X; Y(X)]$ is known (this function will be established shortly), the condition for cracking may be written as:

$$\lim S(X; Y) = -T \quad (22)$$

Naturally, if the tensile strength is high enough so that $-T \leq \hat{S}(X_2; Y=1)$ (Fig. 2) no tension crack will occur.

Applying the second Euler equation (Eq. (15)) to the function g ; using the coordinate transformation (20) and (21) and the expression for r as a function of β (Eq. (19), a first order differential equation for $\bar{S}$ in terms of β is obtained. Further, utilizing Eqs. (8) and (9), this differential equation may be expressed in terms of S and N instead of $\bar{S}$ and $\tilde{C}$ as follows:

$$\frac{dS}{d\beta} - 2\phi_m S + (r \sin \beta - 2N_m) = 0 \quad (23)$$

where $N_m = N/F_s = c/\gamma H F_s$ is the stability number as defined by Taylor

Integrating Eq. (23):

$$S = B \exp(2\phi_m \beta) + \frac{A}{1 + 9\phi_m^2} (\cos \beta + 3\phi_m \sin \beta) \exp(-\phi_m \beta) - N_m \frac{1 - \exp(2\phi_m \beta)}{\phi_m} \quad (24)$$

where B is an integration constant.

It may be verified that

$$\lim_{\psi_m \rightarrow 0} \left(\frac{1 - \exp(2\psi_m \beta)}{\psi_m} \right) = -2\beta \quad (25)$$

Fig. 2. Basic conventions and definitions

Consequently Eq. (24) is valid for $\phi_m=0$ as well as $\phi_m \neq 0$; it is therefore a generalization of the result derived by Baker and Garber (1977 b) where different expressions were presented for $\phi_m=0$ and for $\phi_m \neq 0$. The additional difference between Eq. (24) and the result of Baker and Garber (1977 b) is due to the difference in the definition of the angle β .

Eqs. (19), (24) and (22) are the general results of the variational analysis; Eqs. (19) and (24) give the functional dependence of the slip surface and normal stress distribution on β , while Eq. (22) provides a criterion for the depth of the tension cracks. The general solution depends on nine unknown constants [$X_c, Y_c, N_m, A, B, \beta_1, \beta_2$ (or $X_1; X_2$) ζ and S_2]; these constants are to be determined from the following relations:

a) Two geometrical boundary conditions

$$Y(X=X_1) = \bar{Y}(X=X_1) = 0 \quad (26)$$

$$Y(X=X_2) = 1 - \zeta \quad (27)$$

b) Stress boundary condition $S_2 = S(\beta = \beta_2)$

c) Three equations of equilibrium (horizontal vertical and moments) for the sliding body.

d) Two transversality conditions which fix the points of entry and exit (points 1 and 2 in Fig. 2) of the slip surface (the need for such conditions was discussed by Baker and Garber (1977 b)).

e) The criterion for cracking (Eq. (22)).

It is seen that the number of available equations is the same as the number of unknowns so the formulation is consistent.

DETERMINATION OF A PARTICULAR SOLUTION

In this section the nine relations previously enumerated are combined into a computation scheme which enables the determination of N_m and ξ for given values of ϕ_m, i and T .

a) Geometrical boundary conditions

Introducing the coordinate transformation (20) and (21) into the boundary conditions (26) and (27) and rearranging:

$$A = \frac{-(1-\zeta)}{\Delta[\cos \beta \exp(-\phi_m \beta)]} \quad (28)$$

$$Y_c = A \cos \beta_1 \exp(-\phi_m \beta_1) \quad (29)$$

where the notation $\Delta[f]$ means $f(\beta_2) - f(\beta_1)$. Such a notation will be used throughout this section.

b) Stress boundary condition

Introducing the relation $S_2 = S(\beta_2)$ into Eq. (24) and solving for B :

$$B = \left[S_2 + N_m \left(\frac{1 - \exp(2\phi_m \beta_2)}{\phi_m} \right) - \frac{A}{1 + 9\phi_m^2} (\cos \beta_2 + 3\phi_m \sin \beta_2) \exp(-\phi_m \beta_2) \right] \exp(-2\phi_m \beta_2) \quad (30)$$

c) Equilibrium equations

The equations of equilibrium in the horizontal and vertical directions, and the equation of moment equilibrium may be written as follows (Baker and Garber, 1977 b):

$$\bar{H} = \int_{X_1}^{X_2} [(N + S\phi) - F_s S Y'] dX = 0$$

$$\bar{V} = \int_{X_1}^{X_2} [(N + S\phi) Y' - F_s (\bar{Y} - Y - S)] dX = 0$$

$$\bar{M} = \int_{X_1}^{X_2} [(N + S\psi)(Y - Y'X) - F_s(S(X + Y'Y) - X(\bar{Y} - Y))] dX = 0$$

Transforming these equations, using polar variables (r) (β), utilizing the expressions $r(\beta)$ and $S(\beta)$ given by Eqs. (19) and (24), integrating, and using Eqs. (28) and (29), yields after lengthy, but essentially straight-forward manipulations:

$$\bar{H} = 0 \quad (31)$$

$$\bar{V} = 0 \quad (32)$$

$$\bar{M} = N_m M_2 + M_1 \quad (33)$$

where $\bar{H}$, $\bar{V}$, M_1 and M_2 are given by:

$$\begin{aligned} \bar{H} = N_m \Delta \left[\sin \beta \exp(-\psi_m \beta) - 2 \left(\sin \beta \cosh(\psi_m \beta) - \frac{\cos \beta \sinh(\psi_m \beta)}{\psi_m} \right) \right] \\ + B \Delta [(\cos \beta - \psi_m \sin \beta) \exp(\psi_m \beta)] + \frac{A}{2(1 + 9\psi_m^2)} \\ \times \Delta [(1 + 3\psi_m^2) + (1 - 3\psi_m^2) \cos^2 \beta + 2\psi_m \sin 2\beta] \exp(-2\psi_m \beta) \end{aligned} \quad (34)$$

$$\begin{aligned} \bar{V} = N_m \Delta \left[2 \left(\cos \beta \cos \frac{1}{2}(\psi_m \beta) + \frac{\sin \beta \sinh(\psi_m \beta)}{\psi_m} \right) - \cos \beta \exp(-\psi_m \beta) \right] \\ + B \Delta [(\sin \beta + \psi_m \cos \beta) \exp(\psi_m \beta)] + Y_c \Delta [\sin \beta \exp(-\psi_m \beta)] + \frac{A\psi_m}{1 + 9\psi_m^2} \\ \times \Delta [(3 - 2\cos^2 \beta - 3\psi_m \sin 2\beta) \exp(-2\psi_m \beta)] - X_c/A - \sin \beta_2 \exp(-\psi_m \beta_2) + \frac{\cot(i)}{2A} \end{aligned} \quad (35)$$

$$\begin{aligned} M_1 = X_c/A + 2A \Delta \left[\left(\frac{\cos^3 \beta}{3} + \frac{\psi_m}{1 + 9\psi_m^2} (\sin \beta - 3\psi_m \cos \beta) \right) \exp(-3\psi_m \beta) \right] \\ + Y_c \Delta [\sin^2 \beta \exp(-2\psi_m \beta)] - \sin^2 \beta_2 \exp(-2\psi_m \beta_2) - \frac{\cot(i)}{3A^2} (3X_c - \cot(i)) \end{aligned} \quad (36)$$

$$M_2 = \Delta [(1 - \exp(-2\psi_m \beta))/\psi_m] \quad (37)$$

d) Transversality conditions

The transversality conditions for the present problem were discussed by Baker and Garber (1977 b). On the basis of that discussion it is possible to show that

$$\left(g - Y' \frac{\partial g}{\partial Y'} \right) \Big|_{X=X_i} = 0$$

where X_i may be either X_1 or X_2 .

Using the definition of the function g (Eqs. (11), (12) and (13)), and the coordinate transformation (Eqs. (20) and (21)), these transversality conditions may be written as:

$$[S(\beta)(\sin \beta + \psi_m \cos \beta) + N_m \cos \beta - (\bar{Y} - Y) \sin \beta]_{\beta=\beta_i} = 0 \quad (38)$$

Applying this equation to the point 2, and realizing that at that point $(\bar{Y} - Y) = \zeta$, one obtains:

$$S_2(\sin \beta_2 + \psi_m \cos \beta_2) + N_m \cos \beta_2 - \zeta \sin \beta_2 = 0$$

This equation will be utilized in two ways-either to calculate S_2 where ζ is known (or assumed):

$$S_2 = \frac{\zeta \sin \beta_2 - N_m \cos \beta_2}{\sin \beta_2 + \psi_m \cos \beta_2} \quad (39)$$

or, in combination with the cracking condition $S_2 = -T$, in order to solve for the depth of the tension crack ζ :

$$\zeta = \frac{N_m \cos \beta_2 - T(\sin \beta_2 + \psi_m \cos \beta_2)}{\sin \beta_2} \quad (40)$$

From the physical meaning of ζ , it is clear that if Eq. (40) yields negative values for ζ , it has to be taken as zero.

While it is possible to apply Eq. (38) also to point 1 of Fig. 2, it is more instructive in this case to replace the formal transversality condition by a physical argument which leads to a result which has a clearer physical interpretation. This argument which is essentially due to Taylor (1943) is presented in appendix I. It leads to the result that the critical conditions are obtained when the pole P of the spiral is located on a vertical line through the mid-point M of the slope (Fig. 3). If this requirement implies that $X_1 > 0$, it has to be replaced by the condition $X_1 = 0$ which implies that $X_c = -A \sin \beta_1 \exp(-\psi_m \beta_1)$ (Eq. (20)). This relation can be summarized as follows:

$$X_c = \begin{cases} \frac{\cot(i)}{2} & \text{if } L \leq 0 \\ -A \sin \beta_1 \exp(-\psi_m \beta_1) & \text{if } L > 0 \end{cases} \quad (41)$$

where

$$L = \frac{\cot(i)}{2} + A \sin \beta_1 \exp(-\psi_m \beta_1) \quad (42)$$

e) The cracking condition

The cracking condition may be formulated as a consistency relation between ζ , S_2 and T as follows:

$$\begin{aligned} &\text{IF}[(\zeta=0 \text{ AND } S_2 \geq -T) \text{ OR} \\ &(\zeta \geq 0 \text{ AND } S_2 = -T)] \\ &\text{THEN } K=1: \text{ OTHERWISE } K=0 \end{aligned} \quad (43)$$

where $K=1$ indicates that ζ , S_2 and T are consistent while $K=0$ means that ζ has to be modified.

The first bracket in the IF statement corresponds to the case when the tensile strength is high enough so that no tension cracks appear. The second part requires that at the tip of an existing crack the

Fig. 3. The location of the centre of the critical log spiral surface

Fig. 4. The computation scheme

tensile strength is fully mobilized.

Combining Eqs. (28), (29), (31), (32), (33), (34), (35), (36), (37), (39), (40), (43), and (30), a computation scheme defining a system of two non-linear simultaneous equations in two unknowns $\bar{H}(\beta_1, \beta_2) = 0$ and $\bar{V}(\beta_1, \beta_2) = 0$ is obtained. This computation scheme is shown in Fig. 4.

The solution to this system of non-linear equations was obtained by the extended Newton-Raphson method, using a procedure suggested by Brown (1973). The derivatives $\partial H/\partial \beta_1$; $\partial H/\partial \beta_2$; $\partial V/\partial \beta_1$ and $\partial V/\partial \beta_2$ which are required for the implementation of the Newton-Raphson method were evaluated numerically.

RESULTS

Typical results showing the critical slip surface and normal stress distribution for various conditions are shown in Fig. 5.

Fig. 6 presents N_m (the stability number) as a function of ϕ_m and i (mobilized friction angle and slope inclination respectively), for the case of no tension cracks. The dashed lines on the same figure give the value of T (tensile strength) required in order to maintain the no crack condition. The values of N_m given in Fig. 6 are almost identical to the conventional values based on Taylor's stability chart (Taylor 1948). This is not surprising since as was already found by Taylor (1937) the friction circle, and log-spiral methods give almost identical results.

Fig. 7 presents N_m as a function of ϕ_m and i for the extreme case of a soil with zero tensile strength. The dashed lines on this figure represent the depth of the tension crack $\zeta = z_c/H$ that will develop under such conditions. The largest difference in N_m between Figs. 6 and 7 occurs for $i = 90^\circ$ and $\phi_m = 0$. For such conditions, in the absence of tension cracks, the stability number is $N_m = 0.261$ which is valid provided that the tensile strength $T \geq 0.136$.

Fig. 5. Typical slip surfaces and normal stress distributions

Fig. 6. Stability chart for uncracked slopes

Fig. 7. Stability chart of material with zero tensile strength

For the soil with zero tensile strength, $N_m=0.308$, which is associated with a tension crack developing to a depth of $\zeta_1=0.211$. This difference in stability numbers leads to a difference of up to 20% in the estimate of the critical height of a slope, (Taylor's values being unconservative). It should be noted that the critical slip surface changes when a crack develops in the soil (see Fig.5(d) and (e)).

The line $\overline{ABC}$ appearing in Figs.6 and 7 represent the boundary between the conditions for which the critical slip surface satisfies $X_1=0$ and those for which $X_1<0$. This line does not have the same significance as the corresponding line in Taylor's stability charts which refers to the lowest point on the slip surface rather than the point X_1 . The present classification appears more significant (at least theoretically) since as can be seen from Figs.6 and 7 it corresponds to a discontinuity in the solution.

The stability numbers along the line $\overline{DD'}$ in Figs.6 and 7 are realized on a slip circle passing through infinity. Obviously this is an impossible proposition. Taylor overcame this difficulty by realizing that no homogeneous profile extends to infinity and estimated the effect of bed-rock on the stability of a slope.

One has to realize however that the condition $\phi=0$ is a limiting case only. Even for undrained conditions, the compressibility of water will cause ϕ to have some small, but finite value (say $\phi=0.1^\circ$). The effect of such small values of ϕ on the stability number is shown to be significant for slopes with small inclination (Figs.6 and 7). Fig.5(a) shows that the slip surface for $\phi_m=0.1^\circ$ is finite. It follows that it is not necessary to invoke the existence of bedrock in order to exclude slip circles through infinity.

Fig.7 shows that the maximum depth of the tension crack is $\zeta=0.25$ i.e. the depth of the tension crack never exceeds one quarter of the slope height. This finding is not

inconsistent with the observation of Terzaghi that tension cracks never exceed one half of the slope height (Terzaghi, 1943).

From Fig. 7 it is seen that, as expected, the maximum depth of tension cracks occurs for vertical slopes. Unexpected however is the result that the maximum value of the tension crack is realized for high ϕ value rather than $\phi=0$. It is interesting to see however that the same trend is implied by Eq. (1).

Although Eq. (1) is frequently used for estimating the depth of tension cracks, a more proper interpretation of this equation is that it gives the maximum possible height of a vertical cut. Since the crack itself may be considered as a vertical cut, its depth must be smaller than the value predicted by Eq. (1), or it will collapse. A comparison of z_c

Fig. 8. Effect of cohesion to friction ratio on depth of failure surface

Fig. 9. Effect of tensile strength on stability

values predicted by the present approach with those given by Eq. (1) shows that in every case the cracks predicted by the present approach are possible (stable).

Fig. 8 shows the relative shape of critical slip surfaces for various values of parameter N_f defined as

$$N_f = \frac{c}{\gamma H \phi} = \frac{N_m}{\phi_m} \quad (44)$$

The parameter N_f represents a non-dimensional ratio of cohesion to friction. From Fig. 8 it is seen that as the relative magnitude of c/ϕ decreases, the slip surface becomes shallower so that, at the limit, when $c=0$ (cohesionless soil) the slip surface coincides with the surface or the slope. This observation is a generalization of the well known phenomenon that for an infinite cohesionless slope, slip surface and slope surface coincide. The results shown in Fig. 8 are for the case $T=0$; however identical results are obtained for every value of T . It may easily be shown that the same result will be obtained regardless of the shape of the slope face.

Figs. 6 and 7 represent the two extreme conditions of no cracks, and zero tensile strength (maximum cracks). The dependence of N_m and ζ on intermediate T values is shown in Fig. 9 for two values of ϕ_m . From Fig. 9 it is seen that to a reasonable degree of accuracy, the dependence of both N_m and ζ on T is linear. Such an approximation is certainly justified in view of the uncertainties in the experimental values of tensile strength. This linear behaviour makes it possible to write:

$$N_m(t) = \begin{cases} M_T T + N_T & \text{if } T < T_r \\ N_\zeta & \text{if } T \geq T_r \end{cases} \quad (45)$$

$$\zeta(T) = \begin{cases} \zeta_{\max} \left(1 - \frac{T}{T_r}\right) & \text{if } T < T_r \\ 0 & \text{if } T \geq T_r \end{cases} \quad (46)$$

where

$$M_T = \frac{N_\zeta - N_T}{T_r} \quad (47)$$

and $N_T = N_m(T=0)$ (Fig. 7), $N_\zeta = N_m(\zeta=0)$ (Fig. 6), T_r is the limiting value of T required to prevent cracking (Fig. 6) and $\zeta_{\max}$ is the value of ζ for zero tensile strength (Fig. 7).

Fig. 10. The parameters M_T and M_ζ

Since both N_m and ζ are linear in T , it follows that N_m depends linearly on ζ . In certain situations it may be convenient to consider ζ rather than T as the independent variable. In that case it is easy to show that

$$N_m(\zeta) = M_\zeta + N_\zeta \text{ if } \zeta \leq \zeta_{\max} \quad (48)$$

where

$$M_\zeta = \frac{N_M - N_\zeta}{\zeta_{\max}} \quad (49)$$

Fig. 10 shows the values of M_T and M_ζ as a function of i and ϕ_m . The three Figs. 6, 7 and 10 and Eqs. (45), (46) and (47) enable the determination of N_m and ζ for every combination of the three independent variables ϕ_m , i and T .

APPLICATION

In order to illustrate the use of Figs. 6, 7 and 10, consider the problem of determining the dependence of the factor of safety and depth of tension crack on the slope angle for the following case.

$$H=10 \text{ m}; \phi=20^\circ; c=4 \text{ t/m}^2; \gamma=2 \text{ t/m}^3; t=1 \text{ t/m}^2$$

Assume first that the slope is uncracked and find the factor of safety as a function of i in the usual manner. The results of this analysis are presented in Table 1 where F_ϕ and F_c are the usual factors of safety with respect to cohesion and friction and F_s is taken as $F_s = F_c = F_\phi$.

Having established the function $F_s = F_s(i)$, Fig. 6 is used again to find the factor of

Table 1.

i	$F\phi$	ϕ_m	N_m	F_c	F_s
90	1.00	20	0.181	1.10	1.08
	1.10	18	0.189	1.06	
	1.08	18.5	0.186	1.08	
80	1.00	20	0.148	1.35	1.24
	1.30	15.5	0.165	1.20	
	1.25	16	0.163	1.23	
	1.24	16.5	0.161	1.24	
70	1.30	15.5	0.139	1.44	1.41
	1.40	14.5	0.142	1.41	
60	1.50	13.5	0.122	1.64	1.61
	1.60	13	0.124	1.61	
50	1.70	12	0.107	1.87	1.81
	1.83	11	0.112	1.79	
	1.81	11.2	0.111	1.81	
40	2.00	10	0.098	2.04	2.04
	2.04	10	0.098	2.04	
30	2.40	8.5	0.084	2.38	2.38
	2.38	8.5	0.084	2.38	
20	2.60	8	0.058	3.45	3.04
	3.10	6.5	0.070	2.86	
	3.05	6.8	0.066	3.04	

Table 2.

i	F_s	ϕ_m	T_r	F_t
90	1.08	18.5	0.117	0.43
80	1.24	16.5	0.090	0.56
70	1.41	14.5	0.070	0.71
60	1.61	13	0.058	0.86
50	1.81	11.2	0.051	0.98
40	2.04	10	0.046	1.09
30	2.38	8.5	0.044	1.14
20	3.04	6.8	0.045	1.11

safety with respect to tensile strength $F_t = T/T_r$ ($T = t/\gamma H = 0.05$) as shown in Table 2.

The functions $F_s(i)$ and $F_t(i)$ are plotted in Fig. 11 as the lines ABC' and abc. Obviously these lines are valid only in the range where $F_t > 1$ since they are based on Fig. 6 which is valid for an uncracked slope. The calculations done so far show that in order to prevent cracking the slope angle should be flatter than 50° .

For slope angles steeper than 50° the factor of safety is calculated using Figs. 7, 10 and Eq. (45) as shown in Table 3.

Having established F_s as a function of i for the cracked embankment, the depth of the crack is found using Figs. 6, 7 and Eq. (46) as shown in Table 4.

This F_s as a function of i is plotted as the line BC in Fig. 11. Also plotted in this figure is ζ as a function of i .

Fig. 11 presents the complete analysis of the problem. The difference between the lines BC' and BC represents the effect of cracking on the overall factor of safety. This effect is not large, but at least for the case $i=90$ it may mean the difference between

Fig. 11. Results of example calculation

Table 3.

i	$F\phi$	ϕ_m	M_T	N_T	N_m	F_c	F_s
60	1.50	13.6	-0.098	0.127	0.122	1.64	1.59
	1.64	12.5	-0.100	0.132	0.127	1.59	
	1.58	13.0	-0.099	0.130	0.125	1.60	
	1.59	12.9	-0.099	0.131	0.126	1.59	
70	1.40	14.6	-0.138	0.151	0.144	1.39	1.39
80	1.24	16.4	-0.201	0.178	0.168	1.19	1.20
	1.19	17.0	-0.200	0.176	0.166	1.21	
	1.20	16.9	-0.200	0.176	0.166	1.20	
90	1.00	20.0	-0.328	0.220	0.204	0.98	0.99
	0.99	20.2	-0.328	0.220	0.204	0.99	

Table 4.

i	F_s	ϕ_m	$\zeta_{\max}$	T_r	ζ
60	1.59	12.9	0.081	0.058	0.011
70	1.39	14.6	0.091	0.071	0.027
80	1.20	16.9	0.150	0.089	0.066
90	0.99	20.2	0.235	0.114	0.132

failure and stability of the embankment. The main use of such a figure is, however, to determine the inclination required in order to prevent cracking ($i \cong 50^\circ$) and the depth of crack expected when the conventional factor of safety $F_s = 1.5$ is used (approximately 20 cm). The factor of safety required in order to prevent cracking is 1.86 which appears consistent with usual recommendations.

DISTRIBUTION OF INTERNAL VARIABLES

The variational approach to limit-equilibrium, as presented by Baker and Garber (1977 a, b, 1978) does not deal with the distribution of forces (stresses) inside the sliding mass; only the conditions on the boundary of this mass are considered. Most other

limiting equilibrium approaches are, however, specified in terms of some pattern of force distribution on vertical lines within the sliding mass.

The Morgenstern-Price (1965) method, for example, is defined in terms of the inclination θ of the interslice forces (Fig. 12(a)) while the Janbu method (Janbu, 1973) is defined in terms of the location Y_e of the line of thrust (Fig. 12(a)). In these methods the functions $\theta(X)$ or $Y_e(X)$ are left unspecified, and the engineer makes a reasonable assumption for them. It is of interest, therefore, to investigate the shape of the functions

Fig. 12. Distribution of internal variables

$\theta(X)$ and $Y_e(X)$ which is implied by the variational analysis. This may be done after completion of the solution, by considering the equilibrium of a part of the sliding mass as shown in Fig. 12(a). Typical results showing the distribution of the line of thrust, Y_e , inclination θ , and magnitude E of the forces acting on vertical lines is shown in Figs. 12(b), (c) and (d). The quantity θ_{max} shown in these figures is the maximum inclination which is consistent with the strength properties of the material. This quantity is given by

$$\tan \theta_{max} = \phi_m + \frac{N_m(\bar{Y} - Y)}{(E/\gamma H^2) \cos \theta}$$

The results shown in Fig. 12 indicate clearly that in each of the cases examined the location of the line of thrust is reasonable (within the limitations of this concept discussed before). Notice also how the line of action drops as expected when the tensile strength is reduced (compare Figs. 12(b) and (c)). In every case the inclination of the interslice forces is within the allowable bound, indicating that the factor of safety for

vertical planes is higher than the factor of safety for the critical slip surface (Janbu, 1973).

The function $\theta(X)$ decreases monotonically; the negative θ values appearing in part of Fig. 12(d) indicate that in that zone part of the sliding mass "hangs" from the main body of the slope rather than "leaning" on the remaining lower part of the sliding mass.

CONCLUDING REMARKS

An extension of the variational approach to slope stability problems, which incorporates a finite value of tensile strength has been presented. Based on this approach it was found possible to estimate the depth of tension cracks, and their effect on stability, in a consistent manner. Formulas and charts which enable the determination of the depth of tension cracks and stability number for every combination of i , ϕ_m and t are given. Based on this analysis the following conclusions may be presented:

- 1) The maximum depth of the tension cracks is 25% of the slope height, and this value occurs for vertical slopes.
- 2) The increase in stability number (decrease in critical height) which corresponds to the maximum crack is of the order of 20%. This is valid for cracks which are not filled with water.

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NOTATION

- A, B =constants of integration
 c =cohesion; $c_m=c/F_s$
 $\tilde{c}=c-t\psi$; $\tilde{C}=\tilde{c}/\gamma H$; $\tilde{C}_m=\tilde{C}/F_s$
 E =resultant interslice force
 f_1, f_2, g =auxiliary functions
 G =auxiliary functional
 F, F_s =factor of safety, and minimum factor of safety respectively
 H =height of the slope
 $\bar{H}$ =Resultant horizontal force
 i =Slope inclination
 $L=\frac{\cot(i)}{2}+A \sin \beta_1 \exp(-\phi_m \beta_1)$
 $\bar{M}$ =resultant moment
 $M_1 M_2, M_T, M_\zeta$ =auxiliary functions
 N_m =stability number, $N_m=\frac{c}{\gamma H F_s}$
 $N_f=c/\gamma H \phi$
 r =radius vector in polar coordinate system
 $S=\sigma/\gamma H$
 t =tensile strength
 $T=t/\gamma H$
 T_r =tensile strength required to prevent cracking
 $\bar{V}$ =resultant vertical force
 x =horizontal coordinate
 $X=x/H$

- X_c, Y_c = coordinates of the focus of the log-spiral
 y = vertical coordinate
 $Y = Y/H$
 $\bar{y}$ = equation describing the face of the slope
 $\bar{Y} = \bar{y}/H$
 Y_e = line of thrust of interslice forces
 z_c = depth of tension crack
 β = polar angle
 β_1, β_2 = polar angles of the end points of the slip surface
 γ = unit weight
 $\zeta = z_c/H$
 $\zeta_{\max}$ = a non dimensional depth of tension crack for soil with zero tensile strength
 θ = inclination of interslice forces
 $\theta_{\max}$ = max possible value of θ
 λ_1, λ_2 = Lagrange undetermined multipliers
 σ, τ = normal and tangential stress distribution on the slip surface
 Σ = reduced stress defined by $\Sigma^2 = \sigma + t$
 $\bar{\Sigma}^2 = \Sigma^2 / \gamma H$
 ϕ = angle of internal friction
 ϕ_m = mobilized angle of internal friction
 $\psi = \tan \phi$; $\psi_m = \phi / F_s$

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APPENDIX 1

Consider the situation as shown in Fig. 3, where the pole P of the spiral is located on a vertical line through the mid-point M of the slope face $\overline{BC}$. If the slope $\overline{BC}$ is cut away parallel to itself to the line B_1C_1 , the change in weight ΔW_1 is an upward force acting through the center of gravity of the area removed (G_1). The line of action of the cohesive forces must be the same as before the removal of the area BCC_1B_1 ; thus ΔW_1 must intersect ΔC_1 at the point O_1 . The force ΔP , which is the change in the resultant of the frictional and normal forces along the slip surface $\overline{AD}$, must pass through the point of intersection of ΔW , and ΔC , (O_1) and through the pole P of the spiral (the characteristic property of the log-spiral function). Hence the direction of ΔP_1 is also known, and the force polygon of ΔW_1 , ΔC_1 , and ΔP_1 may be drawn as shown in Fig. 3. From this force polygon it is seen that ΔC_1 acts to the left, which indicates a decrease in the required cohesion. It follows, therefore, that removal of the area BCC_1B_1 has produced a more stable condition.

Similarly, if the surface $\overline{BC}$ is built up to the line B_2C_2 adding the weight ΔW_2 , identical analysis demonstrates that in that case too ΔC_2 acts to the left. It follows, therefore, that moving the mid-point of the slope to either side of the vertical line through the pole of the spiral leads to a greater stability. Hence the critical conditions are realized when the pole of the spiral is located exactly above the mid-point of the slope, i. e.

$$X_c = \frac{\cot(i)}{2} \quad (50)$$

This condition is however not always possible, since it may result in X , being positive, i. e. the slip surface strating above the toe of the slope. Using the coordinate transformation (20) one may write:

$$X_1 = X_c + A \sin \beta_1 e^{-\phi_m \beta_1} = \frac{\cot(i)}{2} + A \sin \beta_1 e^{-\phi_m \beta_1} \quad (51)$$

If $X_1 \leq 0$ than the preceding analysis is correct, and the critical value of X_c is $\cot(i)/2$. If however Eq. (51) yields $X_1 > 0$ this is not possible and the conditions $X_c = \cot(i)/2$ has to be replaced by the requirement $X_1 = 0$ which implies $X_c = -A \sin \beta_1 \exp(-\phi_m \beta_1)$. The preceding discussion may be summarized as:

$$X_c = \begin{cases} \frac{\cot(i)}{A} & \text{if } L \leq 0 \\ -A \sin \beta_1 \exp(2\phi_m \beta_1) & \text{if } L > 0 \end{cases} \quad (52)$$

where

$$L = \frac{\cot(i)}{2} + A \sin \beta_1 \exp(2\phi_m \beta_1) \quad (53)$$

This equation is used instead of the transversality condition applied at the point 1.

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