A COMPARISON BETWEEN TWO THEORIES OF
FINITE STRAIN CONSOLIDATION

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ABSTRACT
This paper compares the theories of finite strain consolidation developed by Gibson, England and Hussey (1967) and Mikasa (1963). It is shown that these theories, though independently developed are quite similar. The only difference is that the theory developed by Mikasa inherently assumes a process of rapid sedimentation in which a layer is instantaneously formed at a constant initial void ratio, while the Gibson–England–Hussey theory is unrestricted with regard to its initial conditions.

Key words: consolidation, consolidation test, land reclamation, ocean soil, pore pressure, settlement, soft ground, wastes

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INTRODUCTION
The conventional theory of consolidation (Terzaghi, 1924) is constrained by three major physical assumptions (Schiffman, 1980). First, the void ratio–effective stress relationship is linear resulting in a constant compressibility ($a_o$) or ($m_o$). Second, the coefficient of permeability ($k$) of the soil is assumed to be a constant. Third, the strains are assumed to be infinitesimal. While this theory has been used in many ways, it is less than satisfactory in a variety of physical situations in which the void ratio is a nonlinear function of the effective stress, or when the permeability is likewise a nonlinear property of the void ratio. Furthermore, conventional theory is not applicable to consolidation processes when the effect of the self-weight of the soil dominates the applied loading effects. All of these factors imply the need for a finite strain theory of consolidation.

Generalizations of consolidation theory have been independently established by Mikasa (1963) and Gibson, England and Hussey (1967). Both of these finite strain generalizations are unrestricted with regard to the debilitating assumptions inherent in conventional theory.

It is the intent of this paper to compare these two theories. The similarities and dissimilarities are shown.

FINITE STRAIN CONSOLIDATION
We start with the formulation of finite strain consolidation as given by Gibson, England

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and Hussey (1967):
\[
\frac{\partial}{\partial z} \left[ g(e) \frac{\partial e}{\partial z} \right] \pm f(e) \frac{\partial e}{\partial t} = \frac{\partial e}{\partial t},
\]
(1 a)

where
\[
g(e) = -\frac{k(e)}{\gamma_w (1+e)} \frac{d \sigma'}{d e},
\]
(1 b)
\[
f(e) = -\left( \frac{\gamma_s}{\gamma_w - 1} \right) \frac{d}{d e} \left[ \frac{k(e)}{1+e} \right].
\]
(1 c)
in which \(e\) is the void ratio, \((\gamma_s)\) and \((\gamma_w)\) are the solid and fluid weights per unit of their own volume respectively, \((k)\) is the coefficient of permeability, \((\sigma')\) is the effective stress, and \((z)\) is a material coordinate encompassing a volume of solids in a volume lying between the datum plane and the Lagrangian coordinate point (McNabb, 1960). The Lagrangian (initial) coordinate \((a)\) is related to the material coordinate \((z)\) by
\[
z(a) = \int_0^a \frac{d \sigma'}{1+e(\sigma', 0)}.
\]
(2)

Eqs. (1) with appropriate boundary and initial conditions and constitutive properties provide the governing relationships from which a solution can be developed.

For reasons of analytical convenience Eqs. (1) are expressed in terms of the initial (undeformed) Lagrangian coordinate \((a)\), and this coordinate is directed with gravity. Then
\[
\frac{\partial}{\partial a} \left[ \frac{k}{1+e} \left( 1+e(a, 0) \right) \frac{\partial e'}{\partial a} - (\gamma_s - \gamma_w) \right] + \frac{\gamma_w}{1+e(a, 0)} \frac{\partial e}{\partial t} = 0.
\]
(3)

In addition to the coordinate transformation, Eq. (3) differs from Eq. (1) in one important aspect. Eqs. (1) assume that the consolidation process is monotonic. Eq. (3) lifts this assumption, and is based only on the balance laws governing equilibrium and continuity of the fluid and soil skeleton.

Both formulations given above are unrestricted with respect to the linearity of the deformation of the soil skeleton, and the change of permeability during consolidation. Furthermore, there are no restrictions on the magnitude of the strain. It is assumed that the fluid is incompressible and both the unit weights of water \((\gamma_w)\) and solids \((\gamma_s)\) are constant quantities.

If one considered a clay layer fully consolidated and in equilibrium under both its own weight and a vertical effective stress \((\sigma')\) acting on the upper surface, the initial vertical effective stress distribution in the layer would be
\[
\sigma'(z, 0) = q' + (\gamma_s - \gamma_w) z.
\]
(4)
The initial void ratio \(e(a, 0)\) would be derived from the appropriate void ratio–effective stress relationship and the transformation from \((z)\) to \((a)\) coordinates given by Eq. (2). However, in cases of dredged fills and mine waste disposal operations the initial void ratio \(e(a, 0)\) is a constant \((e_0)\) with depth. In this case Eq. (3) becomes
\[
\frac{\partial}{\partial a} \left[ \frac{1+e_0 k(e)}{1+e} \frac{\partial e'}{\partial a} - \frac{k(e)}{\gamma_w} \right] + \frac{1}{1+e_0} \frac{\partial e}{\partial t} = 0,
\]
(5 a)

where \((\sigma')\) is the effective unit weight of the clay layer which is a function of the void ratio
\[
\sigma' = \frac{\gamma_s - \gamma_w}{1+e}.
\]
(5 b)

It is noted that the assumption that \(e(a, 0)\) is a constant leads to a constant initial effective stress in Eq. (4).

1 The upper/lower sign is taken if the coordinate direction \((z)\) is measured against/with gravity.
2 The coordinate \((a)\) is identical to the coordinate \((z_0)\) given by Mikasa (1963).
We now assume that the strain of the soil skeleton ($\epsilon$) is a nonlinear function of the effective stress ($\sigma'$). Thus

$$d\epsilon = m_0 \epsilon d\sigma'.$$

(6)

This formulation requires that the consolidation is monotonic. That is, although the stress-strain is nonlinear, the relationship is unique. Furthermore, there are no spatial nonhomogeneities and no intrinsic time effects which result in secondary consolidation. This implies that

$$\frac{\partial \sigma'}{\partial a} = \frac{\partial \epsilon}{\partial a} \frac{d\sigma'}{d\epsilon} = \frac{1}{m_0} \frac{\partial \epsilon}{\partial a}.$$

(7)

In addition a coefficient of consolidation ($c_v$) is defined in the usual manner as

$$c_v = \frac{k}{\gamma_m v},$$

(8)

where ($c_v$) is a function of the void ratio ($\epsilon$) or the strain ($\epsilon$). It is noted parenthetically that for ($c_v$) to be a constant both ($k$) and ($m_0$) or their ratio would have to be constant.

Applying the above constitutive properties to Eq. (5) results in a governing equation in the form

$$\frac{\partial}{\partial a} \left[ \frac{(1+\epsilon_0) c_v(\epsilon)}{1+\epsilon} \frac{\partial \epsilon}{\partial a} - c_v(\epsilon) m_0(\epsilon) \gamma' (\epsilon) \right] + \frac{1}{1+\epsilon_0} \frac{\partial \epsilon}{\partial \epsilon} = 0.$$

(9)

It now remains to relate the void ratio ($\epsilon$) to the strain ($\epsilon$). This is accomplished by use of a logarithmic strain definition in which

$$\epsilon = \log \frac{\xi}{\zeta},$$

(10a)

where the consolidation ratio ($\xi$) is defined as

$$\xi(a, t) = \frac{1+\epsilon_0}{1+\epsilon}.$$  

(10b)

Applying these relationships to Eq. (9) results in

$$\frac{\partial}{\partial a} \left[ c_v \frac{\partial \xi}{\partial a} - c_v m_0 \gamma' \right] = \frac{1}{\xi} \frac{\partial \xi}{\partial \epsilon}.$$  

(11)

Moreover, it is noted that

$$\frac{\partial \epsilon}{\partial a} = \frac{\partial \xi}{\partial a} \frac{d\xi}{d\epsilon} = \frac{1}{\xi} \frac{\partial \xi}{\partial \epsilon}.$$  

(12)

As a result

$$\xi^2 \frac{\partial}{\partial a} \left[ c_v \frac{\partial \xi}{\partial a} - m_0 \gamma' \right] = \frac{\partial \xi}{\partial \epsilon}.$$  

(13)

If the deposit is thin, as in a standard oedometer test, then Eq. (13) reduces to

$$\frac{\partial}{\partial a} \left[ c_v \frac{\partial \xi}{\partial a} \right] = \frac{\partial \xi}{\partial \epsilon}.$$  

(14)

A small amount of manipulation of Eq. (13) results in

$$\xi^2 \left[ c_v \frac{\partial \xi}{\partial a} + \left( \frac{\partial \xi}{\partial a} \right)^2 \frac{d c_v}{d \xi} - \frac{d}{d \xi} (c_v m_0 \gamma') \frac{\partial \xi}{\partial a} \right] = \frac{\partial \xi}{\partial \epsilon}. \quad (15)$$

Both Eqs (14) and (15) are identical to the relationships derived by Mikasa (1963), for conditions without and with self-weight respectively.

CONCLUDING COMMENTS

The above analysis has shown that the Gibson, England and Hussey (1967) theory and the Mikasa (1963) theory differ in their underlying assumptions in only one respect. Mikasa's theory is limited to the case where the deposit consolidates under its own weight, with or without an imposed surface loading, after rapid sedimentation. The Gibson, England, Hussey theory is unrestricted as to its initial condition. It applies equally to rapid sedimentation, slow sedimentation (Schiffman and Cargill, 1981) and to the progress
of consolidation of loaded clay layers (Gibson, Schiffman and Cargill, 1981).

Both theories are identical when self-weight is ignored. In this case, both theories assume a constant initial void ratio.

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