ON MECHANICS OF TWO-DIMENSIONAL RIGID ASSEMBLIES

MASAHIRO AI*

ABSTRACT

This paper investigates the collapse of rigid assemblies under applied forces, by using a two-dimensional model of contacting rigid discs. Expansions are made rigorously for both the frictionless and the Coulomb-frictional contacts, respectively. For the frictionless assemblies, some existing mathematical theorem and an associated calculation procedure are introduced, which are precisely applicable to their mechanics. As for the frictional assemblies, by employing new decompositions of a contact force and its relative displacement, conjugate to each other, and by showing the existence of frictionless rigid assemblies of another kind with common static conditions to them, it is concluded that the limits of their static equilibria do not coincide with the collapse initiations.

Key words : yield, constitutive equation of soil, friction, granular material (IGC : E 0)

INTRODUCTION

While, in practice, soils and other granular materials are dealt with as continua, we can also find, in the past, attempts to analyze them as the original assemblies of particles (eg., Deresiewicz, 1958 ; Horne, 1965, 1969). Recently, the significant dependence of their overall properties on the spatial arrangements of particles has been recognized, with using the concept of so-called "fabric" (Field, 1963 ; Brewer, 1964 ; Oda, 1972), and many theoretical and experimental researches have been made to relate the overall properties to the mechanics of particles (Satake, 1978 ; Cundall and Strack, 1979 ; Christoffersen et al, 1981 ; Mehrabadi et al, 1982 ; Nemat-Nasser and Mehrabadi, 1982 ; Thornton and Barnes, 1982).

The researches of granular masses can be divided into two classes. The first is to assign degrees of freedom to each particle, and to solve the (dynamic) contacting problem for all the grains (eg., Cundall and Strack, 1979 ; Kawai and Takeuchi, 1982). Provided a recent digital computer, this attempt can be attained, numerically, for a fairly large number of frictionless and frictional grains. The second approach is to take averages of the mechanical factors over a representative assembly, and to develop macroscopic relations, by regarding as a continuous body (Christoffersen et al, 1981 ; Mehrabadi et al, 1982 ; Nemat-Nasser and Mehrabadi, 1982). In analyses of the latter, geometrical quantities which represent the effect of the grain arrangement on the overall mechanics are called the "fabric tensors," and,

* Associate Professor, Department of Civil Engineering, Hosei University, Koganei, Tokyo.
Manuscript was received for review on August 6, 1984.

Written discussions on this paper should be submitted before July 1, 1986, to the Japanese Society of Soil Mechanics and Foundation Engineering, Sugayama Bldg. 4F, Kanda Awaji-cho 2-23, Chiyoda-ku, Tokyo 101, Japan. Upon request the closing date may be extended one month.

so far, several definitions have been proposed for them.

This paper intends to analyze the collapse initiation of rigid assemblies (a discrete analysis). The mechanics of granular masses is characterized more dominantly by the mutual discrete motions of grains, rather than by the deformation properties of grains themselves. It is difficult to rigorously develop mechanical relations for such a large number of grains as contained in an actual sample. Although a very large number of grains must be considered to derive realistic relations, the analytical results obtained for an assembly of not so many grains are still useful for both theoretical and practical applications. In this paper, two-dimensional assemblies of finite numbers of rigid discs are put on analysis, rigorously and analytically, with both frictionless and frictional contacts, respectively.

GEOMETRICAL RELATIONS

Consider an assembly of rigid discs in two dimensions, with x and y as the background Cartesian coordinates. The location of each disc is defined by the coordinates of its center, and its orientation is characterized by the angle θ measured from the x -direction to a permanent radial mark on the disc, as shown in Fig. 1. In the following expansions, a boldface letter or a letter in braces means a column vector, unless the contrary is explicitly stated.

Relation between Two Discs: For two typical discs, m' and m , of the assembly, let $\mathbf{R}$

Fig. 1. Geometric relations between two discs

be a vector connecting the center of m' to that of m , and $\psi_{m'}$ and ψ_m be the corresponding angular positions of the discs, measured from the $\mathbf{R}$ -direction. The geometric relation between the two discs is then specified by the set of three parameters, $\{R, \psi_{m'}, \psi_m\}$, where $R=|\mathbf{R}|$; for any rigid displacement of the two discs, these parameters remain constant, and any relative change of the discs would involve at least a change in these parameters. Other parameters may be used to describe the relative position of two discs, but they will by necessity relate uniquely to the parameters stated above by an appropriate transformation.

We define a set of alternative three parameters, as shown in Fig. 2. Let the two discs, assumed to have initially been at contact with their radial marks in $\mathbf{R}$ -direction, be brought to any relative positions through the following three steps: (1) translate the two discs in $\mathbf{R}$ -direction to the final distance R , which has been defined and now called "separation"; (2) rotate m' and m in θ - and the opposite directions, respectively, with the same turning distance z on surfaces of the discs. Let z be called "rolling"; and (3), finally, rotate both of them in θ -direction with the same turning distance $s/2$ on their surfaces. We call twice the distance, s , "sliding." These $\{R, z, s\}$ are related to the spatial positions of m' and m , $\mathbf{X}_{(m)} = \{x, y, \theta\}_{(m)}$ and $\mathbf{X}_{(m')} = \{x, y, \theta\}_{(m')}$, as follows:

$$\left. \begin{aligned} R &= \{(x_m - x_{m'})^2 + (y_m - y_{m'})^2\}^{1/2} \\ z &= \left(\frac{r_{m'}}{2} \psi_{m'} - \frac{r_m}{2} \psi_m \right) \end{aligned} \right\}$$

Separation

Rolling

Sliding

Fig. 2. Parameters defining relative motion of two discs

$$\left. \begin{aligned}
 &= \frac{r_{m'}}{2} \theta_{m'} - \frac{r_m}{2} \theta_m + \frac{r_m - r_{m'}}{2} \alpha \\
 s &= r_{m'} \psi_{m'} + r_m \psi_m \\
 &= r_{m'} \theta_{m'} + r_m \theta_m - (r_m + r_{m'}) \alpha
 \end{aligned} \right\} \quad (1. a \sim c)$$

where $r_{m'}$ and r_m are radii of m' and m , respectively; and

$$\alpha = \arctan \left(\frac{y_m - y_{m'}}{x_m - x_{m'}} \right) \quad (2)$$

When the two discs are in motion with "modes" of displacements, $\dot{\mathbf{X}}_{(m')}$ and $\dot{\mathbf{X}}_{(m)}$, then the associated "mode" of the relative displacements is obtained by taking the differentials in Eq. (1. a-c):

$$\begin{aligned}
 &\left\{ \begin{array}{c} \dot{\mathbf{R}} \\ \dot{\mathbf{z}} \\ \dot{\mathbf{s}} \end{array} \right\} \\
 &= \begin{bmatrix} -\cos \alpha, & -\sin \alpha, & 0 \\ \frac{r_m - r_{m'}}{2R} \sin \alpha, & -\frac{r_m - r_{m'}}{2R} \cos \alpha, & \frac{r_{m'}}{2} \\ -\frac{r_m + r_{m'}}{R} \sin \alpha, & \frac{r_m + r_{m'}}{R} \cos \alpha, & r_{m'} \\ \cos \alpha, & \sin \alpha & 0 \\ -\frac{r_m - r_{m'}}{2R} \sin \alpha, & \frac{r_m - r_{m'}}{2R} \cos \alpha, & -\frac{r_m}{2} \\ \frac{r_m + r_{m'}}{R} \sin \alpha, & -\frac{r_m + r_{m'}}{R} \cos \alpha, & r_m \end{bmatrix} \\
 &\left\{ \begin{array}{c} \left(\begin{array}{c} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{array} \right)_{(m')} \\ \dots \\ \left(\begin{array}{c} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{array} \right)_{(m)} \end{array} \right\} \quad (3)
 \end{aligned}$$

where superimposed asterisk indicates a rate relative to each other in braces; i. e., vector $\{\dot{\mathbf{F}}\}$ means a set of derivatives of F 's with respect to any common parameter.

Relation to Rigid Boundary: We assume that the assembly may have contact with rigid boundaries. For simplicity, let the rigid boundaries be so smooth enough with small curvatures that each disc can have only one contact with each of them, at most.

Fig. 3. Geometric relations to a boundary

Let B be a boundary of this kind, and define orthogonal curvilinear coordinates $\{\xi, \eta\}$ in such a manner that ξ measures the arc length along surface of B , and that η is outer-normal to B (see Fig. 3). The boundary B is then defined by a one-parameter curve $\{x_B(\xi), y_B(\xi)\}$.

For a typical disc m , let ξ^N be the point of intersection of B with a line drawn from the center of m normal to B . The coordinate of this intersection is obtained by the equation

$$\begin{aligned}
 &\{x_m - x_B(\xi^N)\} x_B'(\xi^N) \\
 &+ \{y_m - y_B(\xi^N)\} y_B'(\xi^N) = 0 \quad (4)
 \end{aligned}$$

where prime indicates the differentiation with respect to ξ . By the use of the ξ^N and cosines of the normal direction, $\{-y_B'^N, x_B'^N\}$, the normal distance R_B of the center of m from B is given by

$$R_B = -(x_m - x_B^N) y_B'^N + (y_m - y_B^N) x_B'^N \quad (5)$$

where superscript N indicates the value evaluated at $\xi = \xi^N$.

To characterize the position of disc m relative to the boundary B , imagine that the disc is initially at such a position where it contacts B at origin $\xi=0$ with its radial mark normal to B . To bring the disc to its actual position, consider the following three steps: (1) let m roll on the boundary B until the contact point reaches ξ^N . This defines the "rolling" parameter z_B ; (2) rotate m so that its final angular position is attained. This defines the "slid-

ing" parameter s_B ; and, finally, (3) translate m to its final position. This defines the "separation" parameter R_B . The set of parameters $\{R, z, s\}_B$ defines the position of m relative to the boundary B . With respect to the disc position, we have

$$\begin{cases} z_B = \xi^N \\ s_B = r_m(\theta_m - \pi - \alpha_B^N) + \xi^N \end{cases} \quad (6.a, b)$$

where α_B is given by

$$\alpha_B(\xi) = \arctan\left(-\frac{x_B'(\xi)}{y_B'(\xi)}\right) \quad (7)$$

The associated rate quantities are now given by

$$\begin{aligned} & \begin{pmatrix} \dot{R} \\ \dot{z} \\ \dot{s} \end{pmatrix}_B \\ &= \begin{bmatrix} \cos \alpha_B^N, & \sin \alpha_B^N, & 0 \\ \frac{1}{1-R_B \kappa_B^N} \sin \alpha_B^N, & \frac{-1}{1-R_B \kappa_B^N} \cos \alpha_B^N, & 0 \\ \frac{1-r_m \kappa_B^N}{1-R_B \kappa_B^N} \sin \alpha_B^N, & -\frac{1-r_m \kappa_B^N}{1-R_B \kappa_B^N} \cos \alpha_B^N, & r_m \end{bmatrix} \\ & \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}_{(m)} \end{aligned} \quad (8)$$

where κ_B is the curvature of B , i.e.,

$$\kappa_B(\xi) = x_B'(\xi)y_B''(\xi) - y_B'(\xi)x_B''(\xi) \quad (9)$$

GOVERNING RELATIONS

Inequalities of Compatibility

In an assembly of discs, $m=1, \dots, M$, distance $R_{m,m'}$ of centers of any two discs, m' and m , can not be less than the sum of their radii,

$$\begin{aligned} R_{m,m'} &= \{(x_m - x_{m'})^2 + (y_m - y_{m'})^2\}^{1/2} \\ &\geq r_{m'} + r_m \end{aligned} \quad (10)$$

Similarly, the normal distance of m from boundary B can not be less than the radius of m ,

$$R_B^m = -(x_m - x_B^N)y_B'^N + (y_m - y_B^N)x_B'^N \geq r_m \quad (11)$$

where the equality holds in (10) and (11) when there is actual contact. In the as-

sembly, the contacts are numbered as $j=1, \dots, J$.

When the discs are in motion, relations (3) and (8) for the rates of relative displacements at all the contacts may be rewritten in a unified matrix form as

$$\begin{pmatrix} \dot{n} \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}_1 = [Q(\{X_{(m)}\})] \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}_{(1)} \quad (12)$$

where $\dot{n}$ is the rate of change of the normal contact displacement, i.e., $\dot{n} = -\dot{R}$. Then the normal displacements at contacts must remain nonpositive,

$$\{\dot{n}_j\} = [Q_{(n)}(\{X_{(m)}\})] \{\dot{X}_{(m)}\} \leq \{0\} \quad (13)$$

where $[Q_{(n)}]$ is the partial matrix obtained by collecting relevant rows of $[Q]$, which are associated with the corresponding normal displacements.

Equilibrium Equations

First, as usual, we decompose a contact force into a compressive normal and a shear component, $\mathbf{f} = \{N, S\}$, such that, on rates of contact displacements, $\dot{\mathbf{e}} = \{\dot{n}, \dot{s}\}$, their inner product, $N\dot{n} + S\dot{s}$, represents the real rate of energy dissipation at the contact. The admissibility condition for the reactive force at a frictionless contact is

$$N \geq 0 \quad (14)$$

with S being identically zero. At a contact with the Coulomb friction, one has

$$\begin{aligned} & N \geq 0 \\ & -N \tan \phi \leq S \leq N \tan \phi \end{aligned} \quad (15.a, b)$$

where ϕ is friction angle. Here, inequalities (15.a, b) mean that the contact force is restricted within the convex cone with two generators specified by the friction angle, as shown in Fig. 4. Therefore, representing the contact force vector by the linear combination of two unit vectors along these

Fig. 4. Contact force and displacement

generators, we can employ the corresponding coefficients, namely $\mathbf{f} = \{f^-, f^+\}$, as alternative components of the contact force. Then inequalities (15. a, b) imply

$$\mathbf{f} = \{f^-, f^+\} \geq \{0, 0\} \quad (16)$$

The quantities $\mathbf{f}$ and $\mathbf{f}$ are related by

$$\mathbf{f} = [g]\mathbf{f} \quad (17)$$

where

$$[g] = \begin{bmatrix} \cos \phi & \cos \phi \\ -\sin \phi & \sin \phi \end{bmatrix} \quad (18)$$

Consider a frictional disc assembly at an instant when external forces $\{\mathbf{P}_{(m)}\} = \{(P_x, P_y, M)_{(m)}, m=1, \dots, M\}$ are applied to the discs, where (P_x, P_y) are force components in (x, y) -directions, and M is moment in θ -direction. Initially, the discs are at rest, at positions $\{\mathbf{X}_{(m)}\} = \{(x, y, \theta)_{(m)}, m=1, \dots, M\}$ satisfying the geometrical constraints (10) and (11). Since, under the applied forces, the assembly may not necessarily be in static equilibrium, dynamic equilibrium is now considered with inertial forces and moments taken into account through the application of d'Alembert's principle.

When a "mode" of the discs' accelerations is denoted by $\{\dot{\mathbf{X}}_{(m)}^*\}$, any vector of real accelerations framed in that mode is given in the form $d^2\{\mathbf{X}_{(m)}\}/dt^2 = \beta\{\dot{\mathbf{X}}_{(m)}^*\}$, with $\beta(\geq 0)$ being associated magnitude of the accelerations. By choosing both mode $\{\dot{\mathbf{X}}_{(m)}^*\}$ and magnitude β , we can represent any ac-

celerations of the discs. With zero initial velocities, $\{\dot{\mathbf{X}}_{(m)}^*\}$ gives also the displacement mode after an infinitesimal time-increment. The equation of virtual work at the instant may be written down as

$$\delta\{\mathbf{e}_j\}^T\{\mathbf{f}_j\} - \delta\{\mathbf{X}_{(m)}\}^T(\{\mathbf{P}_{(m)}\} - \beta[W]\{\dot{\mathbf{X}}_{(m)}^*\}) = 0 \quad (19)$$

where $\delta\{\mathbf{e}_j\}$ represents a set of the contact displacements associated to (infinitesimal) virtual displacements $\delta\{\mathbf{X}_{(m)}\}$; and $[W]$ is diagonal mass matrix containing masses and moments of inertia of discs. After introducing relation (12), and from the arbitrariness of $\delta\{\mathbf{X}_{(m)}\}$, we obtain the equations of dynamic equilibrium,

$$[Q_{(ns)}(\{\mathbf{X}_{(m)}\})]^T\{\mathbf{f}_j\} + \beta[W]\{\dot{\mathbf{X}}_{(m)}^*\} = \{\mathbf{P}_{(m)}\} \quad (20)$$

where $[Q_{(ns)}]$ is the partial matrix of $[Q]$ in (12), obtained by collecting all the (n, s) -rows. Further, when the $\{f^-, f^+\}$ -components are employed in place of $\{N, S\}$, introducing the transformation (17), we arrive at

$$[Q_{(-+)}(\{\mathbf{X}_{(m)}\})]^T\{\mathbf{f}_j\} + \beta[W]\{\dot{\mathbf{X}}_{(m)}^*\} = \{\mathbf{P}_{(m)}\} \quad (21)$$

where

$$[Q_{(-+)}] = [G]^T[Q_{(ns)}] \quad (22)$$

and $[G]$ is square matrix containing $[g]_j$, $j=1, \dots, J$, as diagonal elements.

In a frictionless assembly, on the other hand, there is no shear reactions at contacts, and hence there exists no resistances to the disc rotations. We then ignore the degrees of rotational freedom in $\{\mathbf{X}_{(m)}\}$, the external moments in $\{\mathbf{P}_{(m)}\}$, and the shear reactions in $\{\mathbf{f}_j\}$, respectively, and write the equilibrium equations as

$$[Q_{(n)}(\{\mathbf{x}_{(m)}\})]^T\{N_j\} + \beta[w]\{\dot{\mathbf{x}}_{(m)}^*\} = \{\mathbf{p}_{(m)}\} \quad (23)$$

where the lower case $\{\mathbf{x}_{(m)}\}$, $\{\mathbf{p}_{(m)}\}$ and $[w]$ indicate the vectors and the matrix obtained by removing the rotational or moment components from $\{\mathbf{X}_{(m)}\}$, $\{\mathbf{P}_{(m)}\}$ and $[W]$, respectively.

Kinematic Relations

When an assembly of rigid discs is not stationary, each contact force is in a certain relation with the corresponding contact displacement. At a frictionless contact, the separation takes place only if the normal reactive force is zero, and hence

$$Nn^* = 0 \quad (24)$$

is satisfied at any stage. At a contact with the Coulomb friction, on the other hand, the kinematic condition is a little more complicated. In this case, relation (24) holds also for the normal component of the contact force. When the shear component satisfies inequalities in (15. b), then there is no sliding, and when the left (right) equality holds, the sliding may take place in the direction $s^* < 0$ ($s^* > 0$). These restraints can be represented by

$$\left. \begin{aligned} (S - N \tan \phi) s^* &\geq 0 \\ (S + N \tan \phi) s^* &\geq 0 \\ Nn^* &= 0 \end{aligned} \right\} \quad (25 \cdot a \sim c)$$

Now, if the $\{F^-, F^+\}$ -components are to describe the contact force, the above relations are transformed into

$$\left. \begin{aligned} -F^- s^* &\geq 0 \\ F^+ s^* &\geq 0 \\ (F^- + F^+) n^* &= 0 \end{aligned} \right\} \quad (26 \cdot a \sim c)$$

Further, it is possible to define contact displacement rate components $\dot{\mathbf{r}}^* = \{\dot{\mathbf{r}}^-, \dot{\mathbf{r}}^+\}$ conjugate to $\{F^-, F^+\}$ in such a manner that their inner product, $F^- \dot{\mathbf{r}}^- + F^+ \dot{\mathbf{r}}^+$, represents the real rate of energy dissipation at the contact. By introducing Eq. (17) into $\mathbf{F}^T \dot{\mathbf{r}}^* = \mathbf{f}^T \dot{\mathbf{e}}^*$, quantities $\{\dot{\mathbf{r}}^-, \dot{\mathbf{r}}^+\}$ can explicitly be represented in terms of $\{n^*, s^*\}$ as

$$\dot{\mathbf{r}}^* = [\mathbf{g}]^T \dot{\mathbf{e}}^* \quad (27)$$

This is shown in Fig. 4. By replacing $\dot{\mathbf{e}}^*$ by $\dot{\mathbf{r}}^*$ in inequalities (26. a-c), we obtain

$$\left. \begin{aligned} F^- \dot{\mathbf{r}}^- &\geq 0 \\ F^+ \dot{\mathbf{r}}^+ &\geq 0 \\ (F^- + F^+) (\dot{\mathbf{r}}^- + \dot{\mathbf{r}}^+) &= 0 \end{aligned} \right\} \quad (28 \cdot a \sim c)$$

COLLAPSE INITIATION

As stated before, the mechanics of a rigid assembly is governed by the *compatibility conditions*, the *kinematic relations* and the *equilibrium equations* with subsidiary constraints for the reactive forces. These governing equations and/or inequalities are sufficient for calculation of either static or dynamic equilibrium of the assembly under prescribed external forces. A solution with zero accelerations will be called a *static solution*, and one with nonzero accelerations, a *dynamic solution*. The equilibria are now classified as follows: (i) if for a given $\{\mathbf{P}_{(m)}\}$ there exists at least one dynamic solution with $\{\dot{\mathbf{X}}_{(m)}^*\} \neq \{0\}$ and $\beta > 0$, the equilibrium is defined to be *instable*; (ii) if for $\{\mathbf{P}_{(m)}\}$ there exist only static solutions with $\{\dot{\mathbf{X}}_{(m)}^*\} = \{0\}$ (and $\beta = 0$), i. e., if there exists no mode $\{\dot{\mathbf{X}}_{(m)}^*\}$ consistent with any equilibrating contact forces through the kinematic relations, the equilibrium is defined to be *stable*; and (iii) if there are only static solutions, but there exists at least one such static solution as with a collapse mode $\{\dot{\mathbf{X}}_{(m)}^*\} \neq \{0\}$ but zero accelerations, $\beta = 0$, the equilibrium is defined to be *critical*. We refer to both the stable and critical equilibria as *static* ones.

Critical Equilibrium of Frictionless System

There are a number of theorems relating to the duality of linear inequalities, which will be proved useful for analyzing the equilibrium of frictionless rigid assemblies. One is the Minkowsky-Farkas theorem (eg. Hu, 1969). From this theorem, we deduce that exactly one of the following alternatives holds: Either the static equilibrium equations

$$[\mathbf{Q}_{(n)}]^T \{\mathbf{N}_j\} = \{\mathbf{p}_{(m)}\} \quad (29)$$

have a solution for non-negative contact forces $\{\mathbf{N}_j\}$, or the inequalities

$$\left. \begin{aligned} [\mathbf{Q}_{(n)}] \{\dot{\mathbf{x}}_{(m)}^*\} &\leq \{0\} \\ \{\mathbf{p}_{(n)}\}^T \{\dot{\mathbf{x}}_{(m)}^*\} &> 0 \end{aligned} \right\} \quad (30 \cdot a, b)$$

have a solution for disc mode $\{\dot{\mathbf{x}}_{(m)}^*\}$. We

can see that inequality (30.a) is the compatibility conditions (13), and that (30.b) corresponds to decrease in the potential of external forces. Since the frictionless assembly is a conservative system, that is, there is no energy dissipation at contacts, as given in Eq. (24), the existence of a solution of (30.a, b) reveals that the assembly has a dynamic solution under $\{\mathbf{p}_{(m)}\}$. Eventually, the theorem yields nothing beyond the following result: if there exists a static (dynamic) solution, there is no other dynamic (static) solution under the same $\{\mathbf{p}_{(m)}\}$, and then if and only if there exists a static solution, the assembly is physically in static equilibrium with $\{\mathbf{p}_{(m)}\}$. This conclusion will, however, be significant in relation to equilibria of the frictional assemblies. It is easy to see that the set of $\{\mathbf{p}_{(m)}\}$ for which the equilibria are static forms a finite convex cone; for any load vector which is represented by a non-negative linear combination of static loads, the equilibrium is static.

The solution for the critical equilibrium can be obtained through the usual *linear programming* procedure. Given a vector of initial external forces $\{\mathbf{p}_{(m)}^0\}$ and a mode of incremental loading $\{\mathbf{p}_{(m)}^*\}$, the problem of determining the critical state on the linear path $\{\mathbf{p}_{(m)}\} = \{\mathbf{p}_{(m)}^0\} + \rho\{\mathbf{p}_{(m)}^*\}$ is mathematically equivalent to the following linear programming problem.

Primal Problem. Find both non-negative $\{N_j\}$ and (ρ^-, ρ^+) such that

$$\rho = \rho^+ - \rho^- \text{ is a maximum} \quad (31)$$

subject to

$$[Q_{(n)}]^T \{N_j\} - (\rho^+ - \rho^-) \{\mathbf{p}_{(m)}^*\} = \{\mathbf{p}_{(m)}^0\} \quad (32)$$

And, there is to be the following dual problem:

Dual Problem. Find vector $\{\mathbf{x}_{(m)}^*\}$ such that

$$\rho^* = \{\mathbf{p}_{(m)}^0\}^T \{\mathbf{x}_{(m)}^*\} \text{ is a maximum} \quad (33)$$

subject to

$$\left. \begin{aligned} [Q_{(n)}] \{\mathbf{x}_{(m)}^*\} &\leq \{0\} \\ \{\mathbf{p}_{(m)}^*\}^T \{\mathbf{x}_{(m)}^*\} &= 1 \end{aligned} \right\} \quad (34.a, b)$$

Once the two problems are solved by the *simplex* method, from the equilibrium theorem in duality, kinematic conditions (24) at all the contacts are satisfied by the optimal solutions, and from the optimal relation $\rho = -\rho^*$,

$$(\{\mathbf{p}_{(m)}^0\} + \rho\{\mathbf{p}_{(m)}^*\})^T \{\mathbf{x}_{(m)}^*\} = 0 \quad (35)$$

That is, the assembly subject to load $\{\mathbf{p}_{(m)}\} = \{\mathbf{p}_{(m)}^0\} + \rho\{\mathbf{p}_{(m)}^*\}$ is at a critical state with mode $\{\mathbf{x}_{(m)}^*\}$.

Similarly, whether or not the equilibrium is static for given $\{\mathbf{p}_{(m)}\}$ can be distinguished from the solution of the problem: find both non-negative $\{N_j\}$ and ρ^+ such that

$$\rho = \rho^+ \text{ is a minimum} \quad (36)$$

subject to

$$[Q_{(n)}]^T \{N_j\} - \rho^+ \{\mathbf{p}_{(m)}^*\} = \{\mathbf{p}_{(m)}\} \quad (37)$$

If $\rho = 0$ at the optimal state, the assembly is in static equilibrium under $\{\mathbf{p}_{(m)}\}$.

Critical Equilibrium of Frictional System

On a linear path of external forces, $\{\mathbf{p}_{(m)}\} = \{\mathbf{p}_{(m)}^0\} + \rho\{\mathbf{p}_{(m)}^*\}$, consider the infimum of ρ for any of which the frictional assembly has at least one dynamic solution, where the initial equilibrium with $\{\mathbf{p}_{(m)}^0\}$ is assumed to be stable. Since, at the infimum, the assembly has a collapse mode but with zero accelerations of discs, the infimum means the value of ρ at which the equilibrium is critical. The critical equilibrium corresponds to a solution for $\{\mathbf{f}_j\}$, ρ and nonzero $\{\mathbf{x}_{(m)}^*\}$ in the equations of static equilibrium

$$\left. \begin{aligned} [Q_{(-+)}] \{\mathbf{f}_j\} - \rho\{\mathbf{p}_{(m)}^*\} &= \{\mathbf{p}_{(m)}^0\} \\ \text{with subsidiary conditions } \{\mathbf{f}_j\} &\geq \{0\} \end{aligned} \right\} \quad (38)$$

together with compatibility conditions (13) and kinematic relations (26.a-c) or (28.a-c).

Denoting all possible (i.e., satisfying (13)) modes of disc displacements with one degree of freedom by $\{\mathbf{x}_{(m)}^*\}_\delta$, $\delta = 1, \dots, \Delta$, any admissible mode of one or more degrees of freedom can be represented by a nonnegative linear combination of the one-degree modes. Although, strictly speaking, the collapse may take place on a multi-degree mode, for the sake of simplicity, we examine only collapses

of one-degree freedom. In the present system of inequalities for frictional assemblies, it is difficult to develop a systematic solution procedure similar to the simplex method for frictionless ones, but the critical state can be obtained by try-and-error, as discussed below.

After expressing the contact displacements $\{e_j^*\}$ in terms of an assumed mode $\{\dot{X}_{(m)}^*\}_\delta$ through Eq. (12), we use the kinematic relations (26. a-c) to regulate the existence of contact force components $\{\Gamma^-, \Gamma^+\}$ at each contact: if $\{n^*, s^*\} = \{0, 0\}$ both Γ^- and Γ^+ may exist; if $n^* = 0$ and $s^* \neq 0$ then only one of $\{\Gamma^-, \Gamma^+\}$ can exist in such a manner that (26. a, b) hold; and if $n^* < 0$ neither component exists. The contact force components with zero values are eliminated in the equilibrium equations (38). Since the assumed mode $\{\dot{X}_{(m)}^*\}_\delta$ involves a mechanism with one degree of freedom but with an additional variable ρ , the reduced equations can be solved for $\{\Gamma_j^*\}$ and ρ , in general. In this solution, it may occur that $\{\Gamma_j^*\}$ can not be determined uniquely in the sense of redundant forces. However, ρ is unique as long as $\{\dot{X}_{(m)}^*\}_\delta^T \{\dot{P}_{(m)}^*\} \neq 0$. The admissibility of assumed $\{\dot{X}_{(m)}^*\}_\delta$ as a real collapse mode can be examined by observing whether every element of the resulting $\{\Gamma_j^*\}$ is non-negative. If the $\{\dot{X}_{(m)}^*\}_\delta$ is not admissible, the calculation is repeated for next $\{\dot{X}_{(m)}^*\}_\delta$ —until the real collapse mode is encountered.

Pseudo Orthogonal System

The equilibrium equations (38) are not restricted only to the frictional assemblies dealt with in the preceding subsection, but there exists another (imaginary) rigid system with the same equilibrium equations.

As a simple example, consider a point mass on a frictional plane, where the reactive force is restricted within the cone specified by friction angle ϕ , as shown in Fig. 5, and note that the same restriction applies to another point mass which rests at the corner of two frictionless planes inclined at

Frictional Contact

Frictionless Contact

Fig. 5. Associated frictionless contact

$\pm \phi$ from the former plane.

In an assembly of rigid discs, the similar frictionless supports can be imagined in place of the frictional contacts, in such a manner that, at each contact, the corresponding separation n^* and sliding s^* are geometrically restricted by

$$\begin{Bmatrix} \gamma^- \\ \gamma^+ \end{Bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \cos \phi & \sin \phi \end{bmatrix} \begin{Bmatrix} n^* \\ s^* \end{Bmatrix} \leq \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (39)$$

as shown in Fig. 6. Since reactive force components Γ^- and Γ^+ are perpendicular to the directions of γ^+ and γ^- , respectively, the reactive force is restricted also by conditions (16). Then, after replacing all the frictional contacts by the imaginary frictionless supports, Eq. (38) still holds as static equilibrium equations for the replaced assembly. However, as stated before, the compatibility conditions are changed to

$$\{\gamma_j^*\} = [Q_{(-+)}] \{\dot{X}_{(m)}^*\} \leq \{0\} \quad (40)$$

and the associated kinematic relations are given by

$$\begin{cases} \Gamma^- \gamma^- = 0 \\ \Gamma^+ \gamma^+ = 0 \end{cases} \quad (41. a, b)$$

The assembly of rigid discs, governed by (38), (40) and (41. a, b), are now called *pseudo orthogonal system*, compared to the original frictional system governed by (38), (13) and (28. a-c).

It can be seen that each expansion made before for a frictionless assembly with normal

Fig. 6. Frictionless pseudo contact

contacts also apply to the present pseudo orthogonal system only by replacing $[S_{(n)}]$, $\{N_j\}$, $\{n_j^*\}$, $\{P_{(m)}\}$ and $\{x_{(m)}^*\}$ for the former by $[S_{(-+)}]$, $\{F_j\}$, $\{f_j^*\}$, $\{P_{(m)}\}$ and $\{X_{(m)}^*\}$ of the latter, respectively. Then, the critical state can be analyzed by solving the linear programming problem to find the extremal one among loadings for which there exist non-negative $\{F_j\}$ satisfying (38), and the optimal solution of its dual problem gives the associated collapse mode.

Since static equilibrium equations (38) are common to both the frictional and associated pseudo orthogonal assemblies, the optimal value of ρ corresponds to the supremum of ρ for any of which the original frictional assembly has at least a static solution. Let this supremum be called *maximum* value, compared with the critical value.

NUMERICAL EXAMPLE

As an illustrative example, we consider five identical discs, initially located on two planes inclined by $\pm 15^\circ$ from the horizontal plane, as shown in Fig. 7. Let the radius and dead-weight of a disc be taken as units of length and force, respectively, and the gravitational acceleration be denoted by g . With a common frictional angle $\phi = 17.19^\circ$, the discs with the arrangement shown in Fig. 7 are calculated as being in static equilibrium under their dead weights, but when assumed no frictions, they are not.

Six different loading conditions are considered. Additional proportional loads are

Fig. 7. Initial configuration

applied to disc 2, 4 and 5 in accordance with $(\bar{P}_{2x}^*, \bar{P}_{4y}^*, \bar{P}_{5y}^*) = (1, 0, 0)$, $(0, -1, 0)$, $(0, 0, -1)$, $(1, -1, 0)$, $(1, 0, -1)$ and $(0, -1, -1)$, with ρ denoting the proportionality or the load-factor. The critical and maximum values of this load-factor are calculated for each loading condition by regarding the discs as frictional, pseudo orthogonal and frictionless assemblies. The results are given in Fig. 8, in which points marked by $\bullet$, $\times$ and $\circ$ indicate the maximum point as a pseudo orthogonal assembly, the critical point as a frictional one, and the critical point as a frictionless one, respectively. While for the three loading regimes, $(1, 0, 0)$, $(1, -1, 0)$ and $(1, 0, -1)$, the critical values of the load-factor ρ coincide with the maximum values, for the remaining three cases, i.e. $(0, -1, 0)$, $(0, 0, -1)$ and $(0, -1, -1)$, however, the critical values are less than the associated maximum values.

When the discs are regarded to be frictionless, the two critical states are obtained on load path $(1, 0, 0)$, as shown in Fig. 8; this frictionless assembly is then in static equilibrium for loadings between these two critical loads. With $\rho_{cr.} = 0.6603$, the corresponding contact forces and collapse mode are obtained as given in Table 1 and shown in Fig. 9.

As a pseudo orthogonal assembly under loading regime $(0, 0, -1)$, the contact forces

Fig. 8. Critical and maximum points of loading

at the maximum $\rho=1.8470$ are calculated to satisfy equilibrium equations (38); these are given in Table 2, and the corresponding orthogonal collapse mode is shown in Fig. 10. There is, however, no relevant critical mode as a frictional assembly, which is statically consistent with these contact forces at $\rho=1.8470$ through kinematic relations (26. a-c) or (28. a-c). As a frictional one with the same loading regime and with the collapse mode (assumed to be that) shown in Fig. 11, the admissible contact forces and the critical ρ which satisfy both Eqs. (38) and (26. a-c) are obtained as given in Table 2. Similarly, for loading regime $(0, -1, 0)$, the contact forces associated with the critical and maximum loads are calculated as given in Table 3, and the collapse mode

Fig. 9. Collapse mode for $(\bar{P}_{2x}^*, \bar{P}_{4y}^*, \bar{P}_{5y}^*) = (1, 0, 0)$: a frictionless assembly

Table 2. Critical and maximum contact forces for $(\bar{P}_{2x}^*, \bar{P}_{4y}^*, \bar{P}_{5y}^*) = (0, 0, -1)$

Contact No.	Critical Reactions ($\rho_{cr.}=1.1399$)		Maximum Reactions ($\rho_{max}=1.8470$)	
	Γ^-	Γ^+	Γ^-	Γ^+
1	0.0	0.0	0.0	0.0
2	0.1272	0.0	0.7697	0.0
3	0.3146	0.8796	0.2849	0.4655
4	0.0	0.0	0.0	0.4853
5	0.5766	0.1386	0.0	0.5891
6	0.9306	1.3685	1.6427	1.0536
7	0.4822	0.0	0.0303	0.0
8	0.0196	0.6290	0.9119	1.2266
9	1.3489	0.7838	0.9348	0.7541
10	1.8740	1.4361	1.7267	1.8306

Table 1. Critical normal contact forces for $(\bar{P}_{2x}^*, \bar{P}_{4y}^*, \bar{P}_{5y}^*) = (1, 0, 0)$

Contact No.	N
1	0.5977
2	0.6671
3	0.5977
4	0.2588
5	0.2588
6	0.9659
7	1.9319
8	0.0
9	1.3127
10	1.9319

for the critical ρ is obtained as shown in Fig. 12. Further, when at the maximum $\rho=9.5805$ the equations of dynamic equilibrium, (21), are solved on that critical collapse mode, nonzero accelerations of the

Fig. 10. Pseudo collapse mode for $(\bar{P}_{2x}^*, \bar{P}_{4y}^*, \bar{P}_{5y}^*) = (0, 0, -1)$

Fig. 11. Collapse mode for $(\bar{P}_{2x}^*, \bar{P}_{4y}^*, \bar{P}_{5y}^*) = (0, 0, -1)$

discs, as given in Table 4, are obtained together with non-negative contact forces.

DISCUSSION

When the critical state of a rigid assembly

Table 3. Critical and maximum contact forces for $(\bar{P}_{2x}^*, \bar{P}_{4y}^*, \bar{P}_{5y}^*) = (0, -1, 0)$

Contact No.	Critical Reactions ($\rho_{cr.} = 9.4003$)		Maximum Reactions ($\rho_{max} = 9.5805$)	
	Γ^-	Γ^+	Γ^-	Γ^+
1	0.0	0.4153	0.0	0.4645
2	0.3112	0.0	0.4558	0.0
3	0.0384	0.7418	0.0	0.6464
4	0.0	1.9667	0.0	2.0993
5	0.3923	0.0	0.2602	0.0697
6	0.3701	0.7623	0.4397	0.6303
7	0.0	0.0	0.0	0.0
8	2.3906	0.3198	2.5263	0.4183
9	1.5214	0.4027	1.4627	0.3518
10	6.6993	4.3404	6.7041	4.4143

Fig. 12. Collapse mode for $(\bar{P}_{2x}^*, \bar{P}_{4y}^*, \bar{P}_{5y}^*) = (0, -1, 0)$

is required on a prescribed linear path of external forces (proportional loading), the scheme of finding the extremal value of external forces, for which exist admissible contact forces to satisfy the static equilibrium equations, seems reasonable toward the solution. While, as stated before, this scheme can be attained by applying the linear opti-

Table 4. Dynamic equilibrium for $(\bar{P}_{2x}^*, \bar{P}_{4y}^*, \bar{P}_{5y}^*) = (0, -1, 0)$

Contact No.	at Maximum Load ($\rho_{max} = 9.5805$)		Disc No.	Accelerations (in g)		
	Γ^-	Γ^+		x	y	θ
1	0.0	0.4241	(1)	-0.00160	0.00043	0.00166
2	0.3129	0.0	(2)	-0.00160	0.00043	0.00166
3	0.0352	0.7421	(3)	-0.00160	0.00043	-0.00166
4	0.0	2.0016	(4)	-0.00138	-0.00037	0.00143
5	0.3912	0.0	(5)	-0.00138	-0.00037	-0.00143
6	0.3692	0.7628				
7	0.0	0.0				
8	2.4259	0.3104				
9	1.5293	0.3955				
10	6.8042	4.4065				

mization procedure, the physical interpretation of its dual problem leads to the following insight into the mechanics of rigid assemblies :

It is for an orthogonal assembly that the linear optimization ensures determining the critical state. The physical system of frictionless rigid assemblies is in complete correspondence with that mathematical system ; the critical values of external forces and contact forces are obtained as the optimal solution of the primal problem related to the static equilibrium conditions, simultaneously, the associated collapse mode is given by the optimal solution of the dual problem related to the compatibility conditions, and, by the equilibrium theorem, the two optimal solutions satisfy the kinematic relations. From the duality of the two conditions, for a loading larger than the critical value of ρ , the assembly has only dynamic solutions, and, for a loading equal to or less than the critical value, the equilibrium is static.

When a frictional rigid assembly is considered, on the other hand, physically, energy dissipations may take place during sliding at contacts. Mathematically, the static equilibrium and the compatibility conditions are out of the duality. As the infimum of dynamic solutions, we can consider the critical value of loading, for which the equilibrium is static but contains a collapse mode with zero accelerations of discs. At the same time, as the supremum of static solutions, we can calculate the maximum value, by applying the linear optimization to the static equilibrium equations. As shown in the former numerical example, these two values are different, in general. Since satisfying the static equilibrium equations is a necessary condition also for the critical solution, the critical value does not exceed the maximum value. For a loading between these two values, we can not determine whether the equilibrium is static or dynamic.

Consider a simple example (see Fig.13), where disc m is supported by two planes with sufficient friction angles, ϕ and ϕ' , such

that the compressive redundant reaction f_R can exist within the friction angles. For any external force $\mathbf{P}_{(m)} = \{P_x, P_y, M\}_{(m)}$, we can take such a large f_R that reactive forces in static equilibrium exist within the friction angles. There exist, also, such solutions that under certain $\mathbf{P}_{(m)}$ the disc leaves the planes with finite accelerations. Then, the domain of $\mathbf{P}_{(m)}$ for which the equilibria are static depends upon the range of f_R . In the present formulations for perfect rigid discs, we can not determine that range. The solution of this problem is indefinite itself.

Generally speaking, on a linear path of external forces, the critical and maximum values give the infimum and supremum of collapse loads, respectively, associated to various values of redundant forces. In an actual problem, the unique true collapse depends upon the history of external forces and the deformation characteristics of discs, etc. However, it is to be noted that even if a frictional assembly of perfect rigids is in a static equilibrium with external forces beyond the critical value, this equilibrium is physically *instable* for geometrical disturbance of rigids' positions ; for instance, when introduced infinitesimal displacements of rigids in accordance with the collapse mode related to the critical value, the rigids start motion with finite accelerations into the direction of the collapse mode. The critical values are said to regulate the stable domain of external forces among those with static solutions.

Fig. 13. Redundant force

CONCLUSION

For a frictionless rigid assembly, the set of external forces for any of which there exist contact forces satisfying the static equilibrium condition forms a finite (convex) cone with a vertex at the origin in the space of external forces. This finite cone definitely determines the static equilibria of the assembly, and, on its surface, the critical collapses take place into its outer-normal directions, as shown in Fig. 14(a).

As for a frictional rigid assembly, similarly, we can also consider a finite cone of external forces for any of which there exist at least a set of contact forces to satisfy the static equilibrium conditions. But, from the non-duality between the equilibrium and the compatibility conditions, the critical collapses with zero accelerations of rigids do not exist on the (*maximum*) surface of this finite cone, in general. While for a loading out of this cone there never exists a static solution, however, under an interior loading near the surface, the assembly may have both static and dynamic solutions. Within the maximum surface, we can consider a subset of external forces for which the assembly does not have any dynamic solutions, and, on the (*critical*) surface of this subset, the collapses with zero rigid accelerations take place into its outer directions to yield positive energy dissipations with the critical external forces, as shown in Fig. 14(b). Under a loading between the maximum and the critical surfaces, we can not determine whether the assembly of perfect rigids is in static or dynamic equilibrium. If we suppose a static equilibrium in that region, it is unstable for infinitesimal disturbance of rigid positions.

The attainment of this paper is far from to derive sufficient mechanical relations for actual granular materials. But, the present expansions and conclusions give a few of the fundamental mechanical features of rigid assemblies, which are carried by the usual granular materials, and thus might be helpful to understand their phenomena, in both physical and engineering senses.

Fig. 14. Static and dynamic load vectors

ACKNOWLEDGEMENT

This work has been supported by the National Science Foundation under Grant No. CEE-8007764 to Northwestern University, with S. Nemat-Nasser as the Principal Investigator.

REFERENCES

- 1) Brewer, R. (1964) : Fabric and Mineral Analysis of Soils, J. Wiley and Sons.
- 2) Christoffersen, J., Mehrabadi, M. M. and Nemat-Nasser, S. (1981) : "A micromechanical description of granular material behavior," Journal of Applied Mechanics, Vol. 48, No. 2, June, pp. 339-344.
- 3) Cundall, P. A. and Strack, O. D. L. (1979) : "A discrete numerical model for granular assemblies," Géotechnique 29, No. 1, pp. 47-65.

- 4) Deresiewicz, H. (1958) : "Mechanics of granular matter," *Advances in Applied Mechanics* 5, pp. 233-306.
- 5) Field, W. G. (1963) : "Towards the statistical definition of a granular mass," *Proc. 4th A. and N. Z. Conf. on Soil Mech.*, pp. 143-148.
- 6) Horne, M. R. (1965) : "The behaviour of an assembly of rotund, rigid, cohesionless particles. I and II," *Proc. Roy. Soc. London, A* 286, pp. 62-78 and 79-97.
- 7) Horne, M. R. (1969) : "The behaviour of an assembly of rotund, rigid, cohesionless particles. III," *Proc. Roy. Soc. London, A* 310, pp. 21-34.
- 8) Hu, T. C. (1969) : *Integer Programming and Network Flows*, Addison-Wesley.
- 9) Kawai, T. and Takeuchi, N. (1982) : "New discrete models and their application to mechanics of granular materials," In J. T. Jenkins & M. Satake (eds.), *Proc. of the U. S./Japan Seminar on New Models and Constitutive Relations in the Mechanics of Granular Materials*, Ithaca, New York, pp. 151-172.
- 10) Mehrabadi, M. M., Nemat-Nasser, S. and Oda, M. (1982) : "On Statistical description of stress and fabric in granular materials," *Int. J. Num. An. Methods Geomechanics*, Vol. 6, pp. 95-108.
- 11) Nemat-Nasser, S. and Mehrabadi, M. (1982) : "Stress and fabric in granular masses," In J. T. Jenkins & M. Satake (eds.), *Proc. of the U. S./Japan Seminar on New Models and Constitutive Relations in the Mechanics of Granular Materials*, Ithaca, New York, pp. 1-8.
- 12) Oda, M. (1972) : "Initial fabrics and their relationships to mechanical properties of granular material," *Soils and Foundations*, Vol. 12, No. 1, pp. 17-36.
- 13) Satake, M. (1978) : "Constitution of mechanics of granular materials through the graph theory," In S. C. Cowin & M. Satake (eds.), *Proc. of U. S.-Japan Seminar on Continuum-Mechanical and Statistical Approaches in the Mechanics of Granular Materials*, Gakujutsu Bunkai Fukyukai, Tokyo, pp. 47-63.
- 14) Thornton, C. and Barnes, D. J. (1982) : "On the mechanics of granular material," In P. A. Vermeer & H. J. Luger (eds.), *Proc. of IUTAM Symposium on Deformation and Failure of Granular Materials*, Delft, Balkema, pp. 69-77.