TECHNICAL NOTE

A GENERAL FAILURE CRITERION FOR FRICTIONAL AND COHESIVE MATERIALS

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ABSTRACT

The Matsuoka-Nakai failure criterion for a frictional material, and the Von Mises criterion for a cohesive material are expressed in such a way that they are seen to bear similar relationships to the cohesionless Mohr-Coulomb and Tresca criteria respectively. A generalisation of the Matsuoka-Nakai and Von Mises criteria for materials with friction and cohesion is described, which bears a similar relationship to the general Mohr-Coulomb criterion. The new criterion is expressed conveniently in terms of principal stresses, and is also expressed in terms of stress invariants.

Key words: angle of internal friction, cohesion, effective stress, failure, friction, granular material, plasticity, stress, yield (IGC: D6)

INTRODUCTION

The failure of frictional soils is reasonably well modelled in terms of effective stresses by the cohesionless Mohr-Coulomb condition, but it has been shown recently that the effect of the intermediate principal stress can be better modelled by using the Matsuoka-Nakai criterion. For undrained clays it is convenient to use total stress analysis, for which the Tresca criterion (identical to the frictionless Mohr-Coulomb) could be used. Experiments on metals have shown that in this case the Von Mises criterion may be more accurate. In some applications it is necessary to include both friction and cohesion in a single model, and it is common in this case to use the general Mohr-Coulomb criterion. The purpose of this note is to describe a generalisation of the Matsuoka-Nakai and Von Mises criteria, which may be expected to fit the behaviour of soils better than the general Mohr-Coulomb expression.

There is a simple physical interpretation of the Tresca and Mohr-Coulomb expressions, but the physical interpretation of the Von Mises and Matsuoka-Nakai expressions is more complex. The generalisation described here is therefore derived algebraically rather than by use of physical arguments.

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ANALYSIS

The Tresca failure criterion for a cohesive material is often written in the form:

$$\tau_f = c$$

where \(\tau_f\) is the value of the shear stress on a failure plane and \(c\) is the shear strength. In soil mechanics terminology \(c\) is equal to the triaxial compressive strength. The criterion is also often written in terms of the principal stresses:

$$\sigma_1 - \sigma_2 = 2c$$

where \(\sigma_1, \sigma_2\) and \(\sigma_3\) are the principal stresses (using a compressive positive convention) and \(\sigma_1 > \sigma_2 > \sigma_3\). Note that this failure criterion does not depend on the value of the intermediate principal stress \(\sigma_2\). The Tresca condition may also be rewritten as:

$$\frac{(\sigma_1 - \sigma_2)^2}{c^2} = 4$$  (1)

The Von Mises failure condition is known to fit the behaviour of most cohesive materials rather better than the Tresca condition (Taylor and Quinney, 1931). One way of expressing the Von Mises criterion is:

$$\frac{(\sigma_1 - \sigma_2)^2}{c^2} + \frac{(\sigma_2 - \sigma_3)^2}{c^2} + \frac{(\sigma_3 - \sigma_1)^2}{c^2} = 8$$  (2)

where \(c\) is again the shear strength in triaxial compression. Although various physical interpretations of the Von Mises criterion have been suggested, none is as straightforward as the interpretation of the Tresca criterion, which represents a critical value of the maximum allowable shear stress on any plane.

The Mohr–Coulomb failure condition for a purely frictional material is usually written:

$$\tau_f = \sigma_n \tan \phi$$

where \(\sigma_n\) is the normal stress on the failure plane. One way of rewriting the Mohr–Coulomb condition in terms of principal stresses is as follows:

$$\frac{(\sigma_1 - \sigma_2)^2}{\mu^2 \sigma_1 \sigma_3} = 4$$  (3)

where \(\mu = \tan \phi\).

Recently it has been shown by Matsuoka and Nakai (1974) that the failure of many frictional materials is better fitted by their "Spatially Mobilised Plane" criterion, which may be written in terms of principal stresses in the form:

$$\frac{(\sigma_1 - \sigma_2)^2}{\mu^2 \sigma_1 \sigma_3} + \frac{(\sigma_2 - \sigma_3)^2}{\mu^2 \sigma_2 \sigma_3} + \frac{(\sigma_3 - \sigma_1)^2}{\mu^2 \sigma_3 \sigma_1} = 8$$  (4)

Like the Von Mises criterion, any physical interpretation of the Matsuoka–Nakai criterion is less straightforward than the explanation of the Mohr–Coulomb criterion, which represents a critical value of the maximum allowable ratio of shear stress to normal stress on any plane.

Comparison of Eqs. (1) and (2) with Eqs. (3) and (4) demonstrates the similarity of the link between the Tresca and Von Mises conditions for cohesive materials and the Mohr–Coulomb and Matsuoka–Nakai conditions for frictional materials. This similarity has been discussed by Matsuoka (1976).

For materials showing both friction and cohesion the more general Mohr–Coulomb expression applies:

$$\tau_f = c + \sigma_n \tan \phi$$

which may be rewritten in terms of principal stresses in the form:

$$\frac{(\sigma_1 - \sigma_2)^2}{(c + \mu \sigma_1)(c + \mu \sigma_3)} = 4$$  (5)

It is clear that the above equation is a generalisation of Eqs. (1) and (3). When \(\mu = 0\) the Tresca condition is obtained and when \(c = 0\) the purely frictional Mohr–Coulomb condition results. In a similar way Eqs. (2) and (4) may be generalised (Houlsby, 1981) to give:

$$\frac{(\sigma_1 - \sigma_2)^2}{(c + \mu \sigma_1)(c + \mu \sigma_3)} + \frac{(\sigma_2 - \sigma_3)^2}{(c + \mu \sigma_2)(c + \mu \sigma_3)} + \frac{(\sigma_3 - \sigma_1)^2}{(c + \mu \sigma_3)(c + \mu \sigma_1)} = 8$$  (6)

Eq. (6) is a general frictional and cohesive failure criterion which reduces to the Von Mises criterion Eq. (2) for \(\mu = 0\) and the Matsuoka–Nakai criterion Eq. (4) for \(c = 0\).

The Von Mises and Matsuoka–Nakai crite-
ria are often expressed in terms of stress invariants which are defined as:

\[ J_1 = \sigma_1 + \sigma_2 + \sigma_3 \]
\[ J_2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1 \]
\[ J_3 = \sigma_1 \sigma_2 \sigma_3 \]

Defining also the stress deviators:

\[ \sigma_i' = \sigma_i - \frac{1}{3} J_1 \quad i = 1, 2, 3 \]

The invariants of the stress deviators are then written as \( J'_1, J'_2 \) and \( J'_3 \), and it follows that:

\[ J'_1 = \sigma_1' + \sigma_2' + \sigma_3' = 0 \]
\[ J'_2 = \sigma_1' \sigma_2' + \sigma_2' \sigma_3' + \sigma_3' \sigma_1' = J_2 - \frac{1}{3} J'_1 \]
\[ J'_3 = \sigma_1' \sigma_2' \sigma_3' = J_3 - \frac{1}{3} J_1 J_2 + \frac{2}{3} J'_1 \]

The Von Mises condition is most conveniently written in terms of deviator stresses as:

\[ 3 J'_2 + 4 c^2 = 0 \quad (7) \]

The Matsuoka-Nakai expression may be written as:

\[ 9 J_3 - J_2 J_2 + 8 \mu^2 J_3 = 0 \quad (8) \]

Finally Eq. (6) may be expressed in terms of invariants in the form:

\[ 6 c J'_1 + \mu (9 J_3 - J_2 J_2) + 8 (c^2 + c \mu J_1 + c \mu J_2 + \mu J_3) = 0 \quad (9) \]

It can easily be seen that Eq. (9) reduces to Eq. (7) for \( \mu = 0 \) and to Eq. (8) for \( c = 0 \).

There are clearly other ways of generalising the Von Mises and Matsuoka-Nakai criteria. The above formula is one possible formula which includes these criteria as special cases. Inspection of Eq. (6) shows, however, that the more general expression is based on a very simple extension of the previous formulae. The simplicity of the new expression is rather less obvious when expressed in terms of invariants than when expressed in terms of principal stresses.

Ohmaki (1979) described the idea of generalising the Matsuoka-Nakai criterion to include cohesion by moving the Matsuoka-Nakai surface so that the apex is at a point corresponding to a mean stress of \( -\sigma_0 \). The generalisation can then be obtained by replacing the values of the stresses \( \sigma_i \) in the original expression for the surface by \( \sigma_i - \sigma_0 \). It can be shown that the method described in this note corresponds to the same idea, with \( \sigma_0 = c/\mu \), and the yield surface may alternatively be written:

\[ \mu^2 = \frac{J_1 J_2 - 9 J_3 + 2 \sigma_0 (J_1^2 - 3 J_3)}{8 (J_1 + \sigma_0 J_2 + \sigma_0^2 J_1 + \sigma_0^3)} \quad (10) \]

The above expression differs from Ohmaki’s Eq. (1), which is believed to be in error (note that Ohmaki uses a slightly different notation). Eq. (9) is often a more convenient form than Eq. (10), since in the latter case the Von Mises condition can only be obtained by considering the case as \( \mu \) tends to zero and \( \sigma_0 \) tends to infinity with \( \mu \sigma_0 = c \).

**DISCUSSION**

The new criterion is shown in normal stress-shear stress space in Fig.1 for the case \( \sigma_3 = \sigma_0 \) (triaxial compression) or \( \sigma_3 = \sigma_1 \) (triaxial extension). In these two cases the criterion is identical to the Mohr-Coulomb condition. The criterion is shown in the octahedral plane in Fig.2, where it is compared with the Mohr-Coulomb shape. In this plot the shape is identical to the Matsuoka-Nakai criterion; it approaches a circle.
as $\phi$ tends to zero and an equilateral triangle as $\phi$ approaches 90°.

In applications in soil mechanics the Von-Mises criterion is often used in terms of total stresses for the undrained behaviour of clays. To describe the behaviour of sands, or of clays under drained conditions, the failure criterion must be expressed in terms of effective stresses, and a frictional condition, such as the Matsuoka-Nakai criterion, must be used. It is less common to need a condition including both friction and cohesion, but this type of law may be useful in modelling:

(a) the behaviour of unsaturated soils,
(b) the behaviour of cemented soils,
(c) the effect of the curvature of the failure locus for most frictional soils, which results in a reduction in friction angle with increasing stress level, and can be modelled over a limited range of normal stresses by the introduction of a cohesion value.

The new criterion may also be useful in computational work, where it may be more convenient to introduce the Von Mises and Matsuoka-Nakai criteria simply by setting the appropriate parameters to zero, rather than implementing the two criteria as separate models. In computer applications the new criterion has advantages over the general Mohr-Coulomb condition in that firstly it does not involve corners, and secondly it is more easily expressed in terms of invariants.

CONCLUSIONS

A new failure criterion has been described which combines the effects of friction and cohesion. The Matsuoka-Nakai criterion for a frictional material and the Von Mises criterion for a cohesive material are each special cases of the new formula. The proposed criterion bears the same relationship to the general Mohr-Coulomb expression as the Matsuoka-Nakai and Von Mises expressions do to cohesionless the Mohr-Coulomb and Tresca criteria. This relationship is most clearly shown when the criteria are all expressed in terms of principal stresses. Expressions in terms of stress invariants are also given.

The new criterion may be convenient to apply in computing applications, where the simpler criteria may be obtained as special cases. The general criterion may be applicable where it is convenient to model (for a certain stress range) a reduction in effective friction angle with increasing pressure by using an artificial cohesion value. It may also have application to partially saturated soils and cemented soils.

REFERENCES

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