TECHNICAL NOTE

AN UPPER BOUND CALCULATION ON BEARING CAPACITY OF A CIRCULAR FOOTING ON A NON-HOMOGENEOUS CLAY

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ABSTRACT

This note presents an upper bound calculation on bearing capacity factors for a circular footing on clay, of which strength increases linearly with depth. The results are compared with the values obtained by the slip line method as well as the equivalent values for square and strip footings. The values of the bearing capacity factor N_c obtained by the present analysis are found to be very close to those calculated by the slip line method, and also close to those for a square footing. It is confirmed that the shape factor of N_c decreases significantly with the increase of non-homogeneity of soils.

Key words : bearing capacity, cohesive soil, plasticity, shallow foundation, stability analysis (IGC : E3)

INTRODUCTION

In normally or slightly overconsolidated clays, the undrained shear strength, c_u , can often be fitted as a linear increase of strength with depth. The importance of considering this increase of strength in geotechnical design has been pointed out by many research workers.

For the plane strain situation, Gibson and Morgenstern (1962) and Hunter and Shuster (1968) analysed the stability of cut slopes

in clay with undrained shear strength, varying linearly with depth. The bearing capacity of a strip footing resting on such type of clay was analysed by the slip circular method (Nakase, 1966) and by the method of characteristics (slip line method) (Davis and Booker, 1973). For this particular problem centrifuge modelling technique has recently been applied in an attempt to validate the theories (Nakase, Kusakabe and Wong, 1984).

Problems of axial symmetry for such ma-

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terials have also been studied. Britto and Kusakabe (1983) took the linear variation of soil strength into consideration in analysing the stability of axisymmetric excavations. Houlsby and Wroth (1983) applied the slip line method to problems of bearing capacity of a circular footing on such materials.

This note presents an upper bound calculation on bearing capacity factors for a circular footing on clay, of which strength increases linearly with depth. The results are compared with the values obtained by the slip line method as well as the equivalent values for square and strip footings.

Upper bound calculation

In this note only an immediate collapse, that is to say, undrained failure is considered. Under the undrained condition a soil can be treated as a $\phi_u=0$ material, obeying the Tresca yield criterion. The undrained condition means that there exists no volume change during plastic flow. This condition is termed as incompressibility. The equation of the incompressibility for axisymmetric velocity fields can be written as

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{\partial V_z}{\partial z} = 0 \quad (1)$$

where V_r =the velocity component in the radial direction, and V_z =the velocity component in the vertical direction. Any admissible velocity field must satisfy the incompressibility.

The variation of undrained shear strength with depth is assumed to be linear, as is shown in Fig.1. The origin of the axes is chosen as the center of the base of footing and thus the undrained strength at a given depth of z is expressed as

$$c(z) = c_0 + kz \quad (2)$$

in which c_0 =the shear strength at the surface and k =the rate of increase in undrained shear strength with depth.

A non-dimensional parameter kB/c_0 is sometimes useful to describe the variation of soil strength, where B is a representative length of the problem considered, for instance, the width of footing for problems

Fig. 1. Strength profile

Fig. 2. Proposed failure mechanisms

of bearing capacity for the plane strain condition.

Fig. 2 presents newly proposed mechanisms for a smooth footing as well as a rough footing. The mechanism for the smooth footing has two wedges beneath the footing base and thus may be called the Hill type mechanism, whereas the mechanism of the rough footing has a single wedge and may be called the Prandtl type mechanism.

As was demonstrated for a strip footing by Davis and Booker (1973), deformation tends to concentrate only near the surface of clay, due to the fact that the strength increases with depth. This is particularly so, when the value of kB/c_0 becomes large. Geometrical parameters α , β are, therefore,

Table 1. Proposed velocity field for a smooth footing

region	V_r	V_z
I	$F/\tan \alpha$	F
II	zG	$(d-r)G$
III	H	$-H$

$$(d=D/2)$$

$$F = \frac{v}{\tan \alpha} \left(\frac{r \tan \alpha - z}{r} \right)$$

$$G = \frac{v}{r} \frac{[d \sin \alpha - \{(d-r)^2 + z^2\}^{1/2}]}{\{(d-r)^2 + z^2\}^{1/2} \sin^2 \alpha}$$

$$H = \frac{v \{\sqrt{2} d \sin \alpha + d - r - z\}}{2r \sin^2 \alpha}$$

Boundary conditions

$$V_{zI} = v : \text{on OA}$$

$$V_r = V_z = 0 : \text{along ODCB}$$

$$V_{rI} = V_{rII}, V_{zI} = V_{zII} : \text{on AD}$$

$$V_{rII} = V_{rIII}, V_{zII} = V_{zIII} : \text{on AC}$$

Table 2. Proposed velocity field for a rough footing

region	V_r	V_z
I	$F/\tan \alpha$	F
II	zG	$(d-r)G$
III	H	$-H$
IV	0	v

$$F = \frac{v(r \tan \alpha + d \tan \beta - z) \sin \alpha \cos \beta}{r(\tan \alpha + \tan \beta) \sin(\alpha + \beta)}$$

$$G = \frac{v \cos \beta \sin \alpha}{r(\tan \alpha + \tan \beta) \sin \alpha \{(d-r)^2 + z^2\}^{1/2} \cos \alpha \sin(\alpha + \beta)} \times \left[\frac{d \sin \alpha}{\cos \beta} - \{(d-r)^2 + z^2\}^{1/2} \right]$$

$$H = \frac{v \cos \beta \sin \alpha}{2r(\tan \alpha + \tan \beta) \cos \alpha \sin \alpha \sin(\alpha + \beta)} \times \left\{ d + \frac{\sqrt{2} d \sin(\alpha + \beta)}{\cos \beta} - r - z \right\}$$

Boundary conditions

$$V_r = V_z = 0 : \text{along EDCB}$$

$$V_{rI} = V_{rII}, V_{zI} = V_{zII} : \text{on AD}$$

$$V_{rII} = V_{rIII}, V_{zII} = V_{zIII} : \text{on AC}$$

introduced to allow the mechanisms to have more freedom in the process of searching for the critical failure mode. It should be mentioned here that the mechanism of the smooth footing reduces to that proposed by Levin (1955), when $\alpha = 45^\circ$, where he analysed the indentation problem of a circular punch into a uniform material. Although the strength profile is non-homogeneous in this case, the same procedures adopted by Levin can be used to determine the complete set of admissible velocity fields, which satisfy the incompressibility and boundary conditions.

The results are listed in Table 1 for the smooth case and in Table 2 for the rough case, respectively. It can be easily verified that these velocity fields satisfy the boundary conditions as well as the incompressibility. Once an admissible velocity field is found, strain rate components in all regions are readily calculated by the following equations.

$$\left. \begin{aligned} \dot{\epsilon}_r &= \frac{\partial V_r}{\partial r} \\ \dot{\epsilon}_\theta &= \frac{V_r}{r} \\ \dot{\epsilon}_z &= \frac{\partial V_z}{\partial z} \\ 2\dot{\gamma}_{rz} &= \frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r} \end{aligned} \right\} \quad (3)$$

The principal strain rates are, then, determined. The absolutely largest principal strain rate, in this particular case, is found to be

$$|\dot{\epsilon}|_{\max} = \frac{1}{2} [\dot{\epsilon}_\theta^2 + (\dot{\epsilon}_\theta^2 + 4\dot{\gamma}_{rz}^2 - 4\dot{\epsilon}_r\dot{\epsilon}_z)^{1/2}] \quad (4)$$

Velocity jump occurs only along the discontinuity line of AE in the Prandtl type mechanism. The magnitude of the velocity jump is

$$\Delta v = \frac{v \cos \beta}{\sin(\alpha + \beta)} \quad (5)$$

According to Shields and Drucker (1953) an upper bound solution of collapse pressure on the footing p on the Tresca material is calculated from the following equation.

$$\int_S p v dS = 2 \int_{vol} c_u |\dot{\epsilon}|_{\max} d vol + \int_{S_D} c_u \Delta v dS \quad (6)$$

where p = pressure on the footing, v = velocity of the footing, c_u = undrained shear strength, $|\dot{\epsilon}|_{\max}$ = value of absolutely largest principal strain rate, vol = volume domain, Δv = velocity jump across the discontinuity, S_D = surface of discontinuity, and S = contact surface between the footing and the soil.

Substituting the values of $|\dot{\epsilon}|_{\max}$ and Δv given by the Eq. of (4) and (5) into the Eq. (6), the pressure of the footing is, then,

calculated. Dividing each region into 10×10 segments, numerical integration was carried out to obtain the minimum value of p by changing values of α and β , by using a personal computer.

NUMERICAL RESULTS AND DISCUSSIONS

Critical Failure Mode

Geometry of critical failure mechanisms for various value of kB/c_0 is illustrated in Fig. 3. The diameter of footing (D) is taken to be the representative length (B) in this study. This figure demonstrates that the depth of the zone subjected to plastic deformation becomes shallower and the width of the zone narrower with the increase of kB/c_0 value. This is the consistent result with that obtained in the plane strain condition.

It should be also noted that for the case of a rough footing, the region IV (the wedge beneath the footing base) vanishes when kB/c_0 value is more than unity, since the value of geometrical parameter β becomes zero,

Fig. 3. Variation of critical failure surfaces with the value of kB/c_0

Fig. 4. Variation of the plastic zone with the value of kB/c_0

and that the Hill type mechanism becomes the critical failure mode irrespective of the roughness of footing when $kB/c_0 > 1$. In contrast, Chen (1975) obtained the reverse trend, when he studied the bearing capacity of a strip footing in uniform soils. He concluded that the Prandtl type mechanism gives a smaller upper bound than does the Hill type mechanism for great embedded depths, and that the Prandtl type becomes the critical failure mode irrespective of the roughness of footing, if embedded depth exceeds a certain value.

The values of the depth (y) and the width (x) of the plastic zone normalized by the diameter (D) of footing are plotted against kB/c_0 value in Fig. 4. This figure again demonstrates how non-homogeneity of soil affects the size of the plastic zone. The width of the plastic zone at $kB/c_0 = 10$, for instance, is less than half of that in uniform case.

Bearing Capacity Factor

A non-dimensional bearing capacity factor N_c is defined as

$$p/c_0 = N_c$$

where p is the load intensity on the footing and c_0 is the shear strength at the surface of clay. Numerical results of N_c are plotted against kB/c_0 value in Fig. 5. In the figure the values obtained by the slip line method (Houlsby and Wroth) are also given. Since the slip line method is considered to yield a lower bound, the true solution for N_c

(a) smooth footing

(b) rough footing

Fig. 5. Variation of bearing capacity factor N_c values with the value of kB/c_0

value must lie in this relatively narrow range bracketed by the lines of Housby and Worth, and of the present analysis. So the mean value of the two lines may be used for design purposes in engineering point of view.

In Fig. 5 N_c values of a square footing obtained from the upper bound analysis (Ugai, 1985) are also given. These values are very close to those of the present analy-

(a) smooth footing

(b) rough footing

Fig. 6. Variation of shape factor λ_c value with the value of kB/c_0

sis. In fact the values for a square footing falls inbetween the two lines for a circular footing for the case of smooth base. With this result the traditional practice of using the same N_c value for a circular and square footing may be justified as far as a smooth footing is concerned.

Shape Factor

The ratio of N_c for a circular footing to that for a strip footing N_c is referred to as the shape factor (λ_c). Terzaghi and Peck gave this value as 1.3 in their first edition and 1.2 in the second edition of Soil Mechanics in Engineering Practice (1967).

Fig. 6 presents the variation of the shape factor with kB/c_0 taken from the above three analyses. Although there is slight difference in value itself, the general trend of the decrease with the increase of kB/c_0 is consistent among these analyses.

Houlsby and Wroth, and also Ugai has pointed out a rather surprising result that the bearing capacity factor for a circular or square footing is actually smaller than that for a strip footing, where kB/c_0 is significantly large. This figure provides a clear evidence for the need of reconsideration on the value of shape factor for non-homogeneous materials.

In Fig. 6(a) a line of $\lambda_c = 0.733 (N_c)_s^{1/4}$ is also drawn, which is suggested by Chen (1975) when he draw an approximate straight line for the relationship between $\log_{10}(N_c)_c$ (circular) and $\log_{10}(N_c)_s$ (strip) for a uniform material, which was given by Cox, Eason and Hopkins (1961). It is clear from the figure that this relationship does not fit for non-homogeneous cases.

Conclusions

An upper bound calculation on the bearing capacity for a circular footing was made. Following conclusions were obtained;

1) The Hill type mechanism becomes the critical failure mode irrespective of the roughness of footing, when kB/c_0 value is larger than unity.

2) The values of the bearing capacity factor N_c obtained by the present analysis were found to be very close to those calculated by the slip line method (Houlsby and Wroth, 1983), and also close to those for a square footing (Ugai, 1985).

3) Shape factor decreases significantly with the increase of kB/c_0 value. Common practice of using 1.2 for the shape factor should be re-considered.

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