ON DEFORMATION OF GRANULAR MATERIAL IN SIMPLE SHEAR

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ABSTRACT

Paying attention to the important mechanical features peculiar to granular materials : friction and dilatancy, fundamental considerations on the deformation of sand under simple shear condition are made. First, adopting the simple sliding block theory, the two state functions of φ and S are analitically derived and a basic deformation model is formulated. This simple model is found to be a good approximation in the intermediate range of stress ratio. Next, using the experimental data, the entropy of granular materials S is examined. Another deformation model based on the characteristics of the state parameter S can be extended to wider range of stress ratios.

Key words : simple shear, shearing deformation, entropy, sand, granular material (IGC : D6)

INTRODUCTION

For a study of shearing deformation of granular materials, to begin with, the behaviour in simple shear condition must be appreciated. In this case, attention must be focussed on two important features : friction and dilatancy. Paying attention to these fundamental characteristics of granular materials, the author originally proposed a parameter S to measure the degree of shear deformation (Moroto, 1976). The reason to introduce the parameter S is as follows. The plastic work done during shear largely depends on deformation path. Only the form of

$$S = \int \frac{(\text{increment of plastic work done})}{(\text{confining pressure})}$$

is an interesting state parameter for specifying degree of shearing process.

In this paper, adopting a simple friction-dilatancy model, the author analitically derives the parameter S and a function φ . Thus the characteristics of shear deformation is appreciated. Then, this simple deformation model is evaluated by using published experimental data.

Next, based on experimental evidences, the author's state parameter is studied and a more reasonable and realistic deformation model is proposed.

SIMPLE MODEL

Starting with a simple two dimensional model illustrated in Fig.1 where the sample

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Fig. 1. Simple shear under vertical pressure

Fig. 2. Definition of dilatancy angle

is subjected to shear stress τ under normal stress σ , the resulting shear distortion and volume change are expressed by shear strain γ and volumetric strain v respectively. In the sample, a macroscopic shear plane is denoted by AA while the actual movement of particles may occur along the plane $A'A'$ in Fig. 2.

The sliding block analysis, leads to :

$$\phi_\mu = \phi - \nu \quad (1)$$

where

ϕ : apparent angle of shearing resistance

ϕ_μ : intergranular friction angle

ν : dilatancy angle

and

$$\tan \phi = \tau/\sigma \text{ or } \phi = \tan^{-1} (\tau/\sigma) \quad (2)$$

$$\nu = -\frac{dv}{d\gamma} \quad (3)$$

The dilatancy angle ν takes either negative or positive signs depending on contraction and expansion during shear. An essential frictional feature of granular materials subjected to shear force is demonstrated by Eq. (1) (Boden and Tabor, 1954 ; Newland and Alley, 1957 and recently Nemat-Nasser, 1980).

With Eqs. (2) and (3), Eq. (1) can be rewritten

$$\tan^{-1}(\tau/\sigma) = \phi_\mu - \frac{dv}{d\gamma}$$

Thus

$$\frac{\tau}{\sigma} = \tan\left(\phi_\mu - \frac{dv}{d\gamma}\right) \quad (4)$$

and since

$$\frac{dv}{d\gamma} = \tan\left(\frac{dv}{d\gamma}\right)$$

then

$$\frac{dv}{d\gamma} = \frac{-\frac{\tau}{\sigma} + \tan \phi_\mu}{1 + (\tau/\sigma) \tan \phi_\mu} \quad (5)$$

It is interesting to note that Eq. (5) is the same as the expression obtained by Tokue (1979) and Nemat-Nasser (1980) providing some microscopic considerations.

Utilizing the basic equation given by Eq. (5), we define two kinds of state functions S and ϕ . The increment of plastic work done per unit volume of the sample is

$$dW = \sigma dv + \tau d\gamma \quad (6)$$

Substituting Eq. (5) into the above equation gives

$$dW = \sigma F(\tau/\sigma) d\gamma \quad (7)$$

where

$$F(\tau/\sigma) = \frac{(1 + (\tau/\sigma)^2) \tan \phi_\mu}{1 + (\tau/\sigma) \tan \phi_\mu} \quad (8)$$

It has been experimentally well recognized that the shear strain during virgin loading γ depends almost exclusively on the stress ratio (though this is not exactly true), so for a reasonable approximation, it can be expressed :

$$\gamma = \gamma(\tau/\sigma) \quad (9)$$

In this case, Eq. (7) is not a total differential form, but the only form of

$$dS = \frac{dW}{\sigma} = F(\tau/\sigma) d\gamma(\tau/\sigma) \quad (10)$$

becomes total differential. The author originally found out this state parameter (Moroto, 1976, 1983) and called this state parameter S as 'entropy of granular materials'.

On the other hand, definition of the plastic potential leads to

$$\frac{dv}{d\gamma} = -\frac{d\tau}{d\sigma} \quad (11)$$

provided that the stress increments $d\sigma$ and $d\tau$ are located on a constant potential surface.

Substituting Eq. (5) into the above equation yields

$$d\varphi = \frac{d\sigma}{\sigma} + \frac{d(\tau/\sigma)}{F(\tau/\sigma)} = 0 \quad (12)$$

Integrating Eq. (12) results in

$$\varphi = \log_e \sigma + \frac{\tan^{-1}(\tau/\sigma)}{\tan \phi_\mu} + \frac{1}{2} \log_e (1 + (\tau/\sigma)^2) = C \quad (13)$$

It can also be concluded that

$$\frac{d^2 W}{\sigma} = \frac{d\sigma dv + d\tau d\gamma}{\sigma} = d\varphi \cdot dS \quad (14)$$

On the σ - τ stress space, equi- S lines and equi- φ lines can be drawn without any specification of stress paths, as shown schematically in Fig. 3.

Stress-strain formula may therefore be expressed :

$$d\epsilon = \sigma \frac{\partial \varphi}{\partial \sigma} dS \quad (15)$$

where $\epsilon = \epsilon(\gamma, \nu)$ and $\sigma = \sigma(\tau, \sigma)$. The state parameters S and φ may correspond to the

Fig. 3. Equi- S lines and equi- φ lines

yield function and the plastic potential, respectively, in the theory of plasticity for soils. When the sample undergoes sliding distortion, $dS > 0$. The material stability condition ($d^2 W/\sigma \geq 0$ or $d^2 W/\sigma < 0$) will be, therefore, governed by the sign of $d\varphi$.

Naturally, a question arises. To what extent is Eq. (1) sound for actual behaviour of granular materials? This will be examined in the later section. Here, Eq. (5) is rewritten as

$$\frac{dv}{d\gamma} = \frac{-\frac{\tau}{\sigma} + \mu}{1 + \mu \cdot (\tau/\sigma)} \quad (16)$$

where μ is a frictional constant.

PARAMETER TRANSFORMATION

The deformation model previously introduced is to be evaluated by Cole's data (Cole, 1967). For this purpose, we need parameter transformation. The following notations are adopted :

$s = (\sigma_1 + \sigma_3)/2$: mean normal stress

$t = (\sigma_1 - \sigma_3)/2$: maximum shear stress

$d\epsilon_v = d\epsilon_1 + d\epsilon_3$: volumetric strain increment

$d\epsilon_s = d\epsilon_1 - d\epsilon_3$: maximum shear strain

increment

σ_1 : major principal stress

σ_3 : minor principal stress

$d\epsilon_1$: major principal strain

increment

$d\epsilon_3$: minor principal strain

increment

From the geometry of the relevant Mohr's circle (Fig. 4) :

$$\sigma = s + t \cos 2\psi \quad (17)$$

Fig. 4. The Mohr's stress circle

Fig. 5. The Mohr's circle of strain increment

$$\tau = t \sin 2\psi \quad (18)$$

where ψ is the inclination to the horizontal of the plane on which the principal stress acts. Cole performed simple shear test on Leighton Buzzard sand. He found out that the principal axes of stress and of strain increment coincides to each other except the initial stage of a test. Therefore, on the Mohr's circle of strain increment (Fig. 5),

$$\cot 2\psi = \frac{dv}{d\gamma} \quad (19)$$

From Eqs. (17) and (18) :

$$\frac{t}{s} = \frac{\tau/\sigma}{\sin 2\psi - (\tau/\sigma) \cos 2\psi} \quad (20)$$

Substituting for ψ from Eq. (19) :

$$\frac{t}{s} = \frac{(\tau/\sigma) \sqrt{1 + (dv/d\gamma)^2}}{1 - (\tau/\sigma)(dv/d\gamma)} \quad (21)$$

Substituting Eq. (16) into the above equation yields

$$\frac{t}{s} = \frac{(\tau/\sigma) \sqrt{1 + \mu^2}}{\sqrt{1 + (\tau/\sigma)^2}} \quad (22)$$

From the Mohr's circle of strain increment :

$$d\epsilon_v = d\gamma \cdot \cot 2\psi$$

$$d\epsilon_s = d\gamma / \sin 2\psi$$

Then,

$$\frac{d\epsilon_v}{d\epsilon_s} = \cot 2\psi \cdot \sin 2\psi$$

Substituting Eq. (19) into the above equation :

$$\frac{d\epsilon_v}{d\epsilon_s} = \frac{dv/d\gamma}{\sqrt{1 + (dv/d\gamma)^2}} \quad (23)$$

Combination of Eqs. (16) and (23) gives

$$\frac{d\epsilon_v}{d\epsilon_s} = \frac{\mu - (\tau/\sigma)}{\sqrt{(1 + \mu^2)(1 + (\tau/\sigma)^2)}} \quad (24)$$

Let

$$m_0 = \frac{t}{s} + \frac{d\epsilon_v}{d\epsilon_s} \quad (25)$$

Then, from Eqs. (22) and (23) :

$$m_0 = \frac{\mu(1 + \mu \cdot (\tau/\sigma))}{\sqrt{1 + \mu^2} \sqrt{1 + (\tau/\sigma)^2}} \quad (26)$$

The value of m_0 takes almost similar values in the range of $\tau/\sigma = 0.4 - 0.7$. In these deformational stages, therefore, Eq. (25) with constant value of m_0 seems to be a good approximation for this simple model.

COMPARISON OF THEORY WITH EXISTING DATA

From Cole's data,

$$\mu = 0.544 \quad (27)$$

Using this value, the ratios t/s and τ/σ can be obtained as given by Eq. (22). The results are shown in Fig. 6 by solid line. In the initial deformational stage, the observed value deviates from the theoretical line. However, as the shear develops, the theory comes to agree well with the experiment.

With the value of $\mu = 0.544$, Eq. (24) is calculated. Thus obtained theoretical rela-

Fig. 6. t/s and τ/σ relation

Fig. 7. Stress ratio and dilatancy rate relation (dense sand)

Fig. 8. Stress ratio and dilatancy rate relation (medium sand)

Fig. 9. Stress ratio and dilatancy rate relation (loose sand)

tionship between $d\varepsilon_v/d\varepsilon_s$ and t/s is drawn in Figs. 7, 8 and 9. In these figures, the theoretical line almost coincides with the

Fig. 10. Stress-strain data on dense Leighton Buzzard sand by Cole

experimental plots in the intermediate range. Accordingly, it can be recognized that the basic relation given by Eq. (5) is valid at least for the intermediate deformational stage. The coincidence of theory and experiment is better for loosely prepared sample than for other two cases of dense and medium sand.

In the next section, based on the experimental data, a more appropriate deformational model will be proposed.

DEVELOPMENT OF MODEL

The Cole's experimental data are reproduced in Figs. 10 and 11. The deformation rule given by Eq. (9) does not exactly hold since the relation depends on the levels of normal stress to some extent even for samples with similar initial density. Using the Cole's data, the entropy can be calculated by following equation:

Fig. 11. Stress-strain data on loose Leighton Buzzard sand by Cole

Fig. 12. S and γ relation for dense Leighton Buzzard sand

$$S = \int \frac{dW}{\sigma} = v + \int (\tau/\sigma) d\gamma \quad (28)$$

The results are shown in Figs. 12 and 13. As shown in these figures, the relationship between S and γ is represented by a unique line irrespective of the initial densities and the levels of normal stress. It is interesting that the different sets of stress and strain relations shown in Figs. 10 and 11 can be

Fig. 13. S and γ relation for loose Leighton Buzzard sand

unified into almost one representative line of S and γ . This is the fundamental characteristics of the state parameter S . Therefore, we write

$$dS = G(\gamma) d\gamma \quad (29)$$

The slope of S and γ relation is expressed only by a function of γ and almost constant except the final stage of the tests. So we put

$$m = \frac{dS}{d\gamma} \quad (30)$$

This means :

$$\frac{dv}{d\gamma} = m - \frac{\tau}{\sigma} \quad (31)$$

Substitution of Eq. (31) into (21) gives

$$\frac{t}{s} = \frac{(\tau/\sigma) \sqrt{1 + \left(m - \frac{\tau}{\sigma}\right)^2}}{1 - \left(\frac{\tau}{\sigma}\right) \left(m - \frac{\tau}{\sigma}\right)} \quad (32)$$

and from Eqs. (31) and (23) :

$$\frac{d\varepsilon_v}{d\varepsilon_s} = \frac{m - \frac{\tau}{\sigma}}{\sqrt{1 + \left(m - \frac{\tau}{\sigma}\right)^2}} \quad (33)$$

With the value of $m=0.544$, Eqs. (32) and (33) are evaluated and the variation of $d\varepsilon_v/d\varepsilon_s$ and t/s is shown in Fig. 14 which fits the experimental result for all range of deformation except the initial stage of the tests. This may indicate a validity of author's procedure.

Let

Fig. 14. Stress ratio and dilatancy rate relation

$$m_1 = \frac{d\varepsilon_v}{d\varepsilon_s} + \frac{t}{s} \quad (34)$$

Then, from Eqs. (32) and (33) :

$$m_1 = \frac{\left(\frac{\tau}{\sigma}\right) \sqrt{1 + \left(m - \frac{\tau}{\sigma}\right)^2}}{1 - \frac{\tau}{\sigma} \left(m - \frac{\tau}{\sigma}\right)} + \frac{m - \frac{\tau}{\sigma}}{\sqrt{1 + \left(m - \frac{\tau}{\sigma}\right)^2}} \quad (35)$$

Constancy of m_1 is not good. Therefore, the deformation rule given by Eq. (34) with constant value of m_1 will result in some deviations from actual behaviour of granular materials.

Substituting Eq. (31) into (11), the following function is obtained

$$\varphi = \log_e \sigma + \frac{\tau/\sigma}{m} = C \quad (36)$$

Also,

$$\frac{d^2 W}{\sigma} = d\varphi \cdot dS \quad (37)$$

and

$$d\varepsilon = \sigma \frac{\partial \varphi}{\partial \sigma} dS \quad (38)$$

The expression of φ given by Eq. (36) is simpler than Eq. (13). It may be noted that Eqs. (37) and (38) are similar to Eqs. (14) and (15), respectively.

EXAMPLES OF STRAINS CALCULATION

Eq. (36) is better be used rather than Eq. (13) for a realistic strain calculation because of its simplicity and accuracy. From Eqs. (36) and (38), one can easily obtain the strain components as

$$\gamma = \int \frac{dS}{m} \quad \left(\because d\gamma = \sigma \frac{\partial \varphi}{\partial \tau} dS \right) \quad (39)$$

$$v = S - \frac{1}{m} \int \frac{\tau}{\sigma} dS \quad \left(\because dv = \sigma \frac{\partial \varphi}{\partial \sigma} dS \right) \quad (40)$$

Before starting calculation of the above γ and v , we must prepare S vs τ/σ and S vs γ graphs as shown in Figs. 15 and 16, respectively.

For example, the test data for loose sand under $\sigma = 56$ psi will be examined. Fig. 16 presents that

$$S = m(\gamma - \gamma_0)$$

where γ_0 is a constant almost irrespective to σ .

Thus

Fig. 15. Stress ratio and S relation

Fig. 16. S and γ relation Fig. 17. Calculated γ and stress ratio relation Fig. 18. Calculated v and stress ratio relation

$$\gamma = \frac{S}{m} + \gamma_0 \quad (41)$$

Reading the values of S against τ/σ and putting $m=0.520$, Eq. (41) gives the values of γ . The calculated relationship between γ and τ/σ is plotted in Fig. 17.

From Eq. (40) with the integration

$\int (\tau/\sigma) dS$ on the graph of S vs τ/σ in Fig. 15, one can obtain the values of v against τ/σ . This result is plotted in Fig. 18. Figs. 17 and 18 indicate quite close agreement between experimental data and calculated values. This shows a usefulness of the author's model in order to predict the amount of strains. In this procedure, needed are the two parameters of m and γ_0 together with the graph of S vs τ/σ .

COMMENTS

The main point of this paper is to emphasize that the stress dilatancy performance can be expressed by such a flow rule given by Eq. (15) and (38) which are similar to the one in soil plasticity. This basic finding becomes interesting and also important to a sound development of constitutive relations for soils. This kind of study has been done by the author in the triaxial tests on sands in which the Cambridge parameters (p, q, γ, v) were employed (Moroto, 1980). For the simple shear condition, a set of the parameters (σ, τ, γ, v) seems to be convenient in use.

The second result of this paper is that the strength parameter in simple shear can be converted to that in plane strain conditions through Eq. (32). This engineering problem will be reported elsewhere in future.

Strain prediction in the author's method seems to be reasonable but it is not perfect, because the formulation of S vs σ and τ/σ ;

$$S = C(\sigma) \cdot F(\tau/\sigma)$$

has not yet established over the wide ranges of σ and τ/σ .

CONCLUSIONS

- 1) The author's state parameter S plays an important role in considering shearing deformation of granular materials.
- 2) The simple sliding block model is useful for the intermediate deformational stage.
- 3) The author developed a reasonable model based on the experimental evidences,

which can be extended to wider range of stress ratios.

4) The stability condition can be expressed simply in terms of S and φ .

5) The stress-strain formula can be simply written by S and φ .

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