STABILITY ANALYSIS OF CONICAL HEAPS

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ABSTRACT

Presented is a three-dimensional slope stability analysis which is based on the variational limiting equilibrium approach. The characteristic feature of this approach is the absence of arbitrary assumptions with respect to the shape of the potential slip surface. The analysis is applicable to soil with a homogeneous distribution of material strength parameters. The results are presented in the form of two stability charts. One chart makes it possible to estimate the factor of safety for a given problem and the second enables prediction of the volume of moving soil in case failure does occur. This second chart makes it possible to rationally select the design safety factor where cost of potential repair is considered.

The analysis illustrates the importance of the ratio between cohesion and friction in controlling the extent of failure. In particular, it is shown that the factor of safety of purely frictional material stored as a heap is the same as obtained under 2-D configuration.

Key words : failure, landslide, slip surface, slope stability, stability analysis (IGC : E 6/H 4)

INTRODUCTION

Mine products, waste materials and other industrial bulk outputs are frequently stored in piles resembling the geometry of conical heaps. Stability analysis of these cones can be performed using classical limiting-equilibrium (L. E.) procedures commonly employed in geotechnical engineering. The unique feature of heap stability, however, is the 3-D character of the problem.

Leshchinsky and Baker (1986) reviewed the few available methods of 3-D slope stability analysis and concluded that none ap-

pears satisfactory. Some of these methods are limited to purely cohesive soils (Baligh and Azzouz, 1975 ; Ugai, 1985). Others fail to degenerate to known solutions when limiting cases are considered (Hovland, 1977 ; Chen and Chameau, 1983 ; Giger and Krizek, 1975, 1976). Some of the procedures depend on constants which are extremely difficult to estimate (Azzouz and Baligh, 1978 ; Steiner, 1977). All of these methods employ an arbitrary assumed failure mechanism (e.g., a circular cylinder with conical or ellipsoidal end caps).

The analysis presented in this work is

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based on a variational formulation of the L.E. slope stability problem. In this approach, the shape of the slip surface is an unknown function which is determined as part of the solution procedure. Such an approach is particularly valuable in 3-D problems where an intuitive choice for a critical failure mechanism, typically done in conventional L.E. analysis, is much more difficult than in the 2-D case.

The basic statement of the variational L.E. approach was presented by Baker and Garber (1978) and Garber and Baker (1979). For 2-D problems this approach yields results which agree well with predictions by established methods (Baker, 1981). Moreover, it can be shown (Baker and Frydman, 1983; Leshchinsky et al., 1985) that this approach is equivalent to an upper-bound solution in the sense of plasticity where a rigid body movement is considered.

The basic variational approach was extended to 3-D problems by Leshchinsky et al. (1985) and Leshchinsky (1982). The present work utilizes the general solutions presented in (Leshchinsky et al., 1985) and (Leshchinsky, 1982) to derive a complete solution and design charts for the particular problem of homogeneous conical heaps.

In many practical cases instability of conical heaps is likely to be induced by an increase in pore water pressures. Knowledge of these pressures enables one to conduct an effective stress stability analysis. Predicting these pressures, however, might be extremely difficult in a 3-D space. Thus to simplify the presented 3-D stability analysis and its results, pore pressures are ignored (i.e., total stress analysis). Consequently a clear insight into the problem and into the solution behavior is attained without adding the complexities associated with 3-D pore pressures prediction. It should be noted that the necessary general algorithms for the inclusion of pore pressures are given elsewhere (Leshchinsky and Baker, 1986). Results of effective stress analysis will be published as a follow-up to this paper (Leshchinsky and Mullett, 1988). A copy of the 3-D computer

(a) Global geometry

(b) Differential geometry of the slip surface

Fig. 1. Conventions and definitions for the 3-D analysis

program capable of dealing with total and effective stress stability analysis of conical heaps is available upon request from the second author.

VARIATIONAL LIMITING-EQUILIBRIUM

An outline of the formulation developed in (Leshchinsky et al., 1985) and (Leshchinsky, 1982) is presented here for the sake of completeness. The present formulation is restricted to problems possessing at least one plane of symmetry (Leshchinsky and Baker, 1986).

Equilibrium equations for the mass of soil contained between the slip surface and the

slope surface (Fig. 1(a)) can be written in vector form as follows :

$$\mathbf{f} = \iint_D [\sigma \mathbf{n} + \tau |\mathbf{n}| \mathbf{t} - \gamma (\mathbf{R} - \mathbf{r})] dx dy = 0 \quad (1a)$$

$$\mathbf{m} = \iint_D \mathbf{r} \times [\sigma \mathbf{n} + \tau |\mathbf{n}| \mathbf{t} - \gamma (\mathbf{R} - \mathbf{r})] dx dy = 0 \quad (1b)$$

where $\mathbf{f}$ and $\mathbf{m}$ are the resultant force and moment acting on the sliding body ; D is the projection of the sliding body on the horizontal x - y plane (see Fig. 1) ; $\sigma = \sigma(x, y)$ and $\tau = \tau(x, y)$ are the distributions of normal and shear stress acting on the slip surface ; $\mathbf{r} = \mathbf{r}(x, y, z)$ and $\mathbf{R} = \mathbf{R}(x, y, z)$ are the radius vectors defining points on the slip and slope surface, respectively ; γ is the unit weight of the soil above the slip surface ; $\mathbf{t} = \mathbf{t}(x, y)$ is a unit vector tangential to the slip surface pointing in the direction of the resisting shear force ; and $\mathbf{n} = \mathbf{r}_x \times \mathbf{r}_y = \mathbf{n}(x, y)$ is a vector normal to the slip surface, pointing in the direction of the normal stress where $\mathbf{r}_x$ and $\mathbf{r}_y$ are the derivatives of $\mathbf{r}(x, y, z)$ with respect to x and y respectively ; $\times$ is the vector product and $|\mathbf{n}|$ denotes the magnitude of $\mathbf{n}$.

The normal and shear stress are related through the "mobilized" Mohr-Coulomb failure criterion as follows :

$$\tau = \frac{c + \sigma \phi}{F_s} \quad (2)$$

where c is the cohesion ; $\phi = \tan \phi$ and ϕ is the angle of internal friction ; and F_s is the factor of safety quantifying the available margin of safety with respect to the soil shear strength.

By introducing Eq. (2) into Eqs. (1) it can be shown (Leshchinsky et al, 1985) that the factor of safety can be expressed as a functional of three unknown functions, namely, $z = z(x, y)$, $\sigma = \sigma(x, y)$ and $\theta = \theta(x, y)$. The function $z(x, y)$ represents the elevation of the slip surface at (x, y) , and $\theta(x, y)$ defines the direction of elementary shear force at (x, y) as projected into the x - y plane—see Fig. 1(b). The problem is to find three functions $z(x, y)$, $\sigma(x, y)$ and

Fig. 2. The symmetrical problem and its coordinate system

$\theta(x, y)$ that will yield a minimum value of the safety factor and, simultaneously, will satisfy the equations of L. E. (i. e., Eqs. (1) and (2)). This is a variational problem of the "isoperimetric" type (Petrov, 1968 ; Elsgolc, 1962). Details of the solution procedure for the general 3-D case were presented by Leshchinsky et al. (1985).

The scope of the present work is limited to a situation in which both geometry of the slope and distribution of soil properties (γ, c, ϕ) are symmetrical with respect to a vertical plane. A Cartesian coordinate system is selected so that the plane of symmetry coincides with $y=0$ and the z -axis opposes the direction of gravity. Under such conditions only three out of six equilibrium Eqs. (1) are significant. These equations are the force equilibrium in the x and z directions and the moment equilibrium about the y axis. The other three equilibrium equations are satisfied by virtue of symmetry. Moreover, due to symmetry only one half of the potentially sliding body have to be considered (Fig. 2).

Application of standard variational techniques (Euler's equations) to the problem stated above results with differential equations for the unknown functions $z(x, y)$, $\sigma(x, y)$ and $\theta(x, y)$. It can be shown

(Leshchinsky et al, 1985) that the differential equation for the slip surface is

$$\frac{\partial \rho}{\partial \beta} = -\phi_m \sin \alpha \sqrt{\rho^2 + \left[\frac{\partial \rho}{\partial \alpha} \right]^2} \quad (3)$$

where $\rho = \rho(\alpha, \beta)$ is a representation of the slip surface, $z(x, y)$, in spherical coordinates (ρ, α, β) as shown in Fig.2. The center of this spherical system is located at an unknown point (x_c, y_c, z_c) . The mobilized coefficient of friction is denoted as $\phi_m = \phi/F_s$.

Eq. (3) has the following two fundamental solutions

$$\rho = A \exp(-\phi_m \beta) \sin \alpha \quad (4a)$$

$$\rho = A \exp(-\phi_m \beta) / \sin \alpha \quad (4b)$$

where A is an unknown constant. Eq. (4b) represents a cylinder with a log-spiral generator (Fig.3(b)), while Eq. (4a) has the 3-D shape shown in Fig.3(a).

The following in-space combination of the two fundamental solutions (Eq. (4)) also constitutes a solution of Eq. (3) :

$$\rho = \rho(\alpha, \beta) = \begin{cases} A \exp(-\phi_m \beta) \sin \alpha & \text{for } \frac{3\pi}{4} \geq \alpha \geq \frac{\pi}{2} \\ A \exp(-\phi_m \beta) / \sin \alpha & \text{for } \frac{\pi}{2} \geq \alpha \geq \alpha_0 \end{cases} \quad (5)$$

where α_0 is the collection of angles for which the trace of the slip surface is on the symmetry plane ($y=0$), i.e.,

$$\alpha_0 = \cot^{-1} \left[\frac{-y_c}{A \exp(-\phi_m \beta)} \right] \quad (6)$$

Observing Fig.2 one sees that y_c is actually half the length of the cylindrical part of the solution (denoted as l_c). The two fundamental solutions appearing in Eq. (5) are attached smoothly to each other along $\alpha = \pi/2$ (i.e., on either plane $y = -y_c$ or $y = +y_c$). The limits set on α in Eq. (5) correspond to half the symmetrical failing mass in the $-y$ direction. To ensure univalued surface with respect to (x, y) β must be restricted as follows (Leshchinsky et al, 1985).

$$\frac{3\pi}{2} - \phi_m \geq \beta > \frac{\pi}{2} \quad (7)$$

where $\phi_m = \tan^{-1}[\tan \phi / F_s]$.

Eq. (5) describes a family of potential slip

(a) Eq. (4a)

(b) Eq. (4b)

(Note: for clarity of illustration, a positive exponent is used)

Fig. 3. Fundamental modes of failure

surfaces, or, in other words, a failure mechanism. A particular member of this family is specified by the (still unknown) constants F_s, x_c, y_c, z_c, A and a (known) parameter $\phi = \tan \phi$. This derived failure mechanism consists of a central cylindrical part

and two end caps. Essentially, it is an expansion of the 2-D log-spiral function [i.e., expansion of $\rho = A \exp(-\phi_m \beta)$]. The 3-D expansion is obtained through either multiplication or division of the log-spiral by $(\sin \alpha)$; i.e., multiplication produces an end cap (Fig. 3(a)) and division yields the central cylinder (Fig. 3(b)).

Variational extremization of F_s with respect to the function $\theta(x, y)$ (direction of the shear force) yields (Leshchinsky, 1982)

$$\tan \theta = \begin{cases} \frac{-\phi_m \sin 2\alpha}{\phi_m \cos 2\alpha \sin \beta + \cos \beta} & \text{over end caps} \\ 0 & \text{over cylindrical part} \end{cases} \quad (8)$$

Introduction of Eqs. (2), (5) and (8) into the moment equilibrium equation written about the y axis results in (Leshchinsky, 1982)

$$M = \iint_{D_1} \left[\frac{Ac}{F_s} M_{11} - A\gamma M_{12} \right] dx dy + \iint_{D_2} \left[\frac{Ac}{F_s} M_{21} - A\gamma M_{22} \right] dx dy = 0 \quad (9)$$

where D_1 and D_2 are the projections of the cap and cylindrical parts, respectively, into the x - y plane, and

$$M_{11} = \frac{\exp(-\phi_m \beta) \sin^2 \alpha}{(\cos^2 \alpha - \sin^2 \alpha) \cos \beta + \phi_m \sin \beta} \quad (10a)$$

$$M_{12} = \frac{[A \exp(-\phi_m \beta) \sin^2 \alpha \cos \beta - (Z - z_c)] \exp(-\phi_m \beta) \sin^2 \alpha \sin \beta}{\exp(-\phi_m \beta) \sin^2 \alpha \sin \beta} \quad (10b)$$

$$M_{21} = \frac{\exp(-\phi_m \beta)}{\phi_m \sin \beta - \cos \beta} \quad (10c)$$

$$M_{22} = \frac{[A \exp(-\phi_m \beta) \cos \beta - (Z - z_c)] \exp(-\phi_m \beta) \sin \beta}{\exp(-\phi_m \beta) \sin \beta} \quad (10d)$$

Substituting Eqs. (10) into (9) and "solving" for F_s gives

$$F_s = \frac{c}{\gamma} \frac{\iint_{D_1} M_{11} dx dy + \iint_{D_2} M_{21} dx dy}{\iint_{D_1} M_{12} dx dy + \iint_{D_2} M_{22} dx dy} = f(A, x_c, y_c, z_c, F_s | Z(x, y), \phi, c, \gamma) \quad (11)$$

For a given problem (i.e., given slope sur-

face $Z(x, y)$ and soil parameters (γ, ϕ, c)) and a given slip surface (i.e., given (A, x_c, y_c, z_c)) Eq. (11) provides a non-linear relation for the determination of the corresponding factor of safety F_s . This non-linearity is due to the presence of the term $\phi_m = \phi/F_s$ in Eq. (10).

To obtain the factor of safety for a given slope it is still necessary to minimize Eq. (11) with respect to the unknown parameters defining the slip surface, i.e.,

$$F_s = \min_{(A, x_c, y_c, z_c)} f(A, x_c, y_c, z_c, F_s | Z(x, y), \phi, c, \gamma) \quad (12)$$

This minimization also selects the critical slip surface from the family of potential ones defined in Eq. (5). For a given problem, the solution procedure is as follows:

- Assume a value for A, x_c, y_c, z_c and F_s .
- Use Eqs. (10) and (11) iteratively until a consistent value for F_s is obtained. The integration required in Eq. (11) is performed numerically.
- Change the values of A, x_c, y_c, z_c and repeat steps (a) and (b) until a minimum F_s is located. The minimization is done numerically, changing one variable at a time.

The above solution procedure is similar, in principle, to the ones used in conventional 2-D slope stability analyses. In the 3-D problem, however, F_s is minimized with respect to four variables rather than three, and the integration is carried over a two dimensional domain rather than along a line.

Two features of the presented solution deserve a note:

- Eqs. (10), (11) and (12) are valid for any problem with at least one plane of symmetry (i.e., the slope surface shape is still unspecified in Eq. (10)).
- Eq. (11) is independent of the normal stress, i.e., the minimal factor of safety can be obtained without specifying $\sigma(x, y)$. This result is a consequence of the geometrical nature of the failure mechanism (Eq. (5)). This phenomenon is well known in 2-D problems when log-spiral or planar slip surfaces

Fig. 4. Definition of conical problem and modes of failure

(a) 3-D view

(a) Deep-seated failure (Case 3)

(b) Restricted failure (Case 4)

Fig. 5. Three-dimensional view of the critical slip surface ($i=30^\circ$)

are considered. In this sense, the slip surface defined by Eq. (5) is a 3-D generalization of the log-spiral family.

STABILITY OF CONES

The present study deals with the stability of heaps which can be approximated as homo-

(b) Projection of shear resisting forces direction θ

Fig. 6. The case of $i=30$ and $\lambda=0.067$ (shallow failure)

geneous and conical (i.e., the function $Z(x, y)$ is taken as the surface of the cone). The geometry of the problem is defined in Fig. 4. Typical 3-D failure surfaces are illustrated in Figs. 5 and 6. Notice that these surfaces represent one sided failure rather than simultaneous axisymmetrical flow as would be expected in an ideal problem. The justification for selecting this mechanism is that in reality small imperfections in either geometry or distribution of material properties will usually be sufficient to trigger one-sided failure. A similar argument is frequently employed in limit-equilibrium analysis of bearing-capacity of shallow footings (Garber and Baker, 1977).

As shown in Fig. 4, the critical failure

mechanism can be classified into the following four categories :

- Case 1 : Shallow failure in which the slip surface emerges at the toe but its trace is entirely above the foundation surface elevation.
- Case 2 : Intermediate failure in which the slip surface emerges at the toe but part of it is located below the foundation level.
- Case 3 : Deep seated failure in which the slip surface emerges beyond the toe.
- Case 4 : Restricted failure in which the slip surface is, at most, tangent to the foundation that is considered a strong (impenetrable) stratum.

The first three cases of failure are realized naturally for different combinations of material properties and geometry. Case four can be realized only when the extent of the failure surface is limited by a horizontal rigid layer situated at the toe elevation.

At this stage, it is convenient to introduce the following two nondimensional parameters :

$$N = \frac{c}{\gamma H} \quad \text{and} \quad N_m = \frac{N}{F_s} \quad (13a)$$

$$\lambda = \frac{1}{\gamma H} \frac{c}{\phi} = \frac{N_m}{\phi_m} \quad (13b)$$

where H is the height of the cone. N_m is the classical stability number first introduced by Taylor (1948). The parameter λ characterizes the relative contribution of cohesion and friction to the soil strength. This parameter has a major effect on the critical slip surface.

The failure mechanisms shown in Fig.5 corresponds to Case 3 and Case 4. Fig.6(a) illustrates a 3-D view of Case 1. The direction of the shear resisting forces θ , as calculated by Eq.(8), is presented in Fig.6(b). Notice that in the cap region this direction depends on the location (x, y) .

Fig.7 and 8 show, via plan view and cross sections, the effect of material properties on the potential slip surface. Fig.7 indicates that as ϕ increases relative to $c/\gamma H$ (i.e., λ decreases), the critical sliding body becomes

Fig. 7. Failure for various combinations of c and ϕ ($i=30^\circ$)

shallower. In the limit where cohesionless soil is considered, the slip surface coincides with the cone surface. Under this condition, the expression for the factor of safety degenerates to $F_s = \tan i / \tan \phi$. Consequently, in this case there are no 3-D ef-

Fig. 9. Design chart for the determination of the safety factor

fects.

Fig. 9 is a design chart relating the parameters N_m , ϕ_m , λ and i . This relation is the solution of Eqs. (11) and (12). Essentially, this chart is similar to Taylor's stability diagram which pertains to 2-D homogeneous problems. Notice that Case 1 prevails for steep cones with $i \geq 60^\circ$. For cohesive soil and $i < 50^\circ$, however, deep-seated failures (i.e., Case 3) will tend to develop. Unlike the analogous 2-D problem, these deep failures are of finite extent.

The usefulness of Fig. 9 can be best demonstrated through the following example. Given a material possessing $\gamma = 18 \text{ kN/m}^3$, $\phi = 20^\circ$ and $c = 87 \text{ kPa}$. This material has to be stored in a conical pile having height $H = 50 \text{ m}$ and possessing a factor of safety of $F_s = 1.5$. Determine the required inclination of the cone. First, calculate $\phi_m = \tan 20^\circ / 1.5 = 0.243$ and $N_m = 87 / (18 \cdot 50 \cdot 1.5) = 0.064$. Now, using Fig. 9 for the abscissa 0.243 and ordinate 0.064, one gets the corresponding inclination of $i = 50^\circ$.

Fig. 9 can be used also to analyze the stability of an existing cone as demonstrated in the following example. Say that γ , ϕ , c and H are as given in the previous example.

Fig. 10. Design chart for the determination of the volume of potential slide

Determine F_s if the cone is inclined at $i = 60^\circ$. Calculate $\lambda = 87 / (18 \cdot 50 \cdot \tan^2 20^\circ) = 0.266$. Observing that, in fact, each λ represents a constant ratio between the ordinate and the abscissa in Fig. 9, enter the chart along a line for which $\lambda = 0.266$. Its intersection with the curve along which $i = 60^\circ$ is defined by $\phi_m = 0.275$ and $N_m = 0.073$. Thus, the factor of safety is $F_s = \tan \phi / \phi_m = 1.32$ (or, alternatively, $F_s = c / \gamma H N_m = 1.32$).

Fig. 10 presents the relationship between the volume of the potentially sliding mass and ϕ_m for various cone inclinations. The points on this figure are the calculated values. They are included in order to illustrate the extent of computational inaccuracies introduced by the numerical integration and minimization (Eqs. (11) and (12)) performed in calculating the solution. It should be stated that the calculated points defining Fig. 9 are even less scattered than those shown in Fig. 10.

The horizontal axis in Fig. 10 corresponds to $\lambda = 0$ (i.e., purely frictional material) while the vertical coordinate corresponds to

$\lambda = \infty$ (i.e., purely cohesive material). As expected for purely frictional material, the volume of failed soil approaches zero; i.e., the slip and slope surfaces coincide.

Fig. 10 can be used as a tool in selecting a rational value for F_s . Since the volume of a potential slide can be directly related to its cost of repair, one can select F_s and the associated cone geometry based on economical considerations. To demonstrate this approach, the problem stated in the example before is being used again; i.e., $i = 50^\circ$, $H = 50$ m, $\gamma = 18$ kN/m³, $\phi = 20^\circ$, $c = 87$ kPa and $F_s = 1.5$. Since ϕ_m equals 0.242, it follows from Fig. 10 that v_s/v_0 is about 26%. The volume of the given cone is $v_0 = \pi H^3 / (3 \tan^2 i) = 92,165$ m³ and therefore the volume of the potential slide v_s is about 24,000 m³. On the other hand, if one chose to store the same volume of material in a 60° cone, the corresponding height on the cone should be 64.2 m. Using Fig. 9 for $H = 64.2$ m and the same soil properties, it follows that ϕ_m equals 0.322 (i.e., $F_s = 1.13$). Subsequently, the corresponding v_s/v_0 is about 28% and therefore the volume of the potential slide is

25,800 m³; i.e., the trade-off for a decreased factor of safety is a potential increase in the volume of failure and repair cost. Thus, a relation quantifying the repair cost associated with varying the design factor of safety can be formed.

If a 3-D analysis is not available, an approximate way to assess the stability of cones is through a plane-strain type of analysis. In the 2-D approximation, however, one assumes that the height of the slope is constantly H , i.e., the crest is semi-infinite long (see Fig. 11). The geometrical similarity between the 2- and 3-D problems, therefore, is limited to the inclination and the length of the slope face. The stability of the 2-D problem in Fig. 11 was estimated using Baker's (1981) chart. This figure shows the relationship between the ratio of the safety factors of the two problems and the soil's mobilized friction as calculated for the 2-D case. Again, the horizontal axis corresponds to the case $\lambda = 0$ (i.e., purely frictional soil). It is seen from this figure that the degree of conservatism of the 2-D approximation increases as λ increases.

Fig. 11. Comparison of stability predictions by 3-D and an "Equivalent" 2-D analysis

THE PLANE-STRAIN LIMIT

Observing Eqs. (10), (11) and (12) it is clear that the presented solution is applicable to a wide class of symmetrical homogeneous problems where the slope surface function $Z(x, y)$ is known. This generality not only enhances the value of the derived solution but it also enables one to check the performance of the solution as the slope surface approaches a 2-D limit where the results are known. In the engineering sense, a 2-D approximation is justified when (1) the longitudinal dimension of the problem in hand is much longer than the other two dimensions, and (2) the variations in the geometry of the problem along its longitudinal axis are small. Truncated cones can represent approximated 2-D problems. As the radius of the truncated cone increases relative to its height (i.e., R/H increases—see Fig. 12), while its inclination remains constant, a larger portion

Fig. 12. Three-dimensional view of the truncated cone case ($i=60^\circ$; $\lambda=0.433$)

Fig. 13. The effect of R/H in truncated cone on stability predictions

of its arched face may be approximated as a plane parallel to the y -axis. Thus, the geometry of practical 2-D problems can be fitted by truncated cones through proper selection of R , H and i .

Sixty degrees truncated cones with R/H

varying between 1 and 10,000 were analyzed. Fig. 12(a) and 12(b) illustrate typical 3-D views of the failure surfaces. Fig. 13 shows the relationship between $F_s(3-D)/F_s(2-D)$ and R/H for various combinations of material strength (i.e., various λ). For each selected λ in this figure, $F_s(2-D)$ was determined from Baker's chart (Baker, 1981) using H and i of the truncated cone to describe the 2-D geometry of the problem. It can be seen that as R/H increases, the 3-D safety factors predictions approach the plane-strain limit. Simultaneously, the length of the critical surface increases making the end effect contribution negligible. This demonstrates the consistency of the solution when known boundary cases are analyzed.

It is important to note that Baker's (Baker, 1981) solution is based on 2-D variational L.E. analysis and thus is a particular case of the 3-D solution presented here. Furthermore, it was demonstrated (Baker, 1981) that 2-D variational results practically coincide with predictions based on Taylor's classical analysis. This can be used as an indication that the variational approach produces results which agree with predictions of well established methods when conventional problems are analyzed. The fact that the 3-D solution is a rigorous extension of the 2-D variational formulation and that it degenerates to known solutions when limit cases are analyzed, significantly increase the confidence in the present solution.

Finally, it is interesting to note that although the ratio $F_s(3-D)/F_s(2-D)$ approaches unity as R/H increases, the corresponding length of potential failure is finite and distinctive (i.e., see Fig. 12(b)). This phenomenon provides a deterministic explanation for failures of limited extent occurring in nature where pseudo plane-strain conditions exist.

SUMMARY AND CONCLUSIONS

Outlined is a 3-D slope stability analysis which is based on the variational limiting-equilibrium approach. This approach was ap-

plied to study the stability of homogeneous conical heaps assuming that such cones are likely to undergo one sided failure. The analysis results are summarized in Figs. 9, 10 and 11. Figs. 9 and 10 are practical design charts. Fig. 9 is similar to the conventional 2-D stability chart where the factor of safety can be determined directly for a given problem. Fig. 10 enables the prediction of the volume of the potentially sliding mass. This latter chart might be useful when economical considerations, such as potential cost of repair, are used in the selection of a factor of safety in design.

Based on Fig. 9 through 11, the following qualitative observations are made :

—Two dimensional analysis of 3-D problems yields conservative results. The degree of conservatism depends on the nature of the material (i. e., the parameter λ) and the inclination of the cone.

—The most pronounced 3-D effects occur in steep slopes comprised of cohesive soil. The maximum ratio of $F_s(3-D)/F_s(2-D)$ is of the order of 1.6. This effect decreases with λ and for purely frictional soil ($\lambda=0$) there are no 3-D effects, i. e., $F_s(3-D)/F_s(2-D) = \tan i / \tan \phi$.

—The volume of failure also depends on the nature of the soil and the inclination of the cone. For purely frictional materials this volume approaches zero indicating that the failure mechanism consists of rolling down of material along the surface of the cone. In such a case, the consequences of failure may not be critical and therefore it is reasonable to require lower design factor of safety for frictional materials than for cohesive ones.

—The presence of a rigid base may increase the stability and decrease the potential volume of failure of a cone. These, however, are restricted to cases of flat cohesive cones.

Since the presented 3-D solution is general, it was applied to truncated cones to check its predictions when the problem resembles plane-strain conditions. Based on this analysis, the following is concluded.

—The 3-D results degenerate to known

2-D results when the problem approximates plane-strain conditions.

—Even when the problem closely resembles plane-strain conditions, the sliding body extends over well defined length. This provides a quantitative deterministic explanation for some failures in nature.

The 3-D procedure can be extended in several directions ; for example, geometries other than those used here can be analyzed by substituting the proper $Z(x, y)$ into Eq. (10). The only restriction is that the problem should possess at least one plane of symmetry. Other extensions such as earthquake stability can be made via the conventional pseudo-static approach. The procedure can be modified to deal with pore water pressures (e. g., see equations in Ref. 1) and external loads. Such modifications are straightforward since the failure mechanism (Eq. (5)) remains unchanged. The statical portion of the problem (i. e., the equation of moment equilibrium), however, requires proper alterations.

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