A NEW LIMIT EQUILIBRIUM ANALYSIS OF SLOPE STABILITY BASED ON LOWER-BOUND THEOREM

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ABSTRACT

This paper develops a new numerical procedure for the slope stability analysis, by combining the slice method in the limit equilibrium method with the lower-bound approach in the limit analysis method. The procedure provides an appropriate lower-bound solution subject to the collapse mechanism represented by a potential slip surface. First, the safety factor for the shear strength is calculated for the case in which the position of the slip surface is known beforehand. The problem is formulated as an optimization problem for finding a stress field which minimizes the safety factor within the limitations of satisfying the lower-bound conditions. The formulated problem is solved numerically by the mathematical programming technique. The obtained stress field appears physically reasonable. Several case studies have shown that the simplified Janbu method based on the critical slip surface tends to underestimate the safety factor on the safety side, compared with the lower-bound solution by the proposed procedure. Subsequently, the assumption used in each of the representative slice methods, which are necessary to render the stability problem to a statically determinate, is verified by the use of the stress field determined by the proposed procedure. Finally, a procedure for locating the critical noncircular slip surface which minimizes the safety factor within the lower-bound conditions is developed. Many important properties in the slope stability analysis are clarified through the development of this procedure.

Key words: computer application, finite element method, safety factor, slip surface, slope stability, stability analysis (IGC: E6)

INTRODUCTION

When designing geotechnical structures, the stability analysis is still the most powerful means of providing a strength analysis. The methods for stability analysis are classified into three groups, i.e., the limit analysis, slip line, and limit equilibrium methods. Chen (1975) concisely summarized these three methods as follows. The limit
analysis method is based on the lower- and upper-bound theorems in plasticity. The lower-bound solution determined from a distribution of stress, satisfies the equilibrium equations, the stress boundary conditions, and nowhere violates the failure criterion. The upper-bound solution determined by equating the external rate of work to the internal rate of dissipation in an assumed deformation mode, satisfies the velocity boundary conditions, and the strain and velocity compatibility conditions. The solution which satisfies both the lower- and upper-bound conditions, is referred to as the complete solution. The slip line method solves a set of differential equations given by combining the failure criterion with the equations of equilibrium. The limit equilibrium method invokes no kinematical considerations regarding soil behavior; and hence, requires that the shape of the potential slip surface be assumed. Then, the limit equilibrium method reduces the stability problem to one of finding the critical position for the slip surface. In this method, the equations of equilibrium and the failure criterion are satisfied only along the potential slip surface. Neither of these conditions is considered inside or outside of the slip surface. A solution obtained by the use of limit equilibrium method is not necessarily a lower- or an upper-bound solution, but an approximate solution.

None of these methods for stability analysis provides the mathematical solution for a particular problem. This insufficiency accelerated the progress of numerical procedures based on the discretization technique. In the slip line method, numerical procedures using finite difference approximation have been developed and applied to certain types of stability problems. In the limit equilibrium method, a number of slice methods which divide the potential sliding mass into many finite slices, have been proposed and are now being applied quite widely. Subject to the slope stability analysis, Janbu (1957), Morgenstern and Price (1965), Bell (1968), and Madej (1971) developed the slice methods supposing an arbitrary shape of the slip surface. Each of these slice methods provides a means for calculating the safety factor for a known slip surface, but does not furnish a technique for locating the critical slip surface which gives the minimum factor of safety. Celestino and Duncan (1981), Arai and Tagyo (1985a), and Greco and Gulla' (1985) developed practical procedures for determining the location of the critical noncircular slip surface, and made it possible to apply the slice methods over a wide range of slope stability problems. The solution using the slice method which satisfies the equations of equilibrium in each of the slices, seems more reasonable than the other limit equilibrium methods satisfying the equilibrium equations only in the overall sliding mass. However, no slice method is beyond the framework of the limit equilibrium method. In the limit analysis method, many numerical procedures have been proposed which discretize the fields of stress and velocity in a similar manner as in the finite element method (FEM). Fremond and Salençon (1973), Turgeman and Pastor (1982), and Tamura, Kobayashi and Sumi (1984) developed the procedures which combined the upper-bound theorem with the finite element approach. Lysmer (1970), Pastor and Turgeman (1982), and Arai and Tagyo (1985b) proposed the procedures which used the lower-bound theorem together with the discretization technique in FEM. These numerical procedures enabled the application of the limit analysis method to a wide variety of stability problems which have arbitrary geometry and boundary conditions, without considering any skillfull construction of discontinuous fields of stress and velocity being related to the collapse mechanism. This advantage, however, leads to uncertainty in which one cannot clearly evaluate the collapse mechanism or the shape of the failure surface. It is also difficult to make a comparison with the results by the other methods for stability analysis. This defect is due to the limitations of the discretization technique in FEM which is not necessarily suitable for simulating the discontinuous fields of stress.
and velocity. This paper aims at developing a new numerical procedure for the slope stability analysis which combines the slice method with the lower-bound approach. The problem formulation in this paper is somewhat similar to the one employed by Arai and Tagyo (1985 b), except that the present study assumes a potential slip surface while the latter did not consider the failure mechanism. By taking the slip surface into consideration, this paper enables the lower-bound approach to evaluate the collapse mechanism and to make a comparison with the other methods for stability analysis. For instance, in many cases of applying the slice method, a distribution of stress on the whole body except along the slip surface is of interest. A method for investigating this is also presented. Some mechanical properties in the slope stability problem are clarified through the application of the proposed procedure.

SAFETY FACTOR FOR A GIVEN SLIP SURFACE

First, we investigated the problem of finding the safety factor for a known slip surface. The problem of locating the critical slip surface will be discussed later.

Finite Element Meshing

For convenience of illustration, let us consider a simple slope of a homogeneous soil with a zero pore pressure. Fig.1 illustrates the earth slope subjected to analysis. Assume that the location of the potential slip surface is prescribed a priori as shown in Fig.1. The potential slip surface may be of any physically acceptable shape. The potential sliding body is divided into a number of finite slices by vertical lines, and the slip surface is approximated by a series of straight lines. Now each of the slices is further subdivided into several triangular or quadrilateral elements as shown in Fig.2. On this occasion, it is convenient to employ the subdivision system which simplifies an automatic finite element meshing. Note that the region only on the inside of the slip surface is subdivided into finite elements, whereas the lower-bound solution must be statically admissible on the whole region for the problem. The reason will be given later.

Lower-Bound Solution

The conditions required to establish a lower-bound solution are essentially as follows (see Chen, 1975). 1) The stress distribution must everywhere satisfy the equations of equilibrium. 2) The stress field at the boundary must satisfy the stress boundary conditions. In the present problem of slope stability, the stress boundary conditions correspond to the failure condition along a potential slip surface. 3) The stress field must nowhere violate the failure criterion. Equations of equilibrium: Fig.2 includes three types of finite elements, triangular, quadrilateral, and slip surface elements. It is convenient to assume a set of stresses to be constant within each finite element, because stress is an independent variable to be determined by the proposed procedure. In the triangular element, the \([B]\) matrix for

Fig. 1. Slice subdivision system

Fig. 2. Finite element mesh
calculating the strains from nodal displacements is the same as the \([B]\) matrix in the constant strain triangular element in classical FEM. The quadrilateral element is considered to be composed of the four triangular elements as shown in Fig. 3. The \([B]\) matrix in the quadrilateral element is constructed by the superposition of the \([B]\) matrix in each triangular element. The \([B]\) matrix in the slip surface element is the same as the \([B]\) matrix in the joint element used popularly in rock mechanics, with the exception of neglecting the terms of normal strain and moment (see Goodman, 1976). By employing the principle of virtual displacement, the following equations define a set of equivalent nodal forces which is statically in equilibrium with the stress condition in the element.

For the triangular and quadrilateral elements:

\[
\{F_{x1}^T, F_{y1}^T, \ldots\}^T = [B]^T(\sigma_x^T, \sigma_y^T, \tau_{xy}^T)^T A_m
\]

(1)

For the slip surface element:

\[
\{F_{x1}^T, F_{y1}^T, F_{x2}^T, F_{y2}^T\}^T = [B]^T(\sigma_x^T, \tau_{xy}^T)^T L_m
\]

(2)

where \(F_{x1}^T, F_{y1}^T\) and \(F_{x2}^T, F_{y2}^T\) are the horizontal and vertical components of the equivalent nodal forces, \(\sigma_x^T, \sigma_y^T, \tau_{xy}^T\) : horizontal and vertical normal stresses and shear stress in the triangle and quadrilateral elements, \(\sigma_{a}^T, \tau_{a}^T\) : normal and shear stresses in the slip surface element, \(A_m\) : area of the element, \(L_m\) : length of the slip surface element, \(m\) : finite element number, and \(n\) : node number. The equations of equilibrium mean that the sum of the equivalent nodal forces and nodal loads must be equal to zero at each of the nodal points.

For equilibrium in the horizontal direction:

\[
P^n_1 = \sum_{m} F_{xn}^T + G_{xn} = 0
\]

(3)

For equilibrium in the vertical direction:

\[
P^n_2 = \sum_{m} F_{yn}^T + G_{yn} = 0
\]

(4)

where \(G_{xn}\) : horizontal component of the nodal load, and \(G_{yn}\) : vertical component of the nodal load which contains the weight of soil mass.

Failure condition along a slip surface (Stress boundary condition): The potential slip surface means that at any point on the slip surface, the safety factor for shear strength has the same value. When employing the Coulomb failure criterion, this condition is represented as

\[
\begin{align*}
P^m &= P_s - F^m_s = 0 \\
F^m_s &= (c + \sigma_{a}^T \tan \phi) / \tau_{a}^T, m \in S_s
\end{align*}
\]

(5)

where \(P_s\) : overall safety factor for shearing resistance, \(F^m_s\) : safety factor in the slip surface element \(m\), \(c\) : cohesion intercept, \(\phi\) : angle of shearing resistance, and \(S_s\) : set of the slip surface elements. Eq. (5) constitutes a definition of the safety factor. When applying only the above failure condition, the proposed procedure in some cases happens to provide an irregular distribution of safety factor in the neighborhood of the slip surface. This instability may be attributed to the discretization technique of the stress field. The technique describes the equilibrium equations by means of the equivalent nodal forces which are statically in equilibrium with the stress condition in each finite element. Such a system of representing the equilibrium equations may make the stress field unstable in the finite elements neighboring to slip surface, when both the stress field in sliding mass and the stress distribu-
tion along slip surface are unknown. To avoid this instability, all the finite elements in contact with the slip surface, which are shaded in Fig.2, are assumed to have the same safety factor value as the one along the slip surface.

\[ P_s^m = F_s - \tau_s^m = 0, \quad F_s^m = b/a, \quad m \in S_n \]

\[ a = (\sigma_x^m - \sigma_y^m)^2 + (2\tau_x^m)^2 \frac{1}{2} \]

\[ b = (\sigma_x^m + \sigma_y^m)\sin \phi + 2c \cos \phi \]

where \( a, b \): see Fig.4, and \( S_n \) : set of the finite elements neighboring to the slip surface.

From this point of view, the finite elements in the neighborhood of the slip surface are constructed with a somewhat narrow band rather than the other finite elements in the sliding mass.

No-failure condition: Except for the finite elements touching the slip surface, the following no-failure condition must be satisfied for each finite element in the sliding mass.

\[ P_{s,\text{min}}(=b/a) \geq P_s \]  \hfill (7)

When applying this constraint, there are cases where safety factor \( P_s^m \) in the element distant from the slip surface becomes less than that in the element close by the slip surface. This may be contradict to our intuitive thinking. Then we assume as follows.

For each slice:

\[ P_s^m = (F_s^m \text{ in the upper element}) - (F_s^m \text{ in the lower element}) \geq 0 \]  \hfill (8)

This constraint is justified on the grounds that the proposed procedure presents a unique stress field according to the constraint. Furthermore, assuming that the soil mass has no tensile strength,

\[ \sigma_x^m \geq 0, \quad \sigma_y^m \geq 0 \]  \hfill (9)

This constraint is effective to stabilize the iteration behavior in the subsequent numerical analysis.

Problem Formulation

The problem of calculating the safety factor is formulated as an optimization problem which isolates a particular stress field. The independent variables to be determined are the stress components in each finite element \( \sigma_x^m, \sigma_y^m \) and \( \tau_x^m \), normal and shear stresses acting along slip surface \( \sigma_x^m \) and \( \tau_x^m \), and overall safety factor \( F_s \). The constraints to be satisfied are Eqs. (3), (4), (5), (6), (8) and (9). The objective function is to minimize the overall safety factor \( F_s \).

Minimize \( J = F_s \) \hfill (10)

Referring to Eq. (5), this objective function pursues a stress distribution which maximizes the mobilized shear stress against the available shearing resistance both along a slip surface. As shown in Fig.2, the proposed procedure is concerned only with the interior of the potential slip surface. In order to take into consideration the exterior of the slip surface, one must create a boundary which separates the region of analysis from an infinite half space. When assuming a boundary at which the displacements are restricted completely, the stresses at the boundary may be capable of increasing up to any amount of the value. Under such a boundary condition, the objective function defined by Eq. (10) enables the mobilized shear stresses along the slip surface to increase infinitely. As a result, the proposed procedure provides too small safety factor for this case. To avoid such a contradiction, it seems logical to exclude the region on the outside of the slip surface from the discussion in the present stability analysis. Once a stress distribution is determined on the inside of the slip surface, it is not difficult to find a statically admissible stress field on the outside of the slip surface which satisfies the boundary conditions along the slip surface and the conditions at the aforementioned outer boundary. Thus, the proposed procedure is thought to furnish a lower-bound solution on the whole region for the problem, although it deals only with the interior of the slip surface.

Numerical Analysis

To solve the constrained optimization problem formulated above, the proposed procedure employs SUMT (Sequential Unconstrained Minimization Technique) interior point method developed by Fiacco and McCormick.
(1968). This method achieves the minimization of an objective function in the interior of the feasible region by avoiding the boundary which represents the constraints. Such a property is more compatible with the lower-bound approach in which the stress field must be strictly on the inside of the failure criterion. The SUMT interior point method defines the following modified objective function towards the original optimization problem.

Optimization problem: Find \( \mathbf{x}^* \) solving

\[
\text{minimize : } J = F(\mathbf{x}) \tag{11}
\]

subject to

\[
\mathbf{h}(\mathbf{x}) = 0, \quad \mathbf{g}(\mathbf{x}) \geq 0 \tag{12}
\]

Modified objective function:

\[
P(\mathbf{x}, \lambda_k) = F(\mathbf{x}) + \lambda_k \sum_i g_i(\mathbf{x})^{-1} + \lambda_k^{-1/2} \sum_i h_i(\mathbf{x})^2 \tag{13}
\]

where \( \lambda_k \): a positive number referred to as the penalty coefficient. Starting from initial value \( \mathbf{x}_0 \) in the interior of the feasible region, find an unconstrained minimum of \( P(\mathbf{x}, \lambda_k) \) for some \( \lambda_k \). Proceed in this fashion, minimizing \( P(\mathbf{x}, \lambda_k) \) for a strictly monotonically decreasing sequence \( \{\lambda_k\} \), and the sequence of unconstrained minima approaches a local constrained minimum \( \mathbf{x}^* \). The detailed introduction of the procedure is given by Arai and Tagyo (1985 b). The modified objective function for the proposed procedure is defined as

\[
P(\mathbf{x}, \lambda_k) = F_e + \alpha \left[ \lambda_k^{-1/2} \sum_{m \in S} (P_{m}^e)^2 + \lambda_k^{-1/2} \sum_{m \in S} (P_{m}^s)^2 \right] + \lambda_k^{-1/2} \sum_{m \in S} (P_{m}^s)^2 + \lambda_k^{-1/2} \sum_{m \in S} (P_{m}^s)^2 + \lambda_k \sum_{m \in S} (P_{m}^s)^{-1} \tag{14}
\]

where \( \alpha \): a positive constant which adjusts the order of magnitude both of original objective function \( F_e \) and of the penalty terms. In the search for an unconstrained minimum of modified objective function \( P(\mathbf{x}, \lambda_k) \) for a certain \( \lambda_k \), the proposed procedure employs the conjugate gradient technique developed by Fletcher and Reeves (1964). This technique, at first, calculates the gradient of the modified objective function with respect to the independent variables to be determined. The gradient is easily represented in the analytical form. Starting from the trial values of the independent variables, this technique repeats the iteration procedure to search the optimum solution which minimizes the modified objective function. At each iteration step, when the stress components violate Eq. (9), take the boundary value. The detailed introduction of this technique is also given by Arai and Tagyo (1985 b).

Computational Comments

1) When applying the proposed procedure, one must take special care regarding both the penalty coefficient value and the trial values of the independent variables. Many experiments have proved that the following combination provides a reasonable convergence in general cases:

\[
\begin{align*}
\lambda_i &= 1, \quad \lambda_2 = 0.01, \quad \lambda_3 = 0.0001 \\
\alpha (\text{see Eq. 14}) &= 35/N_e \\
\text{trial value : } &\sigma_{\mu} = \gamma h, \\
\sigma_{\mu} &= |\sigma_{\mu}^e(F_{e}^m - \sin \phi) - 2c \cos \phi| / (F_{e}^m + \sin \phi), \quad \tau_{xy}^m = 0
\end{align*}
\tag{15}
\]

where \( N_e \): total number of finite elements, \( \gamma \): unit weight of the soil, and \( h \): depth from the ground surface to the finite element. 2) In SUMT, it is important to control the order of magnitude both for the original objective function and the penalty terms, so that the value of \( \lambda_k \) could be commonly used for a wide-ranging combination between the strength parameters and stress components. For this purpose it is useful to normalize the penalty terms by cohesion \( c \) (see Arai and Tagyo, 1985 b). 3) The minimum calculated by SUMT is a local minimum. It is difficult to ascertain theoretically whether the minimum found is a global minimum or purely a local one. Repeated use of the proposed procedure from some different trial solutions has provided about the same optimal solution. This result appears to exclude the possibility that several other local minima are present. In addition, Arai and Tagyo (1985 b) investigated the errors related to this type of opti-
mization problem.

CASE STUDIES FOR A GIVEN SLIP SURFACE

The proposed procedure is applied to several case studies in which the location of a potential slip surface is known beforehand. In the following Examples 1 through 3, the position of a slip surface was determined by a procedure for locating the critical noncircular slip surface which minimizes the safety factor defined by the simplified Janbu method (see Arai and Tagyo, 1985 a).

Example 1: The first example (see Figs. 1 and 2) considers a simple and homogeneous soil slope having a gentle gradient. The soil parameters are shown in Fig.1. The potential sliding mass in Fig. 1 is divided into 8 slices, all having a width of 6 m. Each slice is further subdivided into 5 triangular or quadrilateral finite elements as shown in Fig. 2. Towards the finite element mesh illustrated in Fig. 2, a computer program automatically provides the coordinates of nodal points and the node numbers constituting each finite element, both of which are easily calculated according to the prescribed number of subdivided finite elements in each slice. Fig. 5 schematically gives the results using the proposed procedure. An inspection of Fig. 5 reveals that the proposed procedure furnishes a physically reasonable and statically admissible stress field. Repeated use of the procedure from some different trial solutions in the iteration scheme has yielded a unique specification of the stress distribution. In Fig. 5 (b), the safety factor in the slip surface element is fractionally different from each other, despite of the constraint by Eq. (5). This constraint requires that all the slip surface elements have the same safety factor value for the shear strength. Such a numerical error is due to the incomplete convergence in the iteration scheme, and due to the limitations of the discretization system in which a set of stresses is assumed constant within each finite element.

In Fig. 5 (b), it is further important to note that a slight difference in the safety factor has arisen between the slip surface elements and the failure elements in the neighborhood of the slip surface, whereas these two types of failed elements must have the same safety factor value according to Eqs. (5) and (6). This difference is caused by the different definitions of safety factor \( F_s \) in Eqs. (5) and (6). That is, Eq. (5) uses the Coulomb failure criterion, while Eq. (6) employs the Mohr-Coulomb failure criterion. To be more precise, strength parameters \( c \) and \( \phi \) have to be adjusted in each criterion so that these two criteria may yield an identical factor of safety. However, such a modification is omitted here, owing to a slight difference in safety factor. On the other hand, the conventional limit equilibrium methods apply Coulomb failure criterion along the slip surface. Thus, a comparison must be made with other limit equilibrium solutions, by the use of the safety factor in the slip surface element. For this purpose, the following new safety factor \( F_a \) is defined by taking the average of the safety factors in

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**Fig. 5. Results in Example 1**
the slip surface elements.

\[ F_s = \text{Average}[F_{s,m} = (c + \sigma_m \tan \phi) / \tau_{m}^{\text{m}}], \quad m \in S_s \]  

(16)

The value of \( F_s \) in Example 1 is given in Fig. 5(b) together with overall safety factor \( F_r \). The latter takes an intermediate value between \( F_s \) and the safety factor in the failure element in the neighborhood of the slip surface.

**Example 2**: To investigate the influence of finite element meshing, the Example 2 (see Fig. 6) studies the same model of the earth slope as in Example 1, while employing a much narrower slice width. The results in this problem are illustrated in Fig. 6. Due to the infinitesimal subdivision system, the obtained stress field appears more continuous and smoother than that found in Example 1. The distribution of safety factor in each finite element is not so different from that in Example 1. Note that the rough finite element meshing in Example 1 produces a smaller value for the overall safety factor.

**Example 3**: The third example (see Fig. 7) investigates a simple and homogeneous slope having a steep gradient. As well as in Examples 1 and 2, the potential slip surface was located by Arai and Tagyo’s procedure. Fig. 7 shows the results by the proposed procedure. Also, in this case, the stress field
SLOPE STABILITY ANALYSIS

Table 1. Comparison with slice methods

<table>
<thead>
<tr>
<th>Example No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<th>5b</th>
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<td>0.919</td>
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<td>—</td>
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<td>0.963</td>
<td>1.984</td>
<td>2.127</td>
<td>2.169</td>
</tr>
</tbody>
</table>


Fig. 9. Results in Example 5

Presented appears quite reasonable.

Examples 4 and 5: The fourth and fifth examples are for comparing the result by the proposed procedure with the existing limit equilibrium solutions. The fourth example (see Fig. 8) investigates a homogeneous earth slope supposing a circular slip surface. This hypothetical model was used from Morgenstern and Price (1965) and Madej (1971). To allow for more convenience in applying the computer program, a slight revision in the slice width is instituted for the original slope model. Example 5 (see Fig.9) considers a slope taken from Sarma (1973) which is a homogeneous earth slope assuming a non-circular slip surface. Fig.9 illustrates the results by the proposed procedure.

A Comparison with the Slice Methods

Table 1 contrasts safety factor $F_a$ by the proposed procedure (see Eq.16) with the safety factors obtained by some slice methods. In comparison with the simplified Janbu method, the proposed procedure overestimates the safety factor in Examples 1 through 4, while the procedure underestimates it in Example 5. Note that in Examples 4 and 5, the proposed procedure gives the safety factor lower than most of the other slice methods. Emphasis is placed upon the fact that in Examples 1 through 3 the potential slip surface is the critical slip surface which minimizes the safety factor defined by the simplified Janbu method. When the critical position of the slip surface is given, the simplified Janbu method gives the safety factor lower than the lower-bound solution by the proposed procedure. The reason is understood as follows. The objective function defined by Eq. (10) pursues a stress field which minimizes overall safety factor $F_a$. In other words, Eq. (10) tries to maximize the mobilized shear stresses against the available shearing resistance along a potential slip surface. Starting from a trial solution, the proposed procedure approaches such a stress field from the inner side of the lower-bound conditions. We cannot exclude the possibility that the procedure finishes the iteration scheme before really maximizing the mobilized stresses. This leads to the possibility that the procedure gives the safety factor higher than the true minimum for a given slip surface. (This view suggests the usefulness of the upper-bound solution in the slope stability analysis, because the solution is expected to underestimate the safety factor on the safety side compared with the lower-bound solution.) This may be the reason why the proposed procedure gives the safety factor higher than the simplified Janbu method, when giving the critical slip surface.
which minimizes the safety factor defined by the simplified Janbu method. However, when assuming the position of slip surface at random, the safety factor obtained by the simplified Janbu method may be neither the minimum for the slope stability problem nor the real solution by the limit equilibrium method. Then the proposed procedure happens to give the safety factor lower than the simplified Janbu method in Examples 4 and 5. By the way, a limit equilibrium solution has been sometimes interpreted as an upper-bound solution, because the solution is not statically admissible but may satisfy the kinematically admissible conditions when assuming a physically reasonable shape of a slip surface. Chen (1975) presented the slope stability solutions based on the upper-bound technique, and concluded that the numerical results for many cases agree well with the existing limit equilibrium solutions. The findings is consistent with the above interpretation concerning the limit equilibrium solution. However, the interpretation is not accepted in Examples 4 and 5, when assuming the location of slip surface at random, because the upper-bound solution in the slope stability analysis is to give the safety factor lower than the lower-bound solution by the proposed procedure as stated previously. These results accentuate the importance of finding the position of the critical slip surface, in case of applying the slice methods. Except the simplified Janbu method, most of the slice methods in Table 1 require many iterations to calculate the safety factor for a given slip surface. This makes it difficult to search the location of the critical slip surface (see Arai and Tagyo, 1985a). At the present stage, the conclusion in this paragraph cannot be confirmed for all the slice methods. However, considering the fact that the slice methods in Table 1 provide about the same safety factor in Examples 4 and 5, the conclusion may appear acceptable for these representative slice methods. It must be examined more exhaustively in a future study.

VERIFICATION OF SLICE METHODS

Each of the conventional slice methods inevitably contains a statically indeterminate problem with respect to the internal forces between the slices. The indeterminacy arises from the lack of knowledge of the stresses obtained in the soil mass. Each of the slice methods employs a certain assumption necessary to render the problem statically determinate. However, the proposed procedure avoids the indeterminacy by trying to find a stress field which minimizes the overall safety factor, and provides a unique stress distribution. This paragraph concerns a comparison between the assumptions used in the representative slice methods and the stress field obtained by the proposed procedure. From the stress field determined by the proposed procedure, the internal forces between the slices are calculated as follows. At first, the equivalent nodal forces using Eqs. (1) and (2) are calculated. The equivalent nodal forces are superposed respectively on the left and right sides of interslice boundary. Fig. 10 demonstrates the horizontal components of superposed nodal forces in Example 1. By taking the moments of these forces about a certain point, the lateral thrust on the side of slice can be found. The vertical shear force on the side of slice is obtained by summing the vertical components of nodal forces respectively on each side of the slice. Fig. 11 illustrates the internal forces calculated by the above procedure. The internal forces on both sides of interslice surface must be consistent each other. The incomplete consist-

![Fig. 10. Horizontal components of nodal forces](image-url)
Fig. 11. Distribution of internal forces

Fig. 12. Janbu's assumption

Fig. 13. Spencer's assumption

Fig. 14. Morgenstern and Price's assumption

Fig. 11. Distribution of internal forces

Fig. 12 plots the position of thrust line shown in Fig. 11. The line of thrust falls approximately 1/3 of the height from the slip surface to the slope surface. The constraint defined by Eq. (9) compels the horizontal component of the stresses to be equal to zero near the extreme right end of the slip surface, while the tensile stresses may actually be acting. This
shear force obtained from Fig. 11. Note that the change in the vertical shear force appears very small especially when assuming a circular slip surface in Example 4. Such a smooth and continuous distribution of the force seems to make the Bishop's assumption mechanically meaningful.

Figs. 12 through 15 prove that each of the assumptions employed in these four slice methods is approximately acceptable in all of the case studies. Morgenstern and Price (1965) and Madej (1971) compared their slice methods with other slice methods and found that the representative slice methods provided about the same safety factor value (see Table 1). Such a result is due to the fact that the assumption used in each slice method is adequate, and due to the mechanical property that in some cases, the safety factor is rather insensitive to varying the relationship between the internal forces as pointed out by Morgenstern and Price (1965).

DETERMINATION OF CRITICAL SLIP SURFACE

Subjected to the homogeneous earth slope shown in Figs. 1 and 7, consider the problem of locating the critical noncircular slip surface which gives the minimum safety factor value defined by Eqs. (5) and (6). Previously, Arai and Tagyo (1985a) developed a numerical procedure to carry out automatically the search for the critical noncircular slip surface which minimizes the safety factor defined by the simplified Janbu Method. The procedure proposed here is essentially founded due to the formulation used in Arai and Tagyo (1985a). That is, all the interslice abcissas shown in Figs. 1 and 2 are stipulated and regarded as known quantities. In this case, a potential slip surface is assumed to intersect the slope surface only on the interslice boundary. Such a formulation requires some trials on which the boundary condition concerning the point of intersection between a potential slip surface and the slope surface is varied methodically. The critical slip surface on a specified boundary condition.
is referred to as a 'proposed' critical slip surface. The global critical slip surface is selected out of the proposed critical slip surfaces. Thus, the search for the critical slip surface having an arbitrary shape, is reduced to a problem of determining the interslice elevations of slip surface $y_t$ on a specified boundary condition concerning the intersection between the slip and slope surfaces. This problem is formulated by slightly modifying the previous problem formulation in this paper such that a series of interslice elevations of slip surface $y_t$ is considered as the unknown variable. In this formulation, the objective function defined by Eq. (10) plays the role of searching the position of the critical slip surface; while in the previous problem formulation, Eq. (10) isolates merely a particular stress field. When handling a series of $y_t$ as independent variables, the proposed procedure does not necessarily yield a kinematically admissible shape of the slip surface. This may be attributed to the high nonlinearity in Eqs. (3) and (4) resulting from the above formulation. The following constraint which avoids such a difficulty must be added to the previous problem formulation.

$$P'_t = (y_{t-1} + y_{t+1})/2 \geq y_t \quad (18)$$

This constraint requires the position of the slip surface to be a downward convex function of the lateral coordinate. Except for the constraint, no restriction is placed at the outset on the shape of the possible slip surface. The computational procedure is the same as in the case where the location of the slip surface is known beforehand.

**Case Studies**

**Example 6**: The sixth example considers a problem of locating the critical slip surface in a gentle earth slope shown in Figs. 1 and 2 (see Example 1). Fig. 16 plots two kinds of proposed critical slip surfaces on the specified boundary conditions. One is obtained by the proposed procedure and the other is given by the Arai and Tagyo (1985a) procedure which employs the simplified Janbu method as a method for stability analysis.

![Fig. 16. Proposed critical slip surfaces in Example 6](image)

![Fig. 17. Proposed critical slip surfaces in Example 7](image)

**Example 7**: The seventh example searches for the position of the critical slip surface in the steep slope shown in Fig. 7 (see Example 3). Fig. 17 compares the results by the proposed procedure with those by the Arai and Tagyo procedure.

In Figs. 16 and 17, except for only one case, safety factor $F_s$ calculated by the simplified Janbu method is always lower than the safety factor $F_s$ which is the lower-bound solution obtained by the proposed procedure (see Eq. 16). The solitary exception in Figs. 16 and 17 may be attributed to the numerical error caused by the proposed procedure. The result in Figs. 16 and 17 confirms the previous conclusion that when the critical slip surface is found, the simplified Janbu method in most cases gives the safety factor lower than the lower-bound solution by the proposed procedure. This makes the use of the slice methods meaningful regarding engineering applications, provided that the position of critical slip surface is determined adequately. Referring to Figs. 16 and 17, the shape of two kinds of critical slip surfaces...
surfaces are appreciably different from each other. Both of these have a preferred pattern of their own. Arai and Tagyo procedure based on the simplified Janbu method gives the location of the critical slip surface deeper than the procedure proposed in this paper. The critical slip surface given by the proposed procedure is close to the failure pattern assumed in the wedge method in which the failure surface is approximated by two or three straight lines. However, we cannot go deep into this subject, due to the limitations of limit equilibrium method. More investigations concerning the shape of the slip surface may require kinematical consideration regarding soil behavior. In addition, it is interesting to note that in Figs. 16 and 17, the two procedures locate the global critical slip surface on about the same boundary condition concerning the intersection between the slip and slope surfaces. This result suggests the possibility of applying the proposed procedure subsequently after determining the boundary condition by the use of the Arai and Tagyo procedure.

CONCLUSIONS

This paper has developed a new numerical procedure for the slope stability analysis, by combining the slice method in the limit equilibrium method with the lower-bound approach in the limit analysis method. The procedure provides an appropriate lower-bound solution subject to the collapse mechanism represented by a potential slip surface. At first, for the case where the position of slip surface is known beforehand, consider the problem of calculating the safety factor which is the ratio of the available shearing resistance to the mobilized shear stresses both along the slip surface. The stress field is discretized in a similar manner as in the finite element displacement approach. The problem of calculating the safety factor is formulated as an optimization problem which isolates a particular stress field. That is, the procedure pursues a stress field which minimizes the safety factor within the limitations of satisfying the lower-bound conditions. The formulated optimization problem is solved numerically by the mathematical programming technique. The obtained stress distribution is statically admissible and appears physically reasonable. Several case studies have proved that the proposed procedure gives the safety factor higher than the simplified Janbu method, when giving the critical slip surface which minimizes the safety factor defined by the simplified method. However, when assuming the position of slip surface at random, there are cases where some slice methods estimate the safety factor higher than the proposed procedure. These results may suggest the possibility that some slice methods tend to underestimate the safety factor on the safety side, provided that the critical slip surface is located adequately. This view accentuates the importance of finding the real critical slip surface when applying the slice methods.

Secondly, the slice methods inevitably contain the statically indeterminate problem with respect to the internal forces between the slices, and must employ a certain assumption necessary to render the stability problem to a statically determinate. Such assumptions in the representative slice methods have been verified by the use of the stress field determined by the proposed procedure. The investigation has confirmed the adequacy of the assumption used in each of the slice methods.

Finally, this paper developed a procedure for locating the critical noncircular slip surface which minimizes the safety factor within the limitations of satisfying the lower-bound conditions. The location of the critical slip surface found by the procedure is appreciably different from the one minimizing the safety factor defined by the simplified Janbu method. In most cases, the safety factor associated with the former is slightly higher than the safety factor associated with the latter. This means repeatedly that the simplified Janbu method based on the critical slip surface tends to underestimate the safety factor on the upper-bound side, compared with the lower-bound solution by the proposed pro-
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procedure.

The computation time required for the total iteration steps in Example 1 is about 7 minutes when using a combination of the personal computer TALOS-68K (CPU: MC 68000, 8 MHz) and softwares CP/M-68K and SVS FORTRAN Compiler. It is easy to extend the proposed procedure to an actual earth slope consisting of layered soil, so long as an appropriate subdivision system of soil mass into finite elements can be devised. Through the development of the proposed procedure, many important properties in the slope stability analysis have been clarified which have been neglected. The application of the procedure to other types of stability problems has promise in bringing about more fruitful results.

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REFERENCES


