DEEP-SEATED FAILURE OF A GRANULAR EMBANKMENT OVER CLAY: STABILITY ANALYSIS

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ABSTRACT

Presented is a method for analyzing the end-of-construction stability of a granular embankment over a clayey foundation with a variable undrained strength. The mechanism utilized allows the embankment to "break" steeply squeezing out the clay underneath. By virtue of the mechanism used, the factor of safety can be determined through a simple minimization process and without resorting to statical assumptions. The formulation is presented in a framework of limit-equilibrium; however, its results can also be shown in the context of limit-analysis.

To demonstrate the method's performance, some results are presented. The factor of safety predictions compare well with those obtained from Spencer's method. Also, stability analysis predictions compare reasonably well with a field case history.

Key words : failure, foundation, plasticity, safety factor, slip surface, slope stability, stability analysis (IGC : E 6/H 4)

INTRODUCTION

It is often necessary to construct a granular embankment over a saturated clayey foundation. In practice, it is assumed that at the end of construction the foundation has not yet begun to consolidate and therefore has not gained strength. The forces that tend to cause failure, however, are at their maximum where they will remain. It is at this stage that the embankment has its minimum stability against deep-seated failure. Subsequently, in design it is common

to assess this stability aspect based on a "total analysis" using the undrained shear strength of the foundation (i.e., c_u and $\phi_u = 0$).

The objective of this paper is to present an analysis method capable of assessing the short-term stability of a granular embankment over a clayey foundation. The analysis is rigorous in the sense that no statical assumptions are necessary to determine the minimum factor of safety. This is a consequence of the utilized failure mechanism which stems from the variational limit-equi-

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librium analysis (Baker and Garber, 1978). For simplicity, however, the analysis is presented in a framework of conventional limit-equilibrium analysis following a format similar to Fellenius' (1936). It should be pointed out that determination of the factor of safety is as simple as in Bishop's procedure (1955).

Comparison of some results with Bishop's and Spencer's (1967) methods shows good agreement in terms of factors of safety. However, when relatively shallow slip surfaces are considered, Bishop's predictions may become unconservative. Consequently, the presented method can be applied to a wider range of problems as compared to Bishop's while maintaining a simple analysis as compared to Spencer's.

FORMULATION OVERVIEW

Fig.1 illustrates a granular embankment defined by its face inclination i , height H , an average unit weight γ and an internal friction angle ϕ . The foundation soil underlying the embankment extends to a depth d and possesses a variable undrained shear strength $c_u(y)$. To simplify the presentation, only the case where $c_u(y)$ changes linearly with depth is explicitly formulated. Subsequently, the strength of the foundation illustrated in Fig.1 is

$$c_u(y) = c_{u0} - s \cdot y \quad (1)$$

where c_{u0} is the undrained shear strength at the top of the foundation (i.e., at $y=0$)

Fig. 1. Notations, conventions and failure mechanism

and s is the rate of shear strength increase with depth. Note in Fig.1 that positive y is upwards thus depth is signified by negative y .

The stability of the embankment against deep-seated failure is analyzed assuming a state of limit-equilibrium. Therefore, the Mohr-Coulomb's strength parameters ϕ and $c_u(y)$ are scaled down so that an artificial state of limit-equilibrium is attained

$$\phi_m = \tan^{-1}(\psi_m) = \tan^{-1}\left[\frac{\tan \phi}{F_s}\right] \quad (2)$$

$$c_m = \frac{c_{u0}}{F_s} \text{ and } s_m = \frac{s}{F_s} \quad (3)$$

where $\psi = \tan \phi$; F_s is the safety margin (i.e., factor of safety); and the subscript m indicates mobilization.

One now seeks the minimal factor of safety for the given structure and for a slip surface which passes through the embankment and its foundation. To formulate the problem, however, the failure mechanism must be postulated. Fig.1 illustrates the selected failure surface: a log-spiral between points (2) and (3), and a circular arc (i.e., a degenerated log-spiral) between points (3) and (1). Note the following: (1) The circular arc emerges at the toe elevation (i.e., $y_1=0$ and $x_1 \leq 0$), (2) Both segments of the slip surface have a common pole at (x_c, y_c) and (3) The extent of penetration is limited by d . This mechanism belongs to the log-spiral family of curves and has implicitly been suggested by Baker and Garber (1978) based on a variational limit-equilibrium analysis extended to layered soil. Theoretically, the selected mechanism is kinematically admissible in the strict framework of limit-analysis. Physically, the mechanism enables the "stiff" embankment to break steeply squeezing out the "soft" foundation material underneath. The moment equilibrium equation written about the failure surface pole (x_c, y_c) can be assembled rigorously (i.e., without resorting to statical assumptions) and explicitly as

$$M = \int_{x_1}^{x_3} [c_m - s_m y] [(y - y_c) - (x - x_c) y'] dx + \int_0^{x_3} \gamma y_c (x - x_c) dx +$$

$$+ \int_0^{x_2} \gamma(\bar{y} - y_c)(x - x_c) dx - \int_{x_3}^{x_2} \gamma(y - y_c)(x - x_c) dx = 0 \quad (4)$$

where $y' = dy/dx$; and $\bar{y}$ is the embankment surface function, i.e.,

$$\bar{y} = 0 \text{ for } x \leq 0 \quad (5)$$

$$\bar{y} = x \tan(i) \text{ for } [H \cot(i)] \geq x \geq 0 \quad (6)$$

$$\bar{y} = H \text{ for } x \geq [H \cot(i)] \quad (7)$$

Note in Eq. (4) that the embankment's shear strength contribution is not explicitly stated. This is a consequence of the log-spiral slip surface; i.e., the resultant of each elemental normal force on this surface and its associated friction must pass through the pole (x_c, y_c) . It should be emphasized, however, that the value of ϕ has a significant effect on the results of the analysis (see Table 1 and Leshchinsky, 1987) since the log-spiral trace is defined by its value.

The parametric equations relating points on the slip surface in the cartesian and polar coordinate systems, which are defined in Fig. 1, are

$$\begin{aligned} x &= x_c + A_2 \exp(-\phi_m \beta) \sin \beta \\ y &= y_c - A_2 \exp(-\phi_m \beta) \cos \beta \end{aligned} \quad \text{for } \beta_2 \geq \beta \geq \beta_1 \quad (8)$$

and

$$\begin{aligned} x &= x_c + A_1 \sin \beta \\ y &= y_c - A_1 \cos \beta \end{aligned} \quad \text{for } \beta_1 \geq \beta \geq -\beta_1 \quad (9)$$

where A_1 and A_2 are constants and β is the independent variable in the polar coordinate system.

Substituting Eqs. (5) through (9) into Eq. (4), transforming dx and y' to the polar coordinate system, carrying out the integration and rearranging the terms gives

$$F_s = -\frac{m_2}{m_1} \quad (10)$$

where

$$\begin{aligned} m_1 &= \gamma A_2^3 \left\{ \frac{\cos^3 \beta}{3} + \frac{\phi_m}{1 + 9 \phi_m^2} (\sin \beta \right. \\ &\quad \left. - 3 \phi_m \cos \beta) \right\} \exp(-3 \phi_m \beta) \Big|_{\beta_1}^{\beta_2} \\ &\quad + \frac{\gamma A_2^2 y_c}{2} \exp(-2 \phi_m \beta) \sin^2 \beta \Big|_{\beta_1}^{\beta_2} \end{aligned}$$

$$+ \gamma H \left\{ \frac{x_c^2}{2} - \frac{A_2^2}{2} \exp(-2 \phi_m \beta_2) \sin^2 \beta_2 \right\} - \frac{\gamma H^2 \cot(i)}{6} [3 x_c - H \cot(i)] \quad (11)$$

$$m_2 = 2 A_1^2 [\beta_1 c_{u0} + s(A_1 \sin \beta_1 - y_c \beta_1)] \quad (12)$$

Examining Eqs. (10) through (12) it can be seen that for a defined problem (i.e., given i , γ , H , d , ϕ , c_{u0} and s) there exist seven unknowns which are needed to define the critical solution: $\min(F_s)$ and the corresponding slip surface parameters x_c , y_c , A_1 , A_2 , β_1 and β_2 . Additional equations can be obtained from geometrical relationships stemming from the selected failure mechanism (for details see Leshchinsky, 1987):

$$\begin{aligned} \eta d [\exp(-\phi_m \beta_1) \cos \beta_1 - \exp(-\phi_m \beta_2) \cos \beta_2] \\ - H \exp(-\phi_m \beta_1) \tan(\beta_1/2) \sin \beta_1 = 0 \end{aligned} \quad (13)$$

$$A_1 = \frac{\eta d}{\tan(\beta_1/2) \sin \beta_1} \quad (14)$$

$$y_c = A_1 \cos \beta_1 \quad (15)$$

$$A_2 = A_1 \exp(\phi_m \beta_1) \quad (16)$$

where the penetration depth of the slip surface is ηd (see Fig. 1) and, therefore, $1 \geq \eta > 0$.

Based on equilibrium considerations and through a straightforward extension of the midpoint argument as presented by Leshchinsky (1987), it can be verified that the center of the critical slip surface lies on a vertical line passing through the middle of the slope face. To comply also with the assumption that the slip surface emerges at the toe elevation, x_1 must be equal, at most, to zero. Consequently, while utilizing Eq. (9), the following can be stated (Leshchinsky, 1987)

$$x_c = \begin{cases} \frac{H \cot(i)}{2} & \text{if } l \leq 0 \\ \frac{\eta d}{\tan(\beta_1/2)} & \text{if } l > 0 \end{cases} \quad (17)$$

where

$$l = \frac{H \cot(i)}{2} - \frac{\eta d}{\tan(\beta_1/2)} \quad (18)$$

There are now eight unknowns needed to define the critical solution: η , $\min(F_s)$, x_c ,

y_c , A_1 , A_2 , β_1 and β_2 , and six available equations: Eqs. (10), and (13) through (17). Hence, F_s can be solved iteratively for any two assumed unknowns. The following steps may be used in a numerical procedure for minimizing F_s :

1. Select values for β_2 and η ($1 \geq \eta > 0$).
2. Assume F_s .
3. Compute $\phi_m = \tan \phi / F_s$.
4. Calculate β_1 , A_1 , y_c , A_2 and x_c utilizing Eqs. (13), (14), (15), (16) and (17), respectively.
5. Calculate F_s using Eqs. (10) through (12).
6. If F_s computed in Step 5 is sufficiently close to its assumed value in Step 2, the factor of safety corresponding to β_2 and η has been determined. Otherwise, use F_s obtained in Step 5 as a new approximation and go to Step 3.
7. Select new values for β_2 and η , and go to Step 2 until $\min(F_s)$ is obtained for all combinations of β_2 and η .

It may be helpful to note that the angle between the tangent to the log-spiral at point (2) and a horizontal line parallel to the x -axis is $(\beta_2 + \phi_m)$. It has been numerically determined that by selecting a tangent's angle of 90° as an initial guess for β_2 [i.e., $\beta_2 = (90^\circ - \phi_m)$] in Step 1 above, the minimization process is greatly facilitated. Note that unlike Leshchinsky (1987), no limit is set on the tangent's angle.

It should be pointed out that one can obtain the explicit form of Eq. (10) through energy formulation following the formal procedures of limit-analysis where rigid body rotation is considered (see Appendix A). In the limit-analysis framework the minimized F_s is considered to be an upper-bound value. Furthermore, through substitution of the failure surface function into Baker and Garber's (1978) safety functional one can verify that it degenerates to Eq. (10). Hence, one realizes that the presented analysis can be introduced also in a framework of variational limit-equilibrium or limit-analysis. The authors prefer the conventional limit-equilibrium presentation since it is widely used in

practice and since its conceptual limitations are known.

APPLICATIONS

The analysis procedure is applied to several hypothetical problems and one case history to demonstrate its performance. To do this effectively, the following nondimensional notation is introduced

$$N_m = \frac{1}{\gamma H} \frac{c_{u_0}}{F_s} \quad (19)$$

and

$$\alpha = \frac{1}{\gamma} \frac{s}{F_s} \quad (20)$$

It should be pointed out that all the results are for embankments with crest widths which contain the critical slip surface. In case the critical slip surface exceeds the embankment's base width, there might be a bearing-capacity problem—see Leshchinsky (1987) for analysis results.

Figs. 2 and 3 are sample stability charts for slopes inclined at 1(V) : 2(H) and 1(V) : 4(H), respectively. Notice that both figures are for a design friction angle $\phi_m = 30^\circ$. To demonstrate the usage of the stability charts, consider the following problem: Given an embankment $H = 3$ m high, inclined at 1 (V) : 2(H), having a unit weight of $\gamma = 18$ kN/m³ and an internal friction angle of 37° .

Fig. 2. Stability chart for slope inclined at 1(V) : 2(H)

Fig. 3. Stability chart for slope inclined at 1(V) : 4(H)

The clayey foundation thickness is $d=3$ m, the undrained shear strength at the top is $c_{u0}=8.15$ kPa and its rate of increase with depth is $s=2.34$ kPa/m. To determine the factor of safety assume $F_s=1.3$; hence, $\phi_m = \tan^{-1}[\tan \phi/F_s]=30^\circ$. Use Fig. 2 : for $d/H=1$ and $\alpha=2.34/(18 \cdot 1.3)=0.10$, it follows that the required $N_m=0.116$. The existing value, however, is $N_m=8.15/(18 \cdot 3 \cdot 1.3)=0.116$. Thus, the assumed $F_s=1.3$ constitutes a solution to the problem; otherwise, a new value for F_s should be assumed and the process repeated.

Comparing Figs. 2 and 3, it is apparent that as the embankment becomes steeper, the required N_m increases, especially for small d/H ratios. Also deserving attention is the fact that as d/H increases, the N_m curves reach a constant value. This is notable and significant as α increases. Constant required N_m signifies a failure surface depth which does not exceed the full thickness of the foundation d/H (i.e., $\eta < 1$) and therefore, d has no effect on the stability in this case. Only when $\alpha=0$ (i.e., homogeneous foundation) the slip surface tends to exceed d . Figs. 4 (a) and 4(b) demonstrate this phenomenon. Since in reality α is likely to be greater than zero, one realizes that the penetration depth of the critical slip surface may naturally be limited without reaching the "hard layer".

Fig. 4. Traces of typical slip surfaces : (a) $d/H=0.1$, and (b) $d/H=1.0$

The presented dimensionless stability charts give some indication of the performance of the analysis. However, in practice many more such charts may be needed since there are many variables in the problem. Therefore, it appears more practical to use a computer program for each specific problem.

COMPARISON OF RESULTS

The problem of a 1(V) : 2(H) slope overlying a foundation of constant undrained shear strength (i.e., $\alpha=0$) was analyzed using the presented method, Bishop's method and Spencer's method. It should be pointed out that in Spencer's method the trace of the critical slip surface, which is of general shape, and its associated factor of safety were determined using Baker's (1987) micro-computer code. His code is based on the dynamic programming technique (Baker, 1980).

Figs. 5 and 6 illustrate the critical slip surface for each method as well as the corresponding factors of safety. Comparing the traces, it is found that when ϕ equals 20° , the presented work and Bishop's method yield similar results. When ϕ equals 40° , the slip surface obtained by the presented work lies over much of its length between

Fig. 5. Comparison of traces of slip surfaces : (a) $\phi=20^\circ$ & $N=0.1453$, and (b) $\phi=40^\circ$ & $N=0.0806$

Fig. 6. Comparison of traces of slip surfaces : (a) $\phi=20^\circ$ & $N=0.1753$, and (b) $\phi=40^\circ$ & $N=0.1506$

Fig. 7. Saint-Alban test embankment : Predicted slip surface

the other two surfaces. Note in Fig. 6(b) (and also in Fig. 7) that the portion of the critical slip surface within the embankment tends to be inclined slightly backwards (i. e., its tangent at the crest is more than 90°). This tendency exists only when failure depths exceeding $d/H=1$ together with high friction angles are considered. One can set an arbitrary upper limit of 90° on the tangent thus avoiding such slip surfaces (Leshchinsky, 1987). However, the authors feel that in the framework of this paper it is more appropriate to present the genuine critical surface without setting physical restrictions on the solution. In any event, an investigation of a few such cases revealed that the factors of safety are similar whether the 90° limit is imposed or not (e. g., for the case in Fig. 6(b) imposing the restriction of the tangent will increase F_s from 1.00 to 1.02 ; in Fig. 7 from 0.96 to 1.00).

Table 1 summarizes the factors of safety for each method. It shows that for a given d/H in the presented method, ϕ has a pronounced effect ; i. e., the required c_{u0} decreases significantly as ϕ increases, especially

Table 1. Summary : Comparison of safety factors

$\frac{d}{H}$	ϕ	$\frac{c_{u0}}{\gamma H}$ $\alpha=0$	Minimum F_s		
			Presented Work	Bishop's Method	Spencer's Method
0.1	20°	0.1453	1.00	1.05	1.03
0.1	40°	0.0806	1.00	1.37	0.99
1.0	20°	0.1753	1.00	0.97	0.96
1.0	40°	0.1506	1.00	0.90	0.93

for small values of d/H . The F_s values are within 10% of one another for each analyzed case, except for $d/H=0.1$ and $\phi=40^\circ$ where Bishop's becomes significantly unconservative. It appears that Bishop method behavior in this case has to do mainly with its failure shape; i.e., a circular arc cannot conform to a horizontal weak thin layer. Conversely, the presented mechanism is of a composite nature, enabling better adaptation to potentially shallow failures under the embankment. In the limit only a translational failure will be feasible thus making both methods useless. However, it seems that the presented method is applicable to a wider range of practical problems as compared to Bishop's while maintaining a simple analysis as compared to Spencer's.

CASE STUDY

La Rochelle, et al. (1974) constructed a granular test embankment over a sensitive clay in Saint-Alban, Quebec, Canada to a height at which deep-seated failure occurred (i.e., $F_s=1.0$). The embankment possessed the following properties: an inclination of $1(V):1.5(H)$ in the failure direction, an average unit weight of $\gamma=120 \text{ lb/ft}^3$, and an internal angle of friction of $\phi=44^\circ$. Its height at failure was $H=12.8 \text{ ft}$.

The undrained shear strength profile of the foundation clay was determined by La Rochelle et al. (1974) using vane shear tests and laboratory UU triaxial tests on undisturbed specimens. Five strength profiles were presented based on various assumptions for the potential strength mobilization in the clay crust. Using the results of the vane tests, three strength profiles were determined using full vane, mid-depth vane (taken at 3 ft), and minimum vane strength hypotheses. These profiles are presented in Table 2, discretizing the continuous curves given by La Rochelle et al. (1974). The UU tests resulted in two strength profiles based upon residual values. Case 1 extrapolated linearly the strength increase from depth of 2 ft to the ground surface. Case 2 assumed a con-

Table 2. Clay strength profile based on vane tests (La Rochelle et al., 1974)

Full		Mid-Depth		Minimum	
$-y \text{ (ft)}$	$c_u \text{ (lb/ft}^2\text{)}$	$-y \text{ (ft)}$	$c_u \text{ (lb/ft}^2\text{)}$	$-y \text{ (ft)}$	$c_u \text{ (lb/ft}^2\text{)}$
0.00	800	0.00	650	0.00	210
2.50	800	3.00	650	6.00	210
4.00	320	4.00	320	7.50	215
4.75	225	4.75	225	15.0	325
6.00	210	6.00	210	18.0	380
7.50	215	7.50	215		
15.0	325	15.0	325		
18.0	380	18.0	380		

Table 3. Clay strength profile based on UU triaxial tests (La Rochelle et al., 1974)

Case 1		Case 2	
$-y \text{ (ft)}$	$c_u \text{ (lb/ft}^2\text{)}$	$-y \text{ (ft)}$	$c_u \text{ (lb/ft}^2\text{)}$
0.00	600	0.00	400
4.00	200	2.00	400
7.20	170	4.00	200
12.0	260	7.20	170
15.0	275	12.0	260
18.0	380	15.0	275
		18.0	380

stant strength equal to the measured value at a depth of 2 ft. Table 3 represents the UU profiles, discretizing the curves given by La Rochelle et al. (1974). As can be seen from Tables 2 and 3, the differences in all five strength profiles are essentially in the crust.

The foundation presented in this paper so far was limited to a single layer with a constant rate of strength change. Using the same failure mechanism as shown in Fig. 1, the formulation was modified to allow for a discretized representation of the foundation. In this representation the full layer was divided into a number of sublayers. The undrained strength at the top and the constant rate of strength change (increase or decrease) within, are input for each sublayer. This modification was implemented in a computer program (Smith, 1988). Utilizing the strength profiles represented in Tables 2 and 3, the embankment at failure was analyzed using the modified formulation. In Fig. 7 the resulting critical surfaces are plotted along with the circular one which was suggested by La Rochelle et al. (1974). Also

Table 4. Comparison of safety factors for Saint-Alban test embankment

Profile Used	Method of Analysis	
	Presented Work	Bishop's (La Rochelle et al., 1974)
UU Case 1	0.96	0.94
UU Case 2	0.93	0.89
Full Vane	1.21	1.20
Mid-Depth Vane	1.16	1.14
Minimum Vane	0.93	0.85

illustrated are the initial and final ground surface profiles. Notice that the slip surfaces for the two UU profiles were nearly identical (the differences are practically unplotable) although the predicted safety factors were not. A similar trend appeared in the three vane strength profiles. Considering the actual factor of safety (i.e., $F_s=1.0$), the measured post-failure ground surface profile, and the penetration depth as compared to the suggested one, it seems that the UU/Case 1 profile produced reasonable results. Furthermore, it can be checked by superimposing La Rochelle et al. (1974) field test results that the predicted slip surface for Case 1 agrees fairly well with the measured data which indicate the location of the actual slip surface. It should be pointed out that firm data is available only to the left side of the toe. Table 4 compares the predicted safety factors based on the presented method with Bishop's as reported by La Rochelle et al. (1974). As is evident from this table, the factors of safety determined for each case are very close for both methods of analysis. This case history provides some insight about the performance of the presented method after it was modified to deal with a realistic foundation strength profile. In this particular case, however, there is no demonstration of a *practical* advantage over Bishop's.

CONCLUSION

A limit-equilibrium method for analyzing the end-of-construction stability of a granular embankment over a clayey foundation is presented. The analysis utilizes a mechanism which stems from the variational

limit-equilibrium allowing the embankment to "break" steeply and enables the clay to be squeezed out. For this mechanism the problem is formulated rigorously. It is shown that the analysis results can also be considered an upper-bound in limit-analysis. Although the explicit formulation is limited to clay with linearly varying strength profile, it can be modified to deal with any given profile. The analysis is limited to deep-seated rotational failure; i.e., translational failure and slide only within the embankment are excluded from consideration.

The following observations regarding the analysis and results were made:

1. The results can be presented in a convenient format of nondimensional design charts. However, because large numbers of charts may be needed, it seems more practical to use a computer program.

2. The penetration depth of the slip surface always equals the clay's prescribed depth when it is homogeneous. When its strength increases with depth, however, shallower slip surfaces result and the clay thickness then may be in excess of the penetration depth of the critical failure surface.

3. The rate of shear strength increase has a significant effect on stability.

4. The embankment's internal angle of friction has a significant effect on stability when shallow failures are considered.

5. For a limited number of cases checked, the F_s predictions of the presented analysis generally compared very well with the results of other methods which are commonly used in practice.

6. When the potential slip surface becomes shallow (i.e., either because of a thin foundation or a rapid rate of increase in its strength), Bishop method tends to be unconservative. For the case checked, however, the presented method deals effectively with this problem apparently because its composite slip surface allows better adaptation to the weak surface.

7. Modification of the method to deal with a foundation of variable strength profile re-

sulted with a predicted factor of safety for the Saint-Alban case which was very close to the actual one. The predicted slip surface appeared to be reasonable.

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APPENDIX A : THE LIMIT-ANALYSIS EQUIVALENCY

Leshchinsky et al. (1985) showed in a general fashion the equivalency existing between the variational procedure and the upper bound of limit-analysis. However, for the sake of brevity the analysis in this paper is presented only in the framework of conventional limit-equilibrium and does not contain the actual variational derivations. Therefore, it may be difficult to accept the equivalency between the presented limit-equilibrium formulation and the limit-analysis. Consequently, the equivalency for the present problem is demonstrated.

Consider the failure mechanism used in this work and represented in Fig. A 1. It consists of a single rigid body defined by the soil surface, a log-spiral segment in the frictional soil (drained conditions) and a log-spiral segment which degenerates to a circular arc in the cohesive soil (undrained conditions).

Fig. A 1. Definitions in limit-analysis

This composite sliding surface defines a kinematically admissible mechanism and therefore suitable for energy formulation in the context of upper bound. In this formulation, one equates the increment of external work, δE , with the increment of energy dissipation δD . It is convenient to define the rigid sliding mass in terms of zone I, in which the drained embankment is contained, and zone II, in which the undrained foundation is contained (Fig. A 1). For a virtual rotation $\delta\theta$, δE in zone I is:

$$(\delta E)_I = \bar{W}_I \cdot (\bar{r} \delta\theta) \quad (A 1)$$

where $\bar{W}_I$ is the weight vector of zone I concentrated at its center of gravity (C.G.), $\bar{r}$ is the radius vector defining C.G. (Fig. A 1) and $\cdot$ signifies a scalar product.

Using the geometry of the problem one can show that the displacement component v of $(\bar{r} \delta\theta)$ in $\bar{W}_I$ direction is

$$v = |\bar{r} \delta\theta| \sin \beta_0 = r \sin \beta_0 \delta\theta \quad (A 2)$$

where β_0 is defined in Fig. A 1. From geometry the term $(r \sin \beta_0)$ equals $(x_0 - x_c)$. Combining this, Eq. (A 2) and the explicit form of the scalar product stated in Eq. (A 1) result with:

$$(\delta E)_I = W_I (x_0 - x_c) \delta\theta \quad (A 3)$$

where x_0 is the ordinate of C.G. (Fig. A 1). The term $W_I (x_0 - x_c)$ is nothing but the moment about the log-spiral pole resulting from the weight of zone I. This moment, however, equal to m_1 (Eq. 11) and therefore

$$(\delta E)_I = m_1 \delta\theta \quad (A 4)$$

It can be verified (e.g. Atkinson, 1981) that the increment of energy dissipated due

to internal stresses during drained plastic collapse is

$$(\delta D)_I = 0 \quad (A 5)$$

Eq. (A 5) implies that a perfectly plastic *frictional* material is neither dissipative nor conservative. This is a consequence of the associative flow rule and its general validity for frictional materials is questionable; however, it is extremely convenient for upper bound computations.

Since β_0 in zone II is zero, the increment of external work here is

$$(\delta E)_{II} = 0 \quad (A 6)$$

The increment of energy dissipation in the undrained zone II is

$$(\delta D)_{II} = - \int_l r \delta\theta (c_m - s_m y) dl \quad (A 7)$$

where dl is a differential length of the circular arc.

Examining Eq. (A 7) it is apparent that integration of the term $[r(c_m - s_m y) dl]$ yields the moment about the pole due to cohesion. Carrying out this integration one gets

$$(\delta D)_{II} = - \frac{m_2}{F_s} \delta\theta \quad (A 8)$$

where m_2 is defined in Eq. (12).

Requiring that $\Sigma(\delta E) = \Sigma(\delta D)$ results in

$$F_s = - \frac{m_2}{m_1} \quad (A 9)$$

Eq. (A 9) is identical to Eq. (10) thus showing the equivalency between the presented limit-equilibrium formulation and limit-analysis.