A CONSTITUTIVE MODEL FOR SOILS EVALUATING
PRINCIPAL STRESS ROTATION AND ITS
APPLICATION TO SOME DEFORMATION
PROBLEMS

Hajime Matsuoka, Yasuyuki Suzuki and Takanori Murata

ABSTRACT

A constitutive model for soils by which the strain increments (dεx, dεy and dτxy) are
directly related to the stress increments (dσx, dσy and dτxy) is proposed. The stress-strain
matrix between the strain increments and the stress increments is expressed in xy-coordinates
fixed arbitrarily. In order to evaluate the influence of rotation of the principal stress axes
on strains, "principal stress rotation tests" with several rotation times, in which the stress
path circles several times along the circumference of a Mohr’s stress circle, are carried out
by a "two-dimensional arbitrary stress apparatus" using a stack of aluminium rods. The
"arbitrary" stress path dependency of strains is also examined by the same apparatus, and
analyzed by the proposed model. The finite element analysis of soil foundation under a uniform
strip load is performed using the proposed model, and the influence of rotation of the principal
stress axes on settlements and lateral displacements is checked by comparing the computed
results under different contribution factors (5% to 100%) of strains due to rotation of the
principal stress axes. It is seen from the computed results that the settlements and lateral
displacements are influenced significantly by the principal stress rotation.

Key words: angle of internal friction, constitutive equation of soil, dilatancy, finite element
method, granular material, sand, special shear test, stress path, stress-strain curve (IGC:
D 6/E 2)

INTRODUCTION

It is natural to consider that the change
in stresses on an arbitrary plane causes strains
in every kind of material. In the simple
theory of plasticity, however, such an idea
is not satisfied because the constitutive rela-
tion is usually formulated by the stress

1) Professor, Department of Civil Engineering, Nagoya Institute of Technology, Gokiso-cho, Showa-
ku, Nagoya 466.

2) Engineer, Nagoya Port Authority, 1-8-21, Irihame, Minato-ku, Nagoya 455 (formerly Graduate
Student at Nagoya Institute of Technology).

3) Engineer, Aichi Prefecture Government, 3-2-1, San-no-maru, Naka-ku, Nagoya 460 (formerly
Graduate Student at Nagoya Institute of Technology).

Manuscript was received for review on March 6, 1989.
Written discussions on this paper should be submitted before October 1, 1990, to the Japanese
Society of Soil Mechanics and Foundation Engineering, Sugayama Bldg. 4F, Kanda Awaji-cho 2-23,
Chiyoda-ku, Tokyo 101, Japan. Upon request the closing date may be extended one month.
invariants such as the principal stresses. For example, the strains caused by the "principal stress rotation test" along a Mohr's stress circle under constant principal stresses cannot be explained by the usual theory of plasticity. Here, using the arbitrary stress apparatus which can apply the arbitrary stresses ($\sigma_x$, $\sigma_y$ and $\tau_{xy}$) directly and independently for a stack of aluminum rod, "principal stress rotation tests" and "arbitrary stress path tests" are performed, and their test results are analyzed by the proposed model which predicts the strains not only due to "consolidation" and "shear", but also due to "principal stress rotation". In order to check the influence of rotation of the principal stress axes on settlements and lateral displacements, the finite element analysis of soil foundation under a uniform strip load is also performed using the proposed model.

A CONSTITUTIVE MODEL DIRECTLY EXPRESSED IN $xy$-COORDINATES

The following equation has been derived from assuming the hyperbolic relationship between the shear-normal stress ratio (\(\tau/\sigma_N\) or $\tau_{xy}/\sigma_N$) and the shear strain ($\gamma_{xy}$) expressed in $xy$-coordinates fixed arbitrarily (Matsuoka et al., 1986).

\[
\gamma_{xy} = \frac{1}{G_0} \cdot \frac{\sin \phi \cdot \sin \phi_{mo} \cdot \sin 2\alpha}{\sin \phi - \sin \phi_{mo}}
\]

(1)

where $G_0$ is the gradient of the initial tangent of the hyperbolic relationship, $\phi$ is the internal friction angle, $\phi_{mo}$ is the mobilized internal friction angle (\(\sin \phi_{mo} = (\alpha_1 - \alpha)/ (\alpha_1 + \alpha_2)\)) and $\alpha$ is the angle of the principal stress direction. As $1/G_0$ (\(=k_0\)) is approximately proportional to the logarithm of the mean principal stress $\sigma_m$, $\gamma_{xy}$ in Eq. (1) is considered to be the function of $\phi_{mo}$, $\alpha$ and $\sigma_m$. By totally differentiating $\gamma_{xy}$ with respect to $\phi_{mo}$, $\alpha$ and $\sigma_m$, \(d\gamma_{xy}^s\) : shear strain increment due to shear ($d\phi_{mo}$), \(d\gamma_{xy}^\delta\) : shear strain increment due to principal stress rotation ($d\alpha$) and \(d\gamma_{xy}^\alpha\) : shear strain increment due to anisotropic consolidation ($d\sigma_m$) have been derived as follows (Matsuoka et al., 1986):

\[
d\gamma_{xy}^s = k_0 \cdot \frac{\sin \phi \cdot \cos \phi_{mo} \cdot \sin 2\alpha}{\sin \phi - \sin \phi_{mo}} \cdot d\phi_{mo} \quad \text{"shear"}
\]

(2)

\[
d\gamma_{xy}^\delta = 2k_0 \cdot \frac{\sin \phi \cdot \sin \phi_{mo} \cdot \sin 2(\alpha + \delta)}{\sin \phi - \sin \phi_{mo}} \cdot d\alpha \quad \text{"principal stress rotation"}
\]

(3)

\[
d\gamma_{xy}^\alpha = k_0 \cdot \frac{\sin \phi \cdot \sin \phi_{mo} \cdot \sin 2\alpha}{\sin \phi - \sin \phi_{mo}} \cdot \frac{d\sigma_m}{\sigma_m} \quad \text{"anisotropic consolidation"}
\]

(4)

It is considered in $d\gamma_{xy}^\alpha$ that the principal stress direction deviates by $\delta$ from the principal strain increment direction. In order to obtain the normal strain increments $d\varepsilon_x$ and $d\varepsilon_y$ from these shear strain increments, the following stress ratio vs. strain increment ratio relation has been introduced (Matsuoka, 1974).

\[
\frac{\tau}{\sigma_N} = \lambda \left( -\frac{d\varepsilon_N}{d\tau} \right) + \mu
\]

(5)

where $\tau/\sigma_N$ is the shear-normal stress ratio on the mobilized plane ($\tau/\sigma_N = \tan \phi_{mo}$), $d\varepsilon_N/d\tau$ is the normal-shear strain increment ratio on the mobilized plane, and ($\lambda$ and $\mu$) are material constants. If Eq. (5) is expressed in $xy$-coordinates, we can get the following expression (Matsuoka et al., 1986).

\[
\frac{d\varepsilon_x}{d\gamma_{xy}} \frac{d\varepsilon_y}{d\gamma_{xy}} = \frac{\mu - \tan \phi_{mo} \cdot \cos \phi_{mo} + \sin \phi_{mo} \cdot \cos 2\alpha}{\lambda} \cdot \frac{2 \cdot \sin 2\alpha}{2 \cdot \sin 2\alpha}
\]

(6)

Combining Eqs. (2), (3) and (4) with Eq. (6), the normal strain increments $d\varepsilon_x$ and $d\varepsilon_y$ for these three kinds of shear strain increments can be obtained. It should be noted that $2\alpha$ in Eq. (6) is changed for $2(\alpha + \delta)$ when the strains due to "principal stress rotation" are calculated by using Eq. (3). In addition to the above-mentioned strain increments, \(d\varepsilon_x^{\text{iso}} = d\varepsilon_y^{\text{iso}}\) : normal strain increments due to isotropic consolidation should be taken into account. They are written in a two-dimensional case as follows:
Fig. 1. Stress changes in Mohr's circle corresponding to (a) isotropic consolidation, (b) anisotropic consolidation, (c) shear and (d) principal stress rotation.

\[
\frac{d\varepsilon_{\sigma}}{d\varepsilon_{\sigma}} = \frac{1}{2} \cdot 0.434 C \cdot \frac{d\sigma_m}{\sigma_m} \quad \text{or} \quad \frac{1}{2} \cdot 0.434 C \cdot \frac{d\sigma_m}{\sigma_m}
\]

(7)

As seen from Eqs. (2)–(7), this constitutive model can predict the strains not only due to "isotropic consolidation", "anisotropic consolidation" and "shear", but also the strains due to "principal stress rotation" (see Fig. 1). As \(d\phi_m, d\sigma_m\) and \(d\sigma_m\) can be expressed by the stress increments \((d\sigma_x, d\sigma_y\) and \(d\tau_{xy}\), the arbitrary strain increments can be directly related to the arbitrary stress increments as follows: (Matsuoka et al., 1986; Matsuoka and Sakakibara, 1987)

\[
\{d\varepsilon_x, d\varepsilon_y, d\tau_{xy}\} = [D]^{-1} \cdot \{d\sigma_x, d\sigma_y, d\tau_{xy}\}
\]

(8)

where \([D]\) is the stress-strain matrix.

**COMPONENTS OF THE STRESS-STRAIN MATRIX**

The components of the matrix \([D]^{-1}\) can be written in the four domains (see Fig. 2) as follows:

\[
[D]^{-1} = \begin{bmatrix}
D_{11} & D_{12} & D_{13} \\
D_{12} & D_{22} & D_{23} \\
D_{13} & D_{23} & D_{33}
\end{bmatrix}
\]

(9)

I. \((d\phi_m \geq 0, d\sigma_m \geq 0)\)

\[
D_{11} = FF \times SS \times A1 + FR \times RR \times B1 + FF \times CA \times C1 + CCI \times C1
\]

\[
D_{12} = FF \times SS \times A2 + FR \times RR \times B2 + FF \times CA \times C1 + CCI \times C1
\]

\[
D_{13} = FF \times SS \times A3 + FR \times RR \times B3 + FF \times CA \times C1 + CCI \times C1
\]

\[
D_{21} = GG \times SS \times A1 + GR \times RR \times B1 + GG \times CA \times C1 + CCI \times C1
\]

\[
D_{22} = GG \times SS \times A2 + GR \times RR \times B2 + GG \times CA \times C1 + CCI \times C1
\]

\[
D_{23} = GG \times SS \times A3 + GR \times RR \times B3 + GG \times CA \times C1 + CCI \times C1
\]

\[
D_{31} = HH \times SS \times A1 + HR \times RR \times B1 + HH \times CA \times C1
\]

\[
D_{32} = HH \times SS \times A2 + HR \times RR \times B2 + HH \times CA \times C1
\]

\[
D_{33} = HH \times SS \times A3 + HR \times RR \times B3 + HH \times CA \times C1
\]

II. \((d\phi_m = 0, d\sigma_m < 0)\)

\[
D_{11} = FF \times SS \times A1 + FR \times RR \times B1 + CSI \times C1
\]

\[
D_{12} = FF \times SS \times A2 + FR \times RR \times B2 + CSI \times C1
\]

\[
D_{13} = FF \times SS \times A3 + FR \times RR \times B3 + CSI \times C1
\]

\[
D_{21} = GG \times SS \times A1 + GR \times RR \times B1 + CSI \times C1
\]

\[
D_{22} = GG \times SS \times A2 + GR \times RR \times B2 + CSI \times C1
\]

\[
D_{23} = GG \times SS \times A3 + GR \times RR \times B3 + CSI \times C1
\]

\[
D_{31} = HH \times SS \times A1 + HR \times RR \times B1 + CSI \times C1
\]

\[
D_{32} = HH \times SS \times A2 + HR \times RR \times B2 + CSI \times C1
\]

\[
D_{33} = HH \times SS \times A3 + HR \times RR \times B3 + CSI \times C1
\]

III. \((d\phi_m < 0, d\sigma_m < 0)\)

\[
D_{11} = FFU \times SSU \times A1 + FR \times RR \times B1
\]
+CSI x C1
D_{t1} = FFU x SSU x A2 + FR x RR x B2
+CSI x C1
D_{t2} = FFU x SSU x A3 + FR x RR x B3
D_{t3} = GGU x SSU x A1 + GR x RR x B1
+CSI x C1
D_{t4} = GGU x SSU x A2 + GR x RR x B2
+CSI x C1
D_{t5} = GGU x SSU x A3 + GR x RR x B3
D_{t6} = HH x SSU x A1 + HR x RR x B1
D_{t7} = HH x SSU x A2 + HR x RR x B2
D_{t8} = HH x SSU x A3 + HR x RR x B3

N. \ (d\phi_m \leq 0, \ \partial a_G \geq 0)
D_{n1} = FFU x SSU x A1 + FR x RR x B1
+CCI x C1
D_{n2} = FFU x SSU x A2 + FR x RR x B2
+CCI x C1
D_{n3} = FFU x SSU x A3 + FR x RR x B3
D_{n4} = GGU x SSU x A1 + GR x RR x B1
+CCI x C1
D_{n5} = GGU x SSU x A2 + GR x RR x B2
+CCI x C1
D_{n6} = GGU x SSU x A3 + GR x RR x B3
D_{n7} = HH x SSU x A1 + HR x RR x B1
D_{n8} = HH x SSU x A2 + HR x RR x B2
D_{n9} = HH x SSU x A3 + HR x RR x B3

\[ HH = \sin 2\alpha \]
\[ FR = \frac{1}{2} \left( \frac{2\mu - \tan \phi_m}{\lambda} \cdot \cos \phi_m \right) 
+ \sin \phi_m + \cos 2(\alpha + \delta) \}
\[ GR = \frac{1}{2} \left( \frac{2\mu - \tan \phi_m}{\lambda} \cdot \cos \phi_m \right) 
+ \sin \phi_m - \cos 2(\alpha + \delta) \}
\[ HR = \sin 2(\alpha + \delta) \]
\[ SS = k_s \frac{\sin^2 \phi \cdot \cos \phi_m}{(\sin \phi \cdot \sin \phi_m)^2} \]
\[ SSU = k_s \frac{\sin^2 \phi \cdot \cos \phi_m}{(\sin \phi \cdot \sin \phi_m)^2} \]
\[ RR = 2k_s \frac{\sin \phi \cdot \sin \phi_m}{\sin \phi \cdot \sin \phi_m} \]
\[ CA = k_s \frac{\sin \phi \cdot \sin \phi_m}{\sin \phi \cdot \sin \phi_m} \]
\[ CCI = \frac{0.434 \cdot C_s}{\sqrt{2(1 + e_0)}} \]
\[ CSI = \frac{0.434 \cdot C_s}{2 \cdot (1 + e_0)} \]
\[ A_1 = \frac{I_1}{\tan \phi_m} \cdot \cos \phi_m \cdot \{ 2 - (I_1/I_2) \cdot \sigma_y \}
A_2 = \frac{I_1}{\tan \phi_m} \cdot \cos \phi_m \cdot \{ 2 - (I_1/I_2) \cdot \sigma_x \}
A_3 = \frac{I_1}{\tan \phi_m} \cdot \cos \phi_m \cdot \{ 2(I_1/I_2) \cdot \tau_{xy} \}
B_1 = \cos 2\alpha \cdot \frac{(\sigma_x - \sigma_y)}{(\sigma_x - \sigma_y)^2}
B_2 = \cos 2\alpha \cdot \frac{\tau_{xy}}{(\sigma_x - \sigma_y)^2}
B_3 = \cos 2\alpha \cdot \frac{1}{(\sigma_x - \sigma_y)}
C_1 = \frac{1}{(\sigma_x + \sigma_y)}
I_1 = \sigma_x + \sigma_y \quad I_2 = \sigma_x \sigma_y - \tau_{xy}\]

In the above equations the letter U means the unloading process. The stress-strain parameters in this constitutive model are \( \phi \), \( C_e/(1 + e_0) \), \( C_e/(1 + e_0) \), \( k_s \), \( k_n \), \( \lambda \), \( \mu \) and \( \delta \). \( \phi \) denotes the internal friction angle in terms of effective stresses, \( C_e \) the compression index, \( C_s \) the swell index and \( e_0 \) the initial void ratio. \( k_s \) is a parameter to be found in Eqs. (2) and (3) or in SS, SSU and RR, which decides the magnitude of the shear strains due to shear and principal stress rotation. It is seen from Eq. (1) that \( k_s \) \((=1/G_s)\) is the gradient of the initial tangent of the relation between \((\sigma_1 - \varepsilon_3)\) and \((\sigma_1 - \varepsilon_3)/(\sigma_1 + \varepsilon_3) = \sin \phi_m \) (in the case of \( \alpha = 45^\circ \)) under a constant mean principal stress \( \sigma_m \).
\( k_e \) is a parameter in Eq. (4) or in \( CA \), which decides the magnitude of the shear strain due to anisotropic consolidation. It can be estimated by the following equation which is derived from a microscopic study of the compression mechanism of granular materials (Matsuoka and Fujii, 1985).

\[
k_e = 0.44 \frac{C_e}{1 + e_0} \tag{10}
\]

\( \lambda \) and \( \mu \) are parameters in Eq. (6) or in \( FF, GG, FFU, GGU, FR \) and \( GR \), which correspond to the gradient and the ordinate intersection of the linear relation between the shear-normal stress ratio and the normal-shear strain increment ratio on the mobilized plane respectively (Matsuoka, 1974). \( \delta \) is the angle between the direction of the principal stress and that of the principal strain increment under rotation of the principal stress axes, and it is assumed to be 30° from the principal stress rotation tests on the stack of aluminium rods (Matsuoka et al., 1986 ; Matsuoka and Sakakibara, 1987).

Even if the coordinate axes are rotated, the strain increments calculated by the above equations have the same values at the same stress state and stress change. So, the calculated strain increments satisfy the characteristics as a tensor.

The constitutive model mentioned here is a two-dimensional one and for isotropic materials. For anisotropic materials the reader may refer to Matsuoka (1982) and Iwata et al. (1986). In order to extend the two-dimensional constitutive model to three dimensions, the superposition of the “two-dimensional” principal strain increments is assumed (Matsuoka et al., 1986 ; Matsuoka and Sakakibara, 1987). The superposition requires the condition that the direction of the principal stress coincides with that of the principal strain increment, but it should be noted that this condition is only approximate in the case of anisotropic materials.

**TWO-DIMENSIONAL ARBITRARY STRESS APPARATUS**

An arbitrary stress apparatus which can apply the arbitrary stresses (\( \sigma_x, \sigma_y \) and \( \tau_{xy} \)) directly has been made for a stack of aluminium rods (mixture of \( \phi 1.6 \) mm and 3.0 mm, 50 mm in length) as a two-dimensional model of granular materials. The normal stress \( \sigma_x \) in the vertical direction and the normal stress \( \sigma_y \) in the horizontal direction are applied by the wires connected with air cylinders, and the shear stress \( \tau_{xy} \) is applied by the loading plate on the sample. These three stresses \( \sigma_x, \sigma_y \) and \( \tau_{xy} \) can be measured by load cells independently (see Fig. 3). The strains \( \varepsilon_x, \varepsilon_y \) and \( \gamma_{xy} \) can be also measured by dial gauges independently.

**PRINCIPAL STRESS ROTATION TESTS WITH SEVERAL ROTATION TIMES**

Fig. 4(a) shows the qualitative pattern of the histogram of the interparticle contact angles expressed in the radial direction, which changes from a circular distribution to an elliptical distribution during shear. Such an elliptical distribution rotates continuously during rotation of the principal stress axes, as shown in Fig. 4(b) (see Oda and Konishi, 1974 ; Matsuoka et al., 1988). From \( \alpha = 0^\circ \) to \( \alpha = 180^\circ \) the elliptical distribution passes in the “virgin” zone in which such structural change has never been experienced, and after \( \alpha = 180^\circ \) the experienced structural change will be repeated. Paying attention to the fact that the shear strain increment due to
principal stress rotation $d\gamma^{xy}$ has the same parameter $k_s$ as the shear strain increment due to shear $d\gamma^{xy}$ (see Eqs. (2) and (3)), the value of $k_s$ is reduced in the experienced structure by measuring the ratio of the tangential gradient $A$ to $B$ of the cyclic stress-strain relationship at the stress ratio $\sin\phi_a=0.25$ which is the same stress ratio as in the principal stress rotation, as shown in Fig.5 ($k_s=0.31\%$ in the virgin structure is reduced to $k_s=0.23\%$ at $\sin\phi_a=0.25$ in the experienced structure). This is based on the similarity of the change in structures during cyclic shearing and principal stress rotation (see Fig.4). Fig.6 shows the shear-normal stress ratio vs. normal-shear strain increment ratio relationship on the mobilized plane ($\alpha=45^\circ+\phi_{mo}/2$) when the stress state circles four times along a Mohr's circle. The broken line in the figure denotes the line of no volumetric strain increment ($de_v=0$). It is seen from Fig.6 that the measured values converge to the line of no volumetric strain increment with the number of rotation times. Therefore, in this analysis the value of the strain increment ratio is assumed to be the intermediate value between the present value and the value at $de_v=0$ of the strain increment ratio. For the 1st cycle $\lambda=0.8$ and $\mu=0.19$ are used, which are usual values for the stack of aluminium rods and the following values are calculated based on the above ideas: $\lambda=1.16$ and $\mu=0.13$ for the 2nd cycle, $\lambda=1.48$ and $\mu=0.084$ for the 3rd cycle, and $\lambda=1.71$ and $\mu=0.046$ for the 4th cycle. Fig.7(a)-(d) shows the results of the principal stress rotation test on the stack of aluminium rods in which the stress state circles four times along a Mohr's circle, and the analytical values by the proposed model. It is seen from this figure that the measured strains in the 2nd to 4th cycles are smaller than in the 1st cycle, and the measured volumetric strain $e_v$ becomes smaller with the number of rotation times. The analytical values based on the above-mentioned ideas explain well such tendency of the measured values. In Fig.7(a)-(d) the initial values of strains are taken to be zero. Fig.8(a) and (b) shows the results of the principal stress rotation tests on the same sample in
which the stress states circle from $2 \alpha = 0^\circ$ to $2 \alpha = 180^\circ$ and $360^\circ$ in the same direction and then the stress states circle reversely to $2 \alpha = 0^\circ$, and the analytical values by the proposed model. It is assumed in this analysis that the strains are not produced during $\Delta(2 \alpha) = 30^\circ$ after the reverse rotation of the principal stress direction, because the measured strains do not change sensitively just after the reverse rotation.

THREE KINDS OF "ARBITRARY" STRESS PATH TESTS

Fig. 9 shows three kinds of "arbitrary" stress paths ($A_1 \rightarrow B_1 \rightarrow B_2$, $A_1 \rightarrow B_2$ and $A_1 \rightarrow A_2 \rightarrow B_2$) represented by Mohr’s stress circles. TEST ① ($A_1 \rightarrow B_1 \rightarrow B_2$) is the test by which the principal stress rotation ($B_1 \rightarrow B_2$) is induced after shearing ($A_1 \rightarrow B_1$). TEST ② ($A_1 \rightarrow B_2$) is the test by which the principal stress rotation and shearing are induced at the same time. TEST ③ ($A_1 \rightarrow A_2 \rightarrow B_2$) is the test by which shearing ($A_2 \rightarrow B_2$) is induced after the principal stress rotation ($A_1 \rightarrow A_2$). It should be noted that the same starting point $A_1$ and the same final point $B_2$ are selected in the three kinds of "arbitrary" stress paths, and the principal stress values at $A_1$ and $A_2$ are the same ($\sigma_1 = 45$, $\sigma_3 = 35$ kPa) and the principal stress values at $B_1$ and $B_2$ are also the same ($\sigma_1 = 50$, $\sigma_3 = 30$ kPa). Therefore, the difference of the above three kinds of "arbitrary" stress paths cannot be expressed in the $\sigma_1 - \sigma_2$ diagram or the $q - p$ diagram ($p = (\sigma_1 + \sigma_2 + \sigma_3)/3$ and $q = \sigma_1 - \sigma_2$). Fig. 10 (a)–(c) shows the measured values by TEST ①, ② and ③ and the analytical values by the proposed model, represented by the relation between the principal stress ratio ($\sigma_1/\sigma_2$) and strains ($\varepsilon_1$, $\varepsilon_2$, $\gamma_{12}$, $\varepsilon_3$, $\varepsilon_1 - \varepsilon_2$, and $\varepsilon_3 = \varepsilon_1 + \varepsilon_2$). The parts where $\sigma_1/\sigma_2$ is constant in Fig. 10 (a) and (c) correspond to the principal stress rotation under constant
principal stresses \((B_1 \rightarrow B_2\) and \(A_1 \rightarrow A_2\) in Fig. 9). In Fig.10 (a), the final calculated values of these strains are indicated by arrows.

Fig. 10 (b) corresponds to the case when the increase in the principal stress ratio (i.e., shearing) and the rotation of the principal stress axes occur at the same time (\(A_1 \rightarrow B_2\) in Fig.9). Table 1 shows the final values of the principal strain \(\varepsilon_1\) caused by the three kinds of “arbitrary” stress path tests. It is seen from Fig.10 (a)-(c) that the values of strains are quite different along the three kinds of “arbitrary” stress paths, though the three kinds of “arbitrary” stress paths become the same in the \(\sigma_1 \sim \sigma_2\) diagram or the \(q \sim p\) diagram. It is seen from Table 1 that the final values of \(\varepsilon_1\) are also different along the three kinds of “arbitrary” stress paths. This means that the “arbitrary” stress path dependency of strains is existent, because the starting point \(A_1\) and the final point \(B_2\) are the same for the three kinds of tests. The analytical values explain well such tendency of the measured values along the three kinds of “arbitrary” stress paths. The parameters used for all these analyses are as follows. For the stack of aluminium rods: \(\phi=\ 24^\circ\), \(\lambda=0.8, \ \mu=0.19, \ k_s=0.31\%\) and \(\delta=30^\circ\).

**CONSIDERATION OF “ARBITRARY” STRESS PATH DEPENDENCY OF STRAINS BY FABRIC CHANGE**

Fig.11 (a)-(c) show the changes in the qualitative patterns of the histograms of the interparticle contact angles expressed in the radial direction, which correspond to TEST ① (\(A_1 \rightarrow B_1 \rightarrow B_2\)), TEST ② (\(A_1 \rightarrow B_2\)) and TEST ③ (\(A_1 \rightarrow A_2 \rightarrow B_2\)) in Fig.9 respectively (see Oda and Konishi, 1974; Matsuoka et al., 1988). The left figure in Fig.11 (a) represents that the distribution of the interparticle contact angles is elongated to the major principal stress direction by shearing and the right figure in Fig.11 (a) shows that the elongated distribution is continuously rotated.

![Fig. 8.](image)

**Fig. 8.** Comparison between measured and calculated strains under principal stress rotation with reverse rotation at (a) \(2\alpha=180^\circ\) and (b) \(2\alpha=360^\circ\)**
Table 1. Measured and calculated final principal strain $\varepsilon_1$ under three kinds of "arbitrary" stress paths

<table>
<thead>
<tr>
<th></th>
<th>Measured</th>
<th>calculated</th>
</tr>
</thead>
<tbody>
<tr>
<td>TEST 1</td>
<td>0.15</td>
<td>0.15</td>
</tr>
<tr>
<td>(A1→B1→B2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>TEST 2</td>
<td>0.13</td>
<td>0.13</td>
</tr>
<tr>
<td>(A1→B2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>TEST 3</td>
<td>0.11</td>
<td>0.12</td>
</tr>
<tr>
<td>(A1→A2→B2)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 10. Measured and calculated strains under three kinds of stress paths (A1→B1→B2, A1→B2, and A1→A2→B2) in Fig. 9

Fig. 11. Qualitative patterns of histograms of interparticle contact angles under three kinds of "arbitrary" stress paths
by principal stress rotation. On the other hand, the left figure in Fig.11 (c) shows that the initial distribution is rotated by principal stress rotation and the right figure in Fig.11 (c) represents that the rotated distribution is elongated to the major principal stress direction by shearing. Fig.11 (b) shows that the distribution of the interparticle contact angles is elongated and rotated at the same time because shearing and principal stress rotation take place at the same time in TEST 2 (A₁ → B₂). It is understood from these figures that even if the starting point A₁ and the final point B₂ are the same in the three kinds of stress paths, the fabric changes differently and so the different final strains are expected to be caused. In other words, it can be understood that the strains due to shearing and principal stress rotation have the "arbitrary" stress path dependency. For example, the left figure in Fig.11 (a) and the right figure in Fig.11 (c) show the same fabric change, though the principal stress axes are different by 45°. On the other hand, the right figure in Fig.11 (a) and the left figure in Fig.11 (c) indicate the different fabric change, so it is clear that the final strains are different under the same principal stress rotation.

APPLICATION TO FINITE ELEMENT ANALYSIS OF SOIL FOUNDATION UNDER STRIP LOAD

In order to check the influence of rotation of the principal stress axes on settlements and lateral displacements, the finite element computations of soil foundation under a uniform strip load (Matsuoka et al., 1977) are also performed using the proposed model. Fig.12 represents the model foundation and its boundary conditions. The soil parameters used for the analyses are for Toyoura sand and are shown as follows: \( \phi = 42^\circ \), \( C_0(1 + e_o) = 0.9\% \), \( C_B(1 + e_B) = 0.5\% \), \( \lambda = 1.1 \), \( \mu = 0.20 \), \( k_s = 0.33\% \) and \( \delta = 30^\circ \). The initial stresses in the foundation are calculated from the unit weight (\( \gamma_s = 15.68 \text{kN/m}^3 \)) and \( K_0 \) value (\( K_0 = 0.5 \)). Fig.13 shows the direction

Fig. 12. Finite element mesh and boundary conditions

Fig. 13. Rotation of principal stress directions in each element

Fig. 14. Computed settlements and lateral displacements
and magnitude of the principal stresses in each element under a uniform strip load of $q=4591$ kPa. It is seen from Fig. 13 that the rotation of the principal stress axes occurs significantly. Therefore, it is important to adopt a constitutive model which can evaluate the influence of rotation of the principal stress axes properly. Fig. 14 represents the computed settlements and lateral displacements under $q=2631$ kPa, in which the solid lines, broken lines, lines with dots and dotted lines correspond to the computed results under the contribution factors of strains due to "principal stress rotation" of 100%, 50%, 10% and 5% respectively. The contribution factors of strains due to "principal stress rotation" mean the multiplying constants of $d_{ys}'$ in Eq. (3). For example, 50% corresponds to $0.5d_{ys}'$. It is seen from Fig. 14 that the settlements and lateral displacements are influenced significantly by principal stress rotation. Especially, the lateral displacements in the case of 5% (dotted lines) are approximately half of those in the case of 100% (solid lines). Fig. 15 (a)-(d) show the computed distributions of local factors of safety (F.S. = $\tan \phi / \tan \varphi_{mo}$) under $q=4591$ kPa in the cases of the contribution factors of principal stress rotation 100%, 50%, 10% and 5%. It is seen from these figures that the zone with low local factors of safety expands laterally in the cases with high
contribution factors of principal stress rotation. On the other hand, in the cases with low contribution factors (10% and 5%) the lateral displacements are smaller under the same load (see Fig.14), so the local factors of safety are not reduced in the lateral direction. As mentioned above, it is important to evaluate the influence of principal stress rotation properly, because the settlements and lateral displacements become smaller and the factors of safety become larger when the influence of rotation of the principal stress axes is ignored. In the usual elasto-plastic constitutive model which does not consider the effect of stress history, the strains due to principal stress rotation are calculated from the elastic constitutive equation, by which the dilatancy and the deviation between the direction of the principal stress and that of the principal strain increment cannot be explained. The method for determining the elastic constants, for example selecting Poisson's ratio ($\nu=0$ or 0.3), also has a considerable effect on the computed results.

CONCLUSIONS

The main results are summarized as follows:

1) A constitutive model for soils by which the strain increments ($d\varepsilon_x$, $d\varepsilon_y$ and $d\gamma_{xy}$) are directly related to the stress increments ($d\sigma_x$, $d\sigma_y$ and $d\tau_{xy}$) is proposed. The stress-strain matrix is expressed in $xy$-coordinates fixed arbitrarily. The model predicts the strains not only due to “consolidation” and “shear”, but also due to “principal stress rotation”.

2) An arbitrary stress apparatus which can apply the arbitrary stresses ($\sigma_x$, $\sigma_y$ and $\tau_{xy}$) directly and independently has been made for a stack of aluminium rods as a two-dimensional model of granular materials. The stresses $\sigma_x$, $\sigma_y$ and $\tau_{xy}$ can be measured by load cells independently, and the strains $\varepsilon_x$, $\varepsilon_y$ and $\gamma_{xy}$ can be also measured by dial gauges independently.

3) Using the arbitrary stress apparatus, “principal stress rotation tests” in which the stress state circles four times along a Mohr's stress circle are carried out to evaluate the influence of rotation of the principal stress axes on strains. The pretty large plastic strains are observed under the principal stress rotation with constant principal stresses, and they are analyzed by the proposed model.

4) Using the same apparatus, three kinds of “arbitrary stress path tests” with the same starting and final stress points are also performed to investigate the arbitrary stress path dependency of strains. The observed strains are quite different along the three kinds of arbitrary stress paths, though the three kinds of arbitrary stress paths become the same in the $\sigma_1$~$\sigma_3$ diagram. All the test results are explained well by the proposed model.

5) To check the influence of rotation of the principal stress axes on settlements and lateral displacements, the finite element computations of soil foundation under strip load are performed using the proposed model. The computed results show that the settlements and lateral displacements are influenced significantly by the principal stress rotation.

REFERENCES


on Numerical Models in Geomechanics, Ghent. pp. 67-78.


