GENERALIZED LIMIT EQUILIBRIUM METHOD AND ITS RELATION TO SLIP LINE METHOD

MEIKETSU ENOKI¹⁾, NORIO YAGI¹⁾, RYUICHI YATABE¹¹⁾
AND EIZABURO ICHIMOTO^{1v)}

ABSTRACT

It has been shown after mathematical and mechanical investigation that Limit Equilibrium Method (LEM) can be considered as a method to obtain the necessary condition of Slip Line Method (SLM), and that the solution can be obtained under the condition that $F_s=1$ and $\partial F_s/\partial\theta=0$, where F_s is the safety factor and θ is the inclination of the plane on which the safety factor is defined.

Next, the generalized LEM (GLEM) is proposed, dealing with the following points: (1) Triangular or quadrangular blocks can be treated; (2) Safety factors are defined also on interblock planes; and (3) All types of plastic problems, slope stability, bearing capacity, and earth pressure are identically formulated. Two situations regarding treatment of the moment equilibrium condition are discussed.

GLEM is applied to well-known problems. The results agree well with those obtained by theoretical methods. GLEM can be used to obtain the distribution of earth pressure or bearing capacity. GLEM, like ordinary LEM, is very effective for the practical problems, but GLEM is more theoretical and it can result in more accurate solutions.

Key words: bearing capacity, earth pressure, limit equilibrium method, slip line method, slope stability (IGC: E0/E3/E5/E6/E13)

INTRODUCTION

There are many method, such as Limit Analysis Method, Slip Line Method (SLM) using theory of plasticity, Finite Element Method, Limit Equilibrium Method (LEM) etc., which

can be used to analyze plastic problems such as slope stability, bearing capacity or earth pressure. Among these, LEM is the oldest method; its origin can be traced back to Coulomb's study on earth pressure. As LEM theory contains some vagueness, many con-

¹⁾ Associate Professor, Department of Civil and Ocean Engineering, Ehime University, Bunkyocho-3, Matsuyama, Ehime 790.

¹¹⁾ Professor, ditto.

¹¹¹⁾ Research Associate, ditto.

^{1v)} Managing Director, Fudo Construction Co. Ltd., Taito-2-1-2, Taito-ku, Tokyo 101.
Manuscript was received for review on May 18, 1989.

Written discussions on this paper should be submitted before January 1, 1992, to the Japanese Society of Soil Mechanics and Foundation Engineering, Sugayama Bldg. 4F, Kanda Awaji-cho 2-23, Chiyoda-ku, Tokyo 101, Japan. Upon request the closing date may be extended one month.

crete methods for a slice system have been proposed by Fellenius (1936), Bishop (1955), Janbu (1957), Spencer (1967), Morgenstern-Price (1965) and others. Though solutions by these methods differ slightly from each other, they are all commonly used to solve practical problems because of their simplicity and effectiveness. It is therefore important to continue to develop LEM and to clarify the theoretical meaning of LEM.

In this paper, mathematical and mechanical meanings of LEM are investigated, and it is clarified that LEM is an approximation method to obtain the necessary condition of SLM. Generalized LEM (GLEM) is then proposed in the same way as the generalized slice method already proposed by the authors (Enoki et al., 1990). The technique used to obtain the solution is refined. The meaning of taking the moment equilibrium is clarified.

Well-known problems, including the bearing capacity of a strip footing, active or passive earth pressure on a rough or smooth vertical wall, are analyzed by GLEM. The results agree well with those obtained by theoretical methods. As evidence that LEM is an approximation method of SLM, the distribution of earth pressure is obtained by GLEM. Practical problems which cannot be easily analyzed by theoretical methods, due to difficulties of boundary condition or inhomogeneity, can be analyzed by GLEM.

RELATION BETWEEN LEM AND SLM

SLM is a method to obtain the solution of the theory of plasticity, and will give a stress field or slip line field which satisfies the equilibrium condition and the failure condition. The equilibrium condition for a two-dimensional problem is written as follows:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0 \quad (1)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \gamma \quad (2)$$

$$\tau_{yx} = \tau_{xy} \quad (3)$$

Although these equations are written in differ-

ential forms, Eqs. (1), (2) and (3) indicate the equilibrium conditions of the x -directional forces, the y -directional forces, and the moment, respectively.

Mohr-Coulomb failure condition is written as follows:

$$\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 = \left(c \cos \phi + \frac{\sigma_x + \sigma_y}{2} \sin \phi \right)^2 \quad (4)$$

The stress field or slip line field which satisfies Eqs. (1) to (4) is generally obtained by SLM using Kötter's expression, but the reader should refer to other papers (Sokolovsky, 1960, etc.) for details of this method. In SLM, the term "slip line" does not mean a discontinuous line of displacement but a surface on which the stresses are in the failure state.

On the other hand, LEM is a method to obtain the force field which satisfies the equilibrium condition of every slice and failure condition on every bottom plane of a slice in a slice (or block) system, as shown in Fig. 1. Particularly, GLEM (as shown later) can obtain the force field which satisfies the equilibrium condition on every block and failure condition on every plane surrounding a block. In LEM, there are also some methods which do not take the moment equilibrium condition into account.

Note that a slope stability problem, in which the safety factor, F_s , is required, can be converted into an active earth pressure problem. In the converted problem, the maximized earth pressure must be zero because no retaining

Fig. 1. A block system in a slope stability problem

Fig. 2. A plastic problem in which the slip line field is known

wall is present in the actual slope problem; This is based on the conditions that the apparent strength parameters are c_a and ϕ_a (where $c_a = c/F_s$, $\tan \phi_a = \tan \phi/F_s$, F_s is actual safety factor) and the apparent safety factor is 1.0.

In LEM, since the force equilibrium equations of a block can be obtained by the integration of stresses along the surrounding planes and by integration of the unit weight on the block area, and since the definition equation of the safety factor (in the case of $F_s = 1.0$) can be obtained by the integration of the failure condition (which will be formulated as Eq. (5)) along the bottom plane of the block, LEM is shown to be a method to obtain the necessary condition of SLM. In other words, if a block is cut from the failing zone of SLM in a manner such that bottom plane is identical to a part of the slip line, as shown in Fig. 2 (a-block, b-block or c-block), the equilibrium equations of forces and the definition equation of the safety factor on the bottom plane in LEM are always satisfied for the block. In the special case of the c-block, the definition equations of the safety factors are satisfied both on the side planes and on the bottom plane of the block.

As LEM can not be used to treat a block with curved surfaces, some error caused by misfit of the slip surface will be inevitable in LEM.

CHARACTERISTICS OF SAFETY FACTOR

In LEM, the safety factor F_s on a plane is commonly defined by the ratio of shear strength S to shear force T , and it is rewritten using stresses, as follows:

$$F_s = \frac{S}{T} = \frac{\int (\sigma \tan \phi + c) dA}{\int \tau dA} \quad (5)$$

The safety factor depends on the inclination of a plane on which it is defined, because stresses depend on the inclination. Here, we will investigate the characteristic of the safety factor defined by stresses, as $F_s = s/\tau = (\sigma \tan \phi + c)/\tau$.

The safety factor on a plane of which inclination differs by δ from the actual slip plane in the failing zone is written as follows:

$$F_s = \frac{1 - \sin \phi \sin(\phi - 2\delta)}{\cos \phi \cos(\phi - 2\delta)} \quad (6)$$

Fig. 3 show the relation between F_s and δ for $\phi = 30^\circ$ and $\phi = 40^\circ$. The important characteristics are that the safety factor has a minimum value, 1.0, in the failing zone for the correct inclination of the plane. The safety factor on a plane in an element which is not in the failing zone is clearly larger than 1.0.

Although the safety factor in LEM is defined as the integral form of stresses along the bottom plane of a block, this characteristic will

Fig. 3. The relation between the safety factor and the inclination of the plane on which the safety factor is defined in the failing zone

Fig. 4. Compression or extension of a two-dimensional specimen under confining pressure σ_x

be maintained; the closer the bottom plane is to the true slip surface, the closer the safety factor will be to the minimum value, 1.0.

The reason why the optimization of the safety factor for the geometry of the main slip surface can give the correct surface in LEM is explained by this characteristic, as follows:

Fig. 4 is a simple problem in which the minimum force F_y is required to cause compression failure of a two-dimensional specimen under the confining pressure σ_x . When the equilibrium of the indicated block and the safety factor defined on the indicated plane are considered, four equations, x and y directional equilibrium equations of the block, $F_s=1$ and $\partial F_s/\partial\theta=0$ (minimization) on the plane, are obtained. Although the details are omitted here, these equations are sufficient to obtain F_y and θ . These equations can be rewritten as $\partial F_y/\partial\theta=0$ (minimization) without losing mathematical validity after the elimination of F_s . If one of the end points of the plane is fixed, $\partial F_y/\partial y=0$ will give the same result, where y is the coordinate of another end point. Since there are two groups of slip planes, the initial value of the variable (θ or y) has influence on the obtained inclination of the plane, but it does not have influence on the minimized F_y . When the maximum force to cause extension failure of the specimen is required, $\partial F_y/\partial\theta=0$ (maximization) will give the solutions. In this case, the direction of τ is reverse

to the compression case. Coulomb's method to obtain earth pressure is relative to this problem.

In this problem a single block or a single plane is used, but in problems where curved slip surfaces are expected by SLM, as shown in Fig. 2, many blocks or many planes must be used to approximate the slip surface. In this case, the problems become more complicated, because the lateral forces acting on the interblock planes must be assumed for the problems to be statically determinate, as mentioned later. However, the characteristics of the safety factor shown in Fig. 3 are helpful to assume the appropriate forces on the interblock planes. When safety factors are defined on interblock planes, they are limited in value from 1.0 to infinitely large. If an interblock plane is completely contained in a failure zone, an arbitrary set of values of the safety factor and the inclination of the plane, as shown in Fig. 3, must be taken. In Fig. 2, for example, the C-interblock plane and $F_s=1$, and the B-interblock plane and $F_s=\infty$, would both be admissible sets. (Note that the B-plane is the principal plane in Rankine's passive zone). Fundamental knowledge about the slip lines of plastic problems, such as bearing capacity, earth pressure and slope failure, will help us to select the proper set. Whitman and Bailey (1967) recognized intuitively this limit of safety factor on interslice planes.

In most of the following analyses, the authors take $F_s=1$ and the corresponding slip plane which belongs to one group of the slip line field.

The range, 1.0 $-\infty$, of safety factors on interblock planes in a bearing capacity problem can be converted into $F_s - \infty$ in a slope problem in which the safety factor on the main slip surface is required, where F_s is the safety factor on the main slip surface. This has already been used to obtain the upper and lower limits of the safety factor for slopes in an earlier paper by the authors (Enoki et al., 1990).

From the formulation of GLEM given later, the following influence of safety factors on interblock planes on the result is predicted:

Underestimation of the safety factors on interblock planes will result in overestimation of the safety factor of slope, underestimation of the active earth pressure, and overestimation of the bearing capacity or passive earth pressure.

From this influence of the safety factors of interblock planes on the solution, GLEM can obtain the upper and lower limits of the solution by assuming the safety factors to be F_s (safety factor on the main slip surface) and ∞ , even if proper sets of inclinations and safety factors of interblock planes can not be found.

In order to obtain approximate solutions of plastic problems with accuracy using the characteristics of the safety factor mentioned here, the following conditions will be required:

- 1) Adequate sets of safety factors and inclinations of interblock planes are assumed.
- 2) The initial slip surface is adequately given for optimization.
- 3) A point either the end point of footing or the bottom point of a wall through which the slip surface passes is given. This condition is not essential to obtain the solution, but its use will save calculation time.

CLASSIFICATION OF PLASTIC PROBLEMS

Although there are many plastic problems to be analyzed, they can be classified into three types, using the model proposed by Xristianovich (Sokolovsky, 1960) and modified by the authors, as shown in Fig. 5.

In the model of a balance, a mass of W_l is weighed on the left side, and friction (maximum $F_{\max}$) will be caused. The mass W_r , weighed on the right side to keep equilibrium, is as follows:

$$W_l - F_{\max} \leq W_r \leq W_l + F_{\max} \quad (7)$$

Here, three types of problems can be made.

Fig. 5. A model of a plastic problem (after Sokolovsky's quotation from Xristianovich)

The first requires maximum mass, $W_{r \max}$, capable to be weighed on the right side. The second requires minimum mass, $W_{r \min}$. The third requires the safety factor F_s when W_r is given and its value is between $W_{r \min}$ and $W_{r \max}$. In the third problem, the safety factor is defined using the mobilized frictional force F_{mob} , as follows:

$$F_s = F_{\max} / |F_{\text{mob}}| \quad (8)$$

Conversely, in the first and the second problems, F_s is 1.0 because the friction is fully mobilized.

Attention should be paid to the direction of the frictional force. It will act downward in the first problem, but upward in the second. In the third, if W_l is heavier than W_r , it will act upward.

The first will correspond to problems of bearing capacity or passive earth pressure, the second will correspond to problems of active earth pressure, and the third will correspond to problems of slope stability.

In the second, when the obtained $W_{r \min}$ is negative, the actual $W_{r \min}$ must be considered to be zero because soil and rock have no tensile strength.

In these problems there is one unknown, $W_{r \max}$ or $W_{r \min}$ under the assumption of $F_s=1$, or F_s under the known value of W_r . It suggests that the number of unknowns is one in these three problems.

Though LEM has been used in practice mainly to analyze slope stability, it is applicable also to analyze bearing capacity or earth pressure under the assumption of $F_s=1$. However, as mentioned above, a slope problem can be considered as an active earth pressure problem, and all plastic problems can be classified into two groups, the first and the second.

COMPARISON OF NUMBERS OF EQUATIONS AND UNKNOWN

Characteristics of problems which can be solved by LEM, and the assumptions required to solve the problems, are clarified in this section by comparison of the number of equa-

Table 1. Comparison of the numbers of equations and unknowns (n : number of blocks)

Equations		Unknowns	
When moment equilibrium is not taken :			
Equilibrium (x -dir.)	n	On base planes	
(y -dir.)	n	Normal forces	$P_i \quad n$
Definition of safety factor	n	Tangential forces	$T_i \quad n$
(On base planes)		Safety factor	$F_s \quad 1$
Definition of safety factor	$n-1$	On interblock planes	
(On interblock planes)		Normal forces	$H_j \quad n-1$
		Tangential forces	$V_j \quad n-1$
		Safety factors	$F_{sj} \quad n-1$
Subtotal	$4n-1$	Subtotal	$5n-2$
When moment equilibrium is added :			
Equilibrium (Moment)	n	On base planes	
		Acting points	$d_i \quad n$
		On interblock planes	
		Acting points	$d'_j \quad n-1$
Total	$5n-1$	Total	$7n-3$

tions with the number of unknowns.

When a block system shown in Fig. 1 is considered for a slope problem, the number of blocks is n , the number of base planes on which a common safety factor is defined is n , and the number of interblock planes on which safety factors are also defined is $n-1$. Table 1 shows a comparison of the numbers of equations and unknowns.

When the moment equilibrium is not taken, $n-1$ unknowns must be given in order to match the number of equations with the number of unknowns.

In the proposed method, $n-1$ safety factors on interblock planes are assumed.

In other slice methods, $n-1$ relations between normal forces and tangential forces on interslice planes are assumed. For example, a ratio of tangential force to normal force is assumed in Spencer's method (1967), and no tangential forces are assumed in Janbu's simplified method (Janbu, et al., 1956) and Bishop's simplified method (1955).

When the moment equilibrium is considered, while n more equations for every block are added, the number of unknowns for the force acting points will increase by $2n-1$. Assumptions regarding the acting points are required for the problem to be statically determinate.

There are two situations in this case. The first is where $n-1$ unknowns (commonly, the acting points of interblock forces) are assumed.

n equations of moment equilibrium are used only to obtain n unknowns of the acting points of forces on base planes. The second is where n unknowns (commonly, the acting points of forces on base planes) are assumed. After elimination of $n-1$ unknowns (the acting points of interblock forces) from n equations of moment equilibrium, the remaining equation will be added to the force equilibrium equations. In this situation a constant (commonly, a factor to relate tangential force with normal force on interblock planes) must be released or changed to an unknown in order to avoid an excess of the number of equations.

FORMULATION OF GENERALIZED LIMIT EQUILIBRIUM METHOD

The generalized slice method for slope stability analysis proposed by Enoki et al. (1990), has the following characteristics:

1. Blocks as well as slices can be treated.
2. Safety factors are defined both on the main slip surface and on interblock planes. When the safety factor, F_{si} defined on interblock planes is assumed as 1.0, the obtained F_s defined on main slip surface is called $F_{s \max}$. When F_{si} is assumed to be equal to the safety factor on the main slip surface, the obtained F_s is called $F_{s \text{ med}}$. When F_{si} is assumed to be infinitely large, the obtained F_s is called $F_{s \min}$.
3. Only the force equilibrium condition is used, because taking the moment equilibrium would increase the number of unknowns (acting points) and hence the problem would become more indeterminate.
4. The equilibrium equations of forces are directly written, and the nonlinear simultaneous equations are solved numerically by computer.
5. Non-circular slip surfaces can be treated as well as circular slip surfaces.

In this section, the generalized slice method is extended so that all types of plastic problems can be expressed in a single formulation. To simplify the formulation, all of the safety factors on interblock planes are assumed to be equal to F_{si} , here.

The equilibrium of the i -th block, shown in

Fig. 6. Equilibrium of the i -th block

Fig. 1 and Fig. 6, surrounded by the i -th base plane, the u -th top plane, the j -th and $j-1$ -th interblock planes, is formulated as follows :

<Equilibrium toward normal direction to the base plane>

$$\begin{aligned} & -H_{j-1} \cos(\theta_{j-1} - \beta_i) + V_{j-1} \sin(\theta_{j-1} - \beta_i) \\ & + H_j \cos(\theta_j - \beta_i) - V_j \sin(\theta_j - \beta_i) \\ & + W_i \cos \beta_i = P_i \end{aligned} \quad (9)$$

<Equilibrium toward tangential direction to the base plane>

$$\begin{aligned} & H_{j-1} \sin(\theta_{j-1} - \beta_i) + V_{j-1} \cos(\theta_{j-1} - \beta_i) \\ & - H_j \sin(\theta_j - \beta_i) - V_j \cos(\theta_j - \beta_i) \\ & + W_i \sin \beta_i = T_i \end{aligned} \quad (10)$$

<Equilibrium of moment around left top point>

$$\begin{aligned} & -P_i \{l_j \cos(\theta_j - \beta_i) - d_i\} + T_i l_j \sin(\theta_j - \beta_i) \\ & + H_j (l_j - d'_j) + H_{j-1} \{l_u \cos(\theta_{j-1} - \theta_u) \\ & - l_{j-1} + d'_{j-1}\} - V_{j-1} l_u \sin(\theta_{j-1} - \theta_u) \\ & - W_i (d_{wi} - l_j \cos \theta_j) = 0 \end{aligned} \quad (11)$$

<Definition of safety factors>

On base plane

$$T_i = k(P_i \tan \phi + c_i) / F_s \quad (12)$$

On interblock planes

$$V_j = m_j (H_j \tan \phi + c_j) / F_{si} \quad (13)$$

$$V_{j-1} = m_{j-1} (H_{j-1} \tan \phi + c_{j-1}) / F_{si} \quad (14)$$

Where k and m_j are parameters to define the directions of tangential forces on bottom and interblock planes. They have the same characteristics as the parameter which appears in SLM. In the sense that LEM obtains the necessary condition of SLM, the characteristics indicated by these parameters are not those of

the displacement field but those of the stress or force field. k and $m_1, \dots, m_{n-1}$ equal to $+1$ for slope and active earth pressure problems, and -1 for bearing capacity and passive earth pressure problems.

If the parameters are considered to indicate the characteristics of the displacement field, $k=+1$ when the blocks slide leftwards, and $m_j=+1$ when the right block of the j -th interblock plane moves downward relative to the left one.

Eqs. (9), (10), (11) and (12) or (13) correspond to Eqs. (1), (2), (3) and (4).

TECHNIQUES TO SOLVE EQUATIONS

In the earlier method described by the authors (Enoki et al., 1990), the equilibrium of moment was not taken. The nonlinear simultaneous equations obtained after substitution of Eqs. (12), (13) and (14) into Eqs. (9) and (10) are solved directly by numerical computation. In this paper, the techniques to obtain solutions without treating nonlinear simultaneous equations will be introduced.

Here, the planes on which V_0 , H_0 and V_n , H_n are applied in Fig. 1, are considered as interblock planes for the sake of convenience.

<Reduction of force equilibrium equations>

By substituting Eqs. (12), (13) and (14) into Eqs. (9) and (10). T_i , V_j and V_{j-1} are eliminated from Eqs. (9) and (10). Elimination of P_i from these equations gives the following equation :

$$-H_{j-1} \cdot f_i + H_j \cdot g_i = h_i \quad (15)$$

Where f_i , g_i and h_i are as follows, and can be determined only by geometry of the i -th block and the safety factors on the base and interblock planes :

$$\begin{aligned} f_i = & \left(-\cos \alpha_{j-1} + \frac{m_{j-1} \tan \phi \sin \alpha_{j-1}}{F_{si}} \right) \frac{k \tan \phi}{F_s} \\ & - \left(\sin \alpha_{j-1} + \frac{m_{j-1} \tan \phi \cos \alpha_{j-1}}{F_{si}} \right) \end{aligned} \quad (16)$$

$$\begin{aligned} g_i = & \left(-\cos \alpha_j + \frac{m_j \tan \phi \sin \alpha_j}{F_{si}} \right) \frac{k \tan \phi}{F_s} \\ & - \left(\sin \alpha_j + \frac{m_j \tan \phi \cos \alpha_j}{F_{si}} \right) \end{aligned} \quad (17)$$

$$h_i = -W_i \left(-\frac{k \tan \phi}{F_s} \cos \beta_i + \sin \beta_i \right) + \frac{k c l_i}{F_s} \\ - \frac{m_{j-1} c l_{j-1}}{F_{si}} \left(-\frac{k \tan \phi}{F_s} \sin \alpha_{j-1} + \cos \alpha_{j-1} \right) \\ + \frac{m_j c l_j}{F_{si}} \left(-\frac{k \tan \phi}{F_s} \sin \alpha_j + \cos \alpha_j \right) \quad (18)$$

Where $\alpha_j = \beta_i - \theta_j$, $\alpha_{j-1} = \beta_i - \theta_{j-1}$.

Now, if i equals to 1 or n , terms having the suffix $j=0$ or $j=n$ are included in Eqs. (16) to (18). The roughness of the footing or wall can be introduced into the analysis by changing the values of c and ϕ included in these terms.

As Eq. (15) can be written for every block, the following simultaneous equations can be obtained:

$$\begin{aligned} -H_0 \cdot f_1 + H_1 \cdot g_1 &= h_1 \\ -H_1 \cdot f_2 + H_2 \cdot g_2 &= h_2 \\ &\vdots \\ -H_{j-1} \cdot f_i + H_j \cdot g_i &= h_i \\ &\vdots \\ -H_{n-1} \cdot f_n + H_n \cdot g_n &= h_n \end{aligned} \quad (19)$$

After substitution and elimination of $H_1, \dots, H_{n-1}$, the following equation can be obtained:

$$H_n = \frac{1}{g_n} h_n + \frac{f_n}{g_n g_{n-1}} h_{n-1} + \frac{f_n f_{n-1}}{g_n g_{n-1} g_{n-2}} h_{n-2} + \dots \\ + \frac{f_n f_{n-1} \dots f_2}{g_n g_{n-1} g_{n-2} \dots g_1} (h_1 + H_0 \cdot f_1) \quad (20)$$

If F_{si} is properly assumed, unknowns in above equation are H_0 , H_n and F_s . In slope problems, H_0 and H_n are given, and F_s is required. In this case Eq. (20) is an equation having an unknown F_s under given H_0 and H_n . In problems of bearing capacity or earth pressure, F_s is 1.0 and H_n is required under a given H_0 . In this case Eq. (20) is an equation having an unknown H_n under given F_s and H_0 .

As Eq. (20) is a nonlinear equation including only a single unknown in every problem, it is easy to obtain the solution numerically for any given F_{si} .

Note that if α_j and l_j can be assumed to be equal to α_{j-1} and l_{j-1} , f_i becomes equal to g_i and the third and fourth terms of h_i in Eq. (18) vanish, and the same equation to the

Fellenius method is then obtained.

<Reduction of moment equilibrium equations>

When the moment equilibrium is taken, and the first situation is taken, $n-1$ unknowns (acting points of interblock forces) are assumed after the solution of Eq. (20) is obtained, and n equations of moment equilibrium are used only to get n unknowns of the acting points of forces on base planes.

When the second situation is taken, n unknowns of the acting points of forces on base planes are assumed to be the midpoints of the base planes, $n-1$ unknowns (acting points of interblock forces) are eliminated from n equations of moment equilibrium as follows:

Note that the terms $H_j d'_j$ and $H_{j-1} d'_{j-1}$ are always included in the equation of moment equilibrium for every block, as shown in Eq. (11). Therefore the unknowns $d'_1, \dots, d'_{n-1}$ will disappear in the equation in which all of the moment equilibrium equations are summed. It is this specific equation which is to be added to the force equilibrium equations mentioned earlier. In this situation, F_{si} should be determined to satisfy Eq. (20) and this equation.

SITUATIONS REGARDING MOMENT EQUILIBRIUM

In the above section two situations regarding the moment equilibrium condition are introduced.

If the first situation is taken, the safety factor in a slope problem can be determined only by the force equilibrium condition, Eq. (20), under the $n-1$ assumptions of interblock forces (or safety factors).

If the second situation is taken, the safety factor can be determined by Eq. (20) and the summed equation under the n assumptions of acting points of forces on bottom planes. From these situations, the slice methods are classified into two groups. For example, while simplified Janbu method is based on the first situation, Spencer method and Morgenstern-Price method are based on the second.

In the second situation, though the moment

equilibrium can be taken into account, additional assumptions on the acting points of forces are required, and it is possible that if the assumptions are not adequate, the characteristic of the GLEM solution which identifies this solution as a necessary condition of SLM may no longer be valid, and other characteristics may appear. For this reason the practical problems to be demonstrated later will be analyzed only from the first situation.

OPTIMIZATION OF SAFETY FACTOR ETC. FOR SLIP SURFACE

In the above sections, the safety factor, bearing capacity or earth pressure is formulated for a given slip surface. In most problems, these values must be optimized for the geometry of the slip surface in order to obtain the actual value.

In slope problems the safety factor is obtained by minimization. In bearing capacity problems, the force capable to be applied to the foundation is obtained by minimization. In earth pressure problems, the active earth pressure is obtained by maximization of the force on a wall and the passive earth pressure is obtained by minimization.

The main slip surface is expressed by connecting the lines of the bottom points of the interblock planes. Thus, to optimize the safety factor or bearing capacity for the slip surface is to optimize it for the coordinates of the bottom points of the interblock planes. In order to determine the main slip surface, the authors use polar coordinates with origin of (x_{0i}, y_{0i}) for the i -th point. The safety factor etc. are, then, optimized for r_i and θ_i under a given (x_{0i}, y_{0i}) , or sometimes only for r_i . Although many methods of optimization are available, the authors used the classical Newton method to optimize the function value.

The following modification will be required to apply the optimization program to GLEM: Decrements or increments of unknowns in one iteration cycle of the optimization should be small, and the initial values of unknowns should not be far from the ones in the result. If this is not the case, the slip surface may

be singular, with zigzag surface.

To increase the number of unknowns more than is necessary would probably not give good results due to the error generated in the numerical calculations.

SLOPE STABILITY

For slope stability problems, $k=1$ and $m_1=m_2=\dots=m_{n-1}=1$ are used. As the application of GLEM to slope problems has been already reported by the authors (Enoki et al., 1990), the details are omitted here.

In slope problems, since there are few problems for which the solutions are theoretically assured to be valid, the authors can not demonstrate comparisons of solutions obtained by SLM and GLEM. However, it is worthwhile to point out that the actual safety factor of a slope for a given geometry of a slice system must be in a range from $F_{s\min}$ to $F_{s\med}$, where $F_{s\min}$ and $F_{s\med}$ are the safety factors defined on the main slip surface, under the assumptions that for $F_{s\min}$ the safety factor defined on interblock planes are infinitely large, and for $F_{s\med}$ the safety factor is equal to that of the main slip surface. Fig. 7 shows the safety factors obtained by the several methods for a given slice system of a

Fig. 7. Safety factors obtained by several methods for a given slice system of a slope problem

Fig. 8. Comparison of bearing capacities and slip surfaces obtained by several methods

Fig. 9. Coefficients of bearing capacity

(a) N_c (b) N_q (c) N_γ

(d) The block system used in this analysis

slope problem. Although some methods give safety factors in a range from $F_{s \min}$ to $F_{s \text{ med}}$, other methods give safety factors larger than $F_{s \text{ med}}$ because of improper assumptions on interslice forces or on the acting points of forces.

Slope problems are the most difficult among plastic problems, because we cannot get information on the slip line field and cannot determine a specific point through which the main slip surface passes.

BEARING CAPACITY

With reference to the slip line fields obtained by SLM for simple problems of bearing capacity, interblock planes which pass through one end of the footing is adopted, and a triangular block system is also adopted in the following analyses. $k=-1$ and $m_1=\dots=m_{n-1}=-1$ are used in Eqs. (16), (17) and (18). F_s and F_{si} are assumed to be 1.0.

The slip surface is determined by adjusting r_i under a given θ_i so that H_n is minimized. H_0 is given as the overburden pressure. No tangential forces are assumed on the plane under surcharge. Hansen (1966) used several methods to obtain the bearing capacity of a smooth strip footing on a ground ($\phi=30^\circ$ and $\gamma=0$) with a uniform surcharge. In Fig. 8, Hansen's result is shown compared with the result of the proposed method. The figure also shows the theoretical value and the slip surface by Prandtl (1921). The proposed method gives the most accurate result among the various slice methods even when the number of blocks is 5.

Figs. 9 (a), (b) and (c) show coefficients N_c , N_q and N_r for bearing capacity obtained by several methods. The solid lines indicate the results obtained by the theory of plasticity. (N_c : Prandtl (1921), N_q : Reisner (1925) and N_r : Vesic (1973)). The figures also show the results of Fellenius and Bishop methods (Yamaguchi (1984)).

Fig. 9 (d) shows the block system used in the analysis.

From these figures GLEM can clearly give more accurate solutions than the other slice

methods even when only 5 blocks are used. Since it is known that the slip line field of active and transient zones is composed of two sets of curved lines in the case where $\gamma \neq 0$, the safety factors assumed to be 1.0 on interblock planes are underestimated, and the bearing capacity may be somewhat overestimated.

Mizuno (1948) used a triangular block system for the transient zone to get an approximate solution of SLM in his bearing capacity problem. His work is full of suggestions useful to the authors.

EARTH PRESSURE

With reference to the slip line fields obtained by SLM for some problems of earth pressure, interblock planes which pass through the top of the wall is adopted, and a triangular block system is also adopted in the following analyses. F_s and F_{si} are assumed to be 1.0.

The formulation for passive earth pressure is identical to that for bearing capacity. The active earth pressure is obtained by maximization of earth pressure and $k=1$, $m_1=\dots=m_{n-1}=1$. It is recognized that for rough wall problems Coulomb's method slightly underestimates the active earth pressure and extremely overestimates the passive earth pres-

Fig. 10. Comparison of passive earth pressure on a rough vertical wall
(a) Passive earth pressure
(b) Slip surfaces and block systems

Fig. 11. Distribution of active earth pressure on a smooth vertical wall

sure. Fig. 10 (a) shows the passive earth pressure on a rough vertical wall ($\delta=0$, 15° and 30°), comparing the results of the calculations by the methods of Coulomb, Sokolovsky (1960) and GLEM.

When a single block is used, it is a matter of course that the earth pressure obtained by GLEM completely agrees with Coulomb's, but with increase of the number of blocks the result of GLEM comes closer to the theoretical result obtained by Sokolovsky. Fig. 10 (b) shows the block systems used in the analysis and the slip surfaces obtained. In GLEM analyses, earth pressure is minimized only for the radii of interblock planes.

As mentioned before, the force field obtained by LEM is a kind of integral form of the stress field obtained by SLM. This means that stresses can be obtained by differentiation of the force field which is obtained by LEM. Fig. 11 shows the distribution of active earth pressure along a smooth vertical wall. To get the distribution, earth pressures for the wall heights 1.0, 0.8, 0.6, 0.4, 0.2 and 0.0 are analyzed by GLEM, and the differences in earth pressures are divided by the differences in wall height. It is clear that the distribution of earth pressure or bearing capacity can be determined by GLEM.

To obtain earth pressure there is no fundamental difference between the methods of Rankine and Coulomb, because Coulomb's method is the simplest case of GLEM.

CONCLUSIONS

The following conclusions have been obtained:

- 1) Limit Equilibrium Method (LEM) is an approximation method to obtain the necessary condition of Slip Line Method (SLM), if the interblock forces are adequately assumed.
- 2) The slip surface obtained by LEM does not represent a discontinuous line of the displacement field; it represents the characteristic line of the stress field, in accordance with item 1) above.
- 3) The safety factor depends on the inclination of the plane on which it is defined. In the failing zone, the safety factor has minimum value, 1.0, for the actual inclination of the plane. LEM can provide an approximate solution by optimization of the bearing capacity etc. because of this characteristic of the safety factor.
- 4) The safety factors defined on interblock planes are limited in value from F_s to ∞ , where F_s is the safety factor defined on the main slip surface. The safety factor and inclination of the interblock plane must be determined as an admissible set.
- 5) Generalized LEM (GLEM) can obtain good solutions which agree with those obtained by theoretical methods.
- 6) GLEM can obtain the distribution of earth pressure or bearing capacity. There is no fundamental difference between Rankine's method and Coulomb's method to obtain earth pressure.

Although GLEM requires knowledge and experience to construct an appropriate block system for a given problem and to give the proper safety factor on the interblock planes, this method will provide a good solution if an adequate set of block system and interblock safety factors is assumed. GLEM is as effective as ordinary LEM, and it will be useful to solve practical problems.

The program to analyze problems demonstrated here has about 700 lines in FORTRAN, including optimization routine (250 lines). Most of the authors' results were obtained using a 16 bit personal computer system (NEC PC-9801 with 8087) within a few minutes of calculation time. Some of the calculations were executed using FACOM M360-AP system at the Ehime University Computer Center.

ACKNOWLEDGEMENT

The authors wish to thank to Mr. Jayalath Edirisinghe for his assistance in preparation of this paper.

NOTATION

- β_i =Inclination angle of base plane of the i -th block from horizontal
 d_i =Length from left end of bottom plane to the acting point of P_i
 d_{wi} =Length from left end of bottom plane to the acting point of W_i
 d'_j =Length from bottom point of interblock plane to the acting point of H_j
 F_s =Safety factor defined on the main slip surface
 F_{si} =Safety factor defined on interblock slip planes (In this formulation, all of the safety factors on interblock slip planes are assumed to be equal to each other)
 $\cdot H_j$ =Normal force on the j -th interblock plane
 k =Parameter to define direction of $T_1, \dots, T_n$; $k=+1$ when blocks slide leftwards, and $k=-1$ when blocks slide rightwards
 l_j =Length of the j -th interblock plane
 l_u =Length of the u -th top plane
 m_j =Parameter to define direction of V_j ; $m_j=+1$ when base planes are concave at point j and when blocks slide leftwards
 P_i =Normal force on the i -th plane of the main slip surface
 T_i =Tangential force on the i -th plane of the main slip surface
 θ_u =Inclination angle of the u -th top plane
 θ_j =Inclination angle of the j -th interblock plane from horizontal

V_j =Tangential force on the j -th interblock plane

W_i =Weight of the i -th block

REFERENCES

- 1) Bishop, A.W. (1955): The use of the slip circle in stability analysis of slopes, *Géotechnique*, Vol. 5, pp. 7-17.
- 2) Enoki, M., Yagi, N. and Yatabe, R. (1990): Generalized slice method for slope stability analysis, *Soils and Foundations*, Vol. 30, No. 2, pp. 14-22.
- 3) Fellenius, W. (1936): Calculation of the stability of earth dams, 2nd Congress on Large Dams, pp. 445-462.
- 4) Hansen, J.B. (1966): Comparison of methods for stability analysis, Bulletin No. 21, Danish Geotechnical Institute.
- 5) Janbu, N. (1957): Earth pressure and bearing capacity calculation by generalized procedure of slices, 4th ICSMFE, Vol. 2, pp. 207-212.
- 6) Mizuno, T. (1948): On the bearing power of soil in a two-dimensional problem, 2nd ICSMFE, Vol. 3, pp. 43-48.
- 7) Mochizuki, A., Mikasa, M. and Katsuda, M. (1986): Two new slice methods for slope stability analysis, JSCE, No. 370/III-5, pp. 261-270 (in Japanese).
- 8) Morgenstern, N.R. and Price, V.E. (1965): The analysis of stability of general slip surface, *Géotechnique*, Vol. 15, pp. 79-93.
- 9) Prandtl, L. (1921): Über die eindringungsfestigkeit plastischer baustoffe und die gestigkeit von schneiden, *Z. Angew. Math. Mech.*, Vol. 1, No. 1, pp. 15-20.
- 10) Reisner, H. (1925): Zum Erddruck Problem, *Proc. 1st Conf. App. Mech.*, pp. 295.
- 11) Sokolovsky, V.V. (1960): *Statics of Soil Media*, Butterworth.
- 12) Spencer, E. (1967): A method to analysis of the stability of embankments assuming parallel interslice forces, *Géotechnique*, Vol. 17, pp. 11-26.
- 13) Vesic, A.S. (1973): Analysis of ultimate loads of shallow foundations, *ASCE*, Vol. 99, No. SM1, pp. 45-73.
- 14) Whitman, R.V. and Bailey, W.A. (1967): Use of computer for slope stability analysis, *ASCE*, No. SM4, pp. 475-498.
- 15) Yamaguchi, H. (1984): *Soil Mechanics*, Gihodo Shuppan, pp. 270 (in Japanese).