MODIFIED STRESS TENSORS FOR ANISOTROPIC BEHAVIOR OF GRANULAR MATERIALS

YOSHIKO TOBITA and EIJI YANAGISAWA

ABSTRACT

Effects of fabric anisotropy on the mechanical behavior of granular materials have become well known. Without the introduction of variables describing fabric anisotropy into the constitutive formulation, some important deformation features of granular materials can not be formulated in a reasonable way. Although the principle of material objectivity and the thermodynamical restrictions in the continuum mechanics give the framework to a class of reasonable constitutive equations associated with anisotropic features, the framework is still too wide to obtain a practical constitutive equation. This fact indicates the necessity of conventional methods within the fundamental restrictions. Modified stress tensors being a consequence of the linear transformation depending on fabric anisotropy from the stress tensor are proposed and discussed, with a particular emphasis on the elasto-plastic behavior of anisotropic granular materials. It is shown that: (a) classical hardening models (isotropic, kinematic, and combined) are obtained as special cases of the present method; (b) the shape change of yield surfaces during plastic deformation is a natural consequence if relevant terms are included in the transformation rule. Anisotropic deformation and strength behaviors of granular materials are formulated based on the modified Drucker–Prager yield condition and the non-associated flow model for two dimensional problems in order to demonstrate the effectiveness of the proposed method.

Key words: anisotropy, constitutive equation of soils, granular material, modified stress tensor, yield (IGC: D6)

INTRODUCTION

Effects of inherent and induced anisotropy on the deformation and failure behavior of granular materials have received much attention in recent years (see, e.g., Oda, 1976; Ishihara and Okada, 1978; Nemat-Nasser and Tobita, 1982; and Wong and Arthur, 1985 among others). Without the introduction of variables describing fabric anisotropy into the constitutive formulation, some recently documented experimental results can not be prop-

1) Associate Professor, Department of Civil Engineering, Hachinohe Institute of Technology, Myo, Hachinohe, 031.
2) Professor, Department of Civil Engineering, Tohoku University, Aoba, Aramaki, Sendai, 980. Manuscript was received for review on December 4, 1989. Written discussions on this paper should be submitted before October 1, 1992, to the Japanese Society of Soil Mechanics and Foundation Engineering, Sugayama Bldg. 4 F, Kanda Awaji-cho 2-23, Chiyoda-ku, Tokyo 101, Japan. Upon request the closing date may be extended one month.
erly taken into account (Tobita, 1989). For example, the non-coaxial and dilatant behavior of sands during the rotation of principal stress directions are typical features (see, e.g., Miura et al., 1986; Gutierrez, 1989 for experimental results). The development of the method taking into account the fabric anisotropy in a proper manner is of importance for further developments in the constitutive equations of granular materials.

When a proper set of variables including anisotropic properties is selected, the most general framework of the constitutive formulation may be obtained by the use of two well known restrictions on the non-linear constitutive equation: (a) the principle of material objectivity; (b) thermodynamical constraints, in particular, the Clausius-Duhem inequality (see, e.g., Malvern, 1969; Eringen, 1971; Tokuoka, 1980; and Kitagawa, 1987). Although these two restrictions are useful in the formal discussion of constitutive formulations, the framework thus obtained is still too wide to obtain a practical constitutive equation; with the result that we still need a convenient method in which fabric anisotropy is embodied in a proper manner.

The aim of this paper is to discuss a convenient and simple method for the development of anisotropic constitutive formulations, with particular emphasis on the anisotropic elastoplastic deformation and strength characteristics. In the method discussed here, the modified stress tensor is used in order to take into account fabric anisotropy. From a physical consideration on the state of granular materials along the inclined plane with respect to principal stress directions, Tobita and Yanagisawa (1985, 1988) and Tobita (1987, 1988) developed a modified stress tensor being of the form

\[ T_{ij} = (F)^{-1} \sigma_{ikj}^{-1} \sigma_{kj} \]  

(1)

where \( \sigma_{ij} \) denotes the Cauchy stress tensor in the small strain theory. \( (F)^{-1} \) is the inverse of \( F_{ik} \); \( F_{ik} \) was called the transformation tensor defined as \( F_{ij} = 2C_{ijkl} \text{tr}(C) \), and \( C_{ijkl} \) is the contact tensor. The physical meanings of the modified stress tensor have been discussed in the above papers; and they are not repeated here. Similar modified stress tensors to Eq. (1) were also proposed by Satake (1983) and Nemat-Nasser and Mehrabadi (1983) following different approaches.

When the principal axes of the transformation and stress tensors are not coincident, the modified stress tensor fails to be symmetric. In the framework of the small strain theory and nonporal material excluding the couple stress, the nonsymmetric part of the modified stress tensor becomes of little importance. We hence consider the symmetric modified stress tensor defined by

\[ T_{ij} = \frac{1}{2} \left[ (F)^{-1} \sigma_{ikj}^{-1} + \sigma_{ikj} (F)^{-1} \right] \]  

(2)

In the following section, we will develop a set of general modified stress tensors based on the representation theorem of isotropic functions. The modified stress tensor defined by Eq. (2) will be called the original modified stress tensor. We will use the notation \( H_{ij} \) to indicate the fabric tensor in general; which can also include the inverse matrix of the physically obtained fabric tensors.

The anisotropic yield function in stress space being of primary importance in the elastoplastic constitutive formulation is obtained by following two steps:

(a) First, we consider a yield function proposed for isotropic materials up to now, e.g., Drucker and Prager (1952), Lade and Duncan (1975), and Matuoka and Nakai (1977) among others in modified stress space

(b) We then substitute the stress tensor and the fabric tensor into the modified stress tensor, the relation of which is given as the fabric dependent linear transformation rule from stress space to modified stress space.

This method was used, in a slightly different manner, by Boehler (1987) for the extension of the von Mises yield condition to include the effects of fabric (structure) anisotropy of plastically incompressible materials. Boehler used the terminology of the transformed stress tensor; we, however, here follow the terminology used in the mechanics of granular materials (see Satake, 1989 for reviews).

After discussing briefly a general linear
transformation rule from stress space to modified stress space based on the representation theorem of isotropic functions, we will observe that the classical hardening (isotropic, kinematic, and combined) rules are obtained as special cases of this method. Restricting our attention to two dimensional problems, we present specific results on the anisotropic extension of the Drucker and Prager yield function and the anisotropic elasto-plastic behavior of granular materials in order to demonstrate the effectiveness of the method.

Throughout this paper we mainly use the indicial notation with respect to a rectangular Cartesian coordinate system $x_i$ with corresponding orthonormal bases $e_i$ ($i=1, 2, 3$). The summation convention over repeated indices in a term is always operated unless otherwise indicated. Direct notation for the trace operation e.g., $tr(T) = T_{kk}$, $tr(He) = H_{kk} \sigma_{kk}$ etc. are also used partly. In the Appendix, the definitions of important parameters used in this paper are given. The usual sign convention in the continuum mechanics that tensile stresses and strains are regarded as positive is employed in this paper; since the opposite sign convention that compression is taken as positive commonly used in soil mechanics leads to confusion when we are concerned with other tensorial quantities in addition to stresses and strains.

GENERAL MODIFIED STRESS TENSOR

In this section we consider the general modified stress tensor which is obtained by the fabric dependent linear transformation from stress space. The principle of material objectivity (or material frame indifference) requires that the equation expressing physical relations must be invariant under the change in observers, which is found to be an equivalent statement that the tensor function must be an isotropic function of considered tensorial variables (see, Yatomi and Nishihara, 1984; Boehler, 1987; and Tobita, 1989, for more details).

For a symmetric second order tensor function $F_{ij}$, for example,

$$T_{ij} = F_{ij}(\sigma_{ki}, H_{kl})$$

$F_{ij}$ is called an isotropic function, if and only if it satisfies the equation

$$Q_{\alpha i} F_{ij}(\sigma_{kl}, H_{kl}) = F_{a\beta}(Q_{\alpha i} \sigma_{ij} Q_{\beta j}, Q_{\alpha i} H_{ij} Q_{\beta j})$$

(4)

to all orthogonal tensors $Q_{\alpha i}$ (i.e., $Q_{\alpha i} Q_{\beta i} = \delta_{\alpha \beta}$), where $\delta_{\alpha \beta}$ is the Kronecker delta ($=1$ for $\alpha=\beta$, $=0$ in others). The representation theorem of isotropic tensor functions leads to the form of

$$T_{ij} = \sum k a_k (G_k)_{ij}$$

(5)

where $a_k$ are isotropic scalar functions satisfying

$$f(Q_{\alpha i} \sigma_{ij} Q_{\beta j}, Q_{\alpha i} H_{ij} Q_{\beta j}) = f(\sigma_{ij}, H_{ij})$$

(6)

to any orthogonal tensors $Q_{\alpha i}$. $a_k$ are functions of basic invariants formed by $\sigma_{ij}$ and $H_{ij}$. $(G_k)_{ij}$ are irreducible generators. The list of these invariants and generators is available in the works by Wang (1970), Smith (1971), Boehler (1987), and for a concise list Tobita (1989).

We require the transformation rule to be linear on the stress tensor for the sake of mathematical simplicity, we consider

$$T_{ij} = L_{ijkl} (H) \sigma_{kl}$$

(7)

and it is a class of isotropic tensor functions. The use of the representation theorem of isotropic functions leads to the form

$$T_{ij} = [a_{0} tr(\sigma) + a_{2} tr(\sigma H) + a_{4} tr(H^2 \sigma)] \delta_{ij}$$

$$+ [a_{0} tr(\sigma) + a_{2} tr(\sigma H) + a_{6} (H^3 \sigma)] H_{ij} + [a_{0} tr(\sigma) + a_{2} tr(\sigma H) + a_{4} tr(H^2 \sigma)] H_{ik} H_{kl} + a_{0} \sigma_{ij} + a_{1} (\sigma_{ik} H_{kj} + \sigma_{kj} H_{ik})$$

$$+ a_{1} (\sigma_{ik} H_{kj} + H_{ij} \sigma_{kl}) + a_{1} (\sigma_{ik} H_{kj} + H_{ij} \sigma_{kl})$$

(8)

where $a_k$ are isotropic scalar functions of three invariants of the fabric tensor obtained from the characteristic equation: $I_1$, $I_2$, $I_3$ (they are defined as $H_{kk}$, $(1/2) (H_{ii} H_{jj} - H_{pq} H_{pq})$, $(1/3) (H_{kk} H_{kk} H_{kk})$ respectively. Similar invariants are also used for the modified stress tensor); or equivalently functions of $tr(H)$, $tr(H^2)$, $tr(H^3)$. The symmetric second order tensor specified by Eq.(8) will be called the general modified stress tensor.

SIMPLIFIED TRANSFORMATION RULES

The mathematical form of the general modified stress tensor is rather complex for practical purposes; and simplified transfor-
formation rules still preserving important features are required. One of the important features of the method discussed here is the fact that classical hardening rules can be obtained as special cases. In this chapter we discuss two simple transformation rules leading to classical hardening rules. It will also be discussed that the original modified stress tensor, irrespective of its simple form, can account for shape changes of the yield surface involved in the general modified stress tensor.

Isotropic Transformation

In Eq. (8), we are concerned with the following transformation rule:

$$T_{ij} = a_{10} tr(\sigma) \delta_{ij} + a_{10} \sigma_{ij}$$  \hspace{1cm} (9)

In this case, when we select the coordinate system to be coincident with the principal axes of the stress tensor, the corresponding equation becomes

$$T_a = a_{10} tr(\sigma) + a_{10} \sigma_a (a=1,2,3)$$  \hspace{1cm} (10)

where $\sigma_a$ denotes the principal values of $\sigma_{ij}$. Eq.(10) indicates that the principal axes of the stress tensor is unaltered after the isotropic transformation. By applying the trace operation to Eq.(10), we have

$$tr(T) = (3a_{10} + a_{10}) tr(\sigma)$$  \hspace{1cm} (11)

The deviatoric part of the modified stress tensor is defined by

$$T^d_{ij} = T_{ij} - \frac{1}{3} tr(T) \delta_{ij}$$  \hspace{1cm} (12)

and it becomes

$$T^d_{ij} = a_{10} s_{ij} : s_{ij} = \sigma_{ij} - \frac{1}{3} tr(\sigma) \delta_{ij}$$  \hspace{1cm} (13)

$s_{ij}$ denotes the stress deviator. Without loss of important features of the modified stress tensor, one may set $a_{10} = 1$, since in most cases we are concerned with the relative magnitudes of the modified stress tensor. We thus have

$$tr(T) = (3a_{10} + 1) tr(\sigma)$$

$$T^d_{ij} = s_{ij}$$  \hspace{1cm} (14)

where $a_1$ is an isotropic scalar function of $tr(H)$, $tr(H^2)$, $tr(H^3)$.

The isotropic transformation rule hence yields the modified stress tensor with following features:

(a) The principal axes of the modified stress tensor are identical with those of the stress tensor: i.e., no rotation of principal stress directions occurs by this transformation;

(b) The deviatoric part of the modified stress tensor may be set to be equal to the stress deviator.

A simpler case may be obtained by setting

$$T_{ij} = a_{10} \sigma_{ij}$$  \hspace{1cm} (14)

which is just a scalar multiplication of all components of the stress with an equal magnitude. The transformation rule specified by Eq.(14) was called “the homothetic isotropic transformation” by Boehler (1987).

Translational Transformation

Adding from Eq.(8) to the isotropic transformation, we have

$$T_{ij} = a_{10} tr(\sigma) \delta_{ij} + a_{10} \sigma_{ij}$$

$$+ a_{10} tr(\sigma) H_{ij} + a_{10} tr(\sigma) H_{ik} H_{kj}$$  \hspace{1cm} (15)

It is apparent that the principal axes of the modified stress are altered after this transformation from those of the stress tensor unless the coaxiality between the stress and fabric tensors is presumed. Applying the trace operation to Eq.(15), we have

$$tr(T) = (3a_{10} + a_{10} + a_1 tr(H))$$

$$+ a_{10} tr(H^2)) tr(\sigma)$$  \hspace{1cm} (16)

The deviatoric part of the modified stress tensor becomes by direct calculation

$$T^d_{ij} = a_{10} (\sigma_{ij} - \frac{1}{3} tr(\sigma) \delta_{ij})$$

$$+ a_{10} tr(\sigma) (H_{ij} - \frac{1}{3} tr(H) \delta_{ij})$$

$$+ a_{10} tr(\sigma) (H_{ik} H_{kj} - \frac{1}{3} tr(H^2) \delta_{ij})$$  \hspace{1cm} (17)

Letting $a_{10}$ be equal to 1 again, we have

$$T^d_{ij} = s_{ij} - a_{ij}$$

$$a_{ij} = - tr(\sigma) [a_1 H_{ij}^d + a_1 (H^2)^d_{ij}]$$  \hspace{1cm} (18)

in which the quantities with superscript $d$ denotes the deviatoric part of the corresponding
tensor. (Note that $a_{ij}$ has a dimension of the stress tensor, since the fabric tensor can be normalized to be non dimensional quantity).

We thus have the modified stress tensor, by the translational transformation rule, with following features:

(a) The principal axes of the modified stress tensor in the translational transformation are rotated with respect to those of the stress tensor unless the coaxiality between $\sigma_{ij}$ and $H_{ij}$ is satisfied;

(b) The deviatoric part in modified stress tensor is given in a form that the stress deviator is subtracted by the magnitude of $a_{ij}$.

As is well known in the kinematical hardening model assumption, the translational rule due to Eq. (18) can lead to the rotation and the translation of yield functions in deviatoric space; however, it can not yield a shape change during plastic deformation. The key role of the shape change in the present method using modified stresses is played by the terms of mixed invariants, e.g., $tr(\mathbf{H}\sigma)$, $tr(\mathbf{H}^2\sigma)$ (when associated generators are not $\delta_{ij}$) as well as mixed generators, e.g., $\sigma_{ik}H_{kj} + H_{ik}\sigma_{kj}$, $\sigma_{ik}H_{kj}H_{ij} + H_{ik}H_{kj}\sigma_{ij}$. This property can be ensured by direct calculation as will be discussed in the following examples, and we here omit the formal discussion of this property. It is worth noting that the original modified stress tensor is defined by the mixed generators in the simplest form; and it yields to the shape change of yield surfaces depending on the relative magnitudes between the stress and fabric tensors.

**ANISOTROPIC YIELD FUNCTION BASED ON MODIFIED STRESS METHOD**

**Fundamental Procedure**

In order to obtain an anisotropic yield function, we first consider the isotropic yield function, being of frequent use in the isotropic hardening model, in *modified stress space*. It is expressed in terms of three invariants of the modified stress tensor $I^1$, $I^2$, $I^3$ (they are defined in similar forms to those of the fabric tensor), or equivalently by the form

$$f(tr(T), tr(T^3), tr(T^5)) = C$$  \hspace{1cm} (19)

We then replace the modified stress tensor in terms of the stress tensor and the fabric tensor by a simple substitution of the transformation rule specified by

$$T_{ij} = L_{ijkl}(\mathbf{H})\sigma_{kl} \hspace{1cm} (20)$$

into the isotropic yield function: Eq. (19)

Following these two steps we obtain an anisotropic yield function in ordinary stress space, the properties of which depend on the function from of $f$ in modified stress space as well as on the transformation rule. The resultant function may be of the form

$$f^*(tr(\sigma), tr(\sigma^2), tr(\sigma^3),
tr(\mathbf{H}), tr(\mathbf{H}^2), tr(\sigma\mathbf{H}), \ldots) = C \hspace{1cm} (21)$$

The effectiveness of the method will be judged by the degree of coincidence with experimental observations. We hereafter call the method using the modified stress tensors for the expression of anisotropic properties of materials as the "modified stress method". It is worth noting that the modified stress method is also applicable to the tensor function as will be used in the definition of an anisotropic elastic relation (Eq. (55)).

Similar but a slightly different method was used to obtain the anisotropic failure criterion of granular materials by Oda and Nakayama (1989). Cowin (1986) also discussed anisotropic failure criteria taking into account the fabric anisotropy.

**Invariance of Anisotropic Yield Function**

The yield function used in the elasto-plasticity model must be an isotropic scalar function with respect to its arguments (a set of selected tensorial variables). In order to verify the invariance property of the anisotropic yield function based on the modified stress method, we rewrite Eq. (19) in the form

$$f(Q_{ij}T_{ij}Q_{ij}) = f(T_{ij}) \hspace{1cm} (22)$$

It is well known that Eq. (22) is an equivalent statement to Eq. (19) if material is considered to be isotropic with respect to $T_{ij}$. Since the modified stress tensor is obtained as a consequence of the representation theorem of isotropic functions on the symmetric tensor
valued functions, we have
\[ Q_{ij} T_{ij} = T_{ij} (Q_{ij} \sigma_{ij} Q_{ij}, Q_{ij} H_{ij} Q_{ij}) \]  
(23)
Substituting from Eq. (23) into Eq. (22), we have
\[ f^*(Q_{ij} \sigma_{ij} Q_{ij}, Q_{ij} H_{ij} Q_{ij}) = f^*(\sigma_{ij}, H_{ij}) \]  
(24)
Eq. (24) is the mathematical expression of the invariance property of scalar functions with respect to \( \sigma_{ij} \) and \( H_{ij} \). We hence understand that the anisotropic yield function obtained through the modified stress method always keeps the invariance property.

**Classical Hardening Models**

The mathematical formulation of classical hardening models may be classified into three fundamentals (see, e.g., Desai and Siriwandane, 1984): (a) isotropic hardening; (b) kinematic hardening; and (c) combined (isotropic+kinematic) hardening. Although the discussions in what follows can be formally generalized so as to be applicable to any types of yield functions proposed for granular materials, we are hereafter concerned with the yield functions with the feature of linear dependency on the hydrostatic pressure taking the mathematical form
\[ f(s_{ij}) - \alpha P = 0 \]  
(25)
or equivalently, we can take the form
\[ f(J^s, J^s) - \alpha P = 0 \]  
(26)
In Eq. (25), \( s_{ij} = s_{i} - \alpha_{ij} \); \( \alpha_{ij} \) is the so called back stress tensor with \( tr(\alpha) = 0 \). \( P \) is the hydrostatic pressure defined as \( P = -\frac{1}{3} tr(\sigma) \). and is positive in compression. \( J^s = (1/2) \sum_{i=1}^{n} s_{ij} s_{ij} \) and \( J^s = (1/3) (s_{ii} s_{ii} s_{ii} \alpha_{ij} \alpha_{ij} \alpha_{ij}) \) are second and third invariants of \( s_{ij} \). Many of the proposed yield functions take the form of Eq. (26). The application of the modified stress method to other types (e.g., of Cap type) can be performed in an exactly similar manner to the following illustrative examples.

**Isotropic Hardening**

Isotropic hardening model can be obtained by the isotropic transformation rule. Substitution from Eq. (13) into the isotropic yield function in modified stress space
\[ f(T^s_{ij}) - \alpha P^* = 0 ; \quad P^* = -\frac{1}{3} tr(T) \]  
(27)
yields
\[ f(s_{ij}) - \alpha (3a_1 + 1) P = 0 \]  
(28)
In the above equations \( \alpha \) is a hardening parameter being a function of e.g., an accumulated equivalent shear strain. Eq. (28) is a mathematical expression of general isotropic hardening models.

The isotropic hardening model is also obtained by the homothetic isotropic transformation (Eq. (14)). In this case we have
\[ f(s_{ij}) - \alpha P = 0 \]  
(29)
Eq. (29) also shows an isotropic hardening model when the yield function hardens in modified stress space (see, Path A in Fig. 1.

![Stress Space](image-url)

**Fig. 1. Hardening behavior of yield functions based on the modified stress method:**
- **Path A:** The isotropic transformation rule (Eq. (9)) yields the isotropic hardening;
- **Path B:** The translational transformation rule (Eq. (15)) yields the combined hardening rule;
- **Path C:** The transformation rules including the mixed invariants and mixed generators yields the shape change.
for a schematic representation of the above discussion.

The difference between the isotropic and homothetic isotropic transformation becomes apparent when we consider an ideal plastic material (with no hardening). In this case the homothetic isotropic transformation rule yields an ideal plastic material, while the isotropic transformation rule shows hardening if the fabric tensor shows an evolution during elasto-plastic deformation, because of the fact that \( a_1 \) changes its value as a result of the change of fabric tensor.

Since kinematic hardening models in deviatoric stress space is of less importance in the discussions of the yield function for frictional granular materials, we next consider the combined hardening model.

**Combined Hardening Model**

Applying the translational rule (Eqs. (16), (18)) to Eq. (27), we have

\[
f(s_{ij} - a_{ij}) - \alpha KP = 0 \tag{30}
\]

where \( a_{ij} = - \{a_j \text{tr}(\sigma) H_{ij} + a_i \text{tr}(\sigma) (H^2)^{ij}\} \) and \( K = \{3a_1 + a_j \text{tr}(H) + a_i \text{tr}(H^2)\} \). Eq. (30) may be considered to be a class of the general combined hardening model (see Path B in Fig. 1).

Another transformation rules different from isotropic and translational rules can also lead to the shape changes of yield surfaces in stress space as a consequence of the introduction of mixed invariants and generators. The property of shape changes in yield surfaces is a necessary feature for further developments of the constitutive formulations along the spirit of elasto-plasticity (see Path C in Fig. 1).

In following sections, we will discuss various properties of the modified Drucker–Prager yield condition and the elasto–plastic behavior of the non–associative model in which the original modified stress tensor is used for the illustrative purpose.

**STRENGTH ANISOTROPY OF GRANULAR MATERIALS**

As an illustrative example, we are hereafter concerned with the case specified by: (a) Drucker–Prager yield condition; (b) the original modified stress tensor; and (c) two dimensional problems. Drucker–Prager yield condition for cohesionless isotropic granular materials takes the form

\[
\sqrt{(s_{ij}s_{ij})/2} - \alpha P = 0 \quad (i,j=1,2) \tag{31}
\]

where \( \alpha \) is a scalar valued function of plastic deformation history; and \( P = -\text{tr}(\sigma)/2 \) for two dimensional problems.

The original modified stress tensor is expressed by

\[
T_{ij} = \frac{1}{2}(\sigma_{ik} H_{kj} + H_{ik} \sigma_{kj}) \tag{32}
\]

where \( H_{ij} = (F)_{ij}^{-1} \) denotes the inverse of the transformation tensor. The corresponding representation in the coordinate system being coincident with the principal axes of the fabric tensor becomes in the component form

\[
T_{11} = h_1 \sigma_{11}, \quad T_{22} = h_2 \sigma_{22}, \\
T_{12} = \frac{1}{2}((h_1 + h_2) \sigma_{12}) \tag{33}
\]

We introduce a new stress system for a convenience of discussion, which takes the form

\[
P = -\frac{1}{2}(\sigma_{11} + \sigma_{22}), \quad Q_1 = \frac{1}{2}(\sigma_{11} - \sigma_{22}), \quad Q_2 = \sigma_{12} \tag{34}
\]

These stress measures are found to show orthogonality in three dimensional representation as shown in Fig. 2 (see, Anand, 1983, for the proof), which is very convenient for the graphical representation of two dimensional problem. The corresponding stress measures in modified stress space is distinguished by adding the superscript asterisk, we thus define

\[
P^* = -\frac{1}{2}(T_{11} + T_{22}), \quad Q_{1}^* = \frac{1}{2}(T_{11} - T_{22}), \\
Q_{2}^* = T_{12} \tag{35}
\]

Substitution from Eq. (33) into Eq. (35) yields the following form:

\[
P^* = H_p P - H_q Q_1, \\
Q_{1}^* = -H_q P + H_p Q_1, \\
Q_{2}^* = H_p Q_2 \tag{36}
\]

where \( H_p \) denotes \((h_1 + h_2)/2\); \( H_q = (h_1 - h_2)/2\).
In the new stress system, Drucker–Prager yield condition can read

$$\sqrt{\left[\left(Q_1^*\right)^2 + \left(Q_2^*\right)^2\right]} - \alpha^*P^* = 0$$  \hspace{1cm} (37)

Substitution from Eq. (36) into Eq. (37) yields

$$\sqrt{\left(-H_0P+H_QQ_1\right)^2 + \left(H_0P-H_QQ_1\right)^2} - \alpha^*\left(H_0P-H_QQ_1\right) = 0$$  \hspace{1cm} (38)

In the sequel, $H_0$ is set to be 1 for simplicity. Assuming that the principal axes of the stress tensor are coincident with those of the fabric tensor, we have

$$\left(-H_0P+Q\right) - \alpha^*(P-H_0Q) = 0$$  \hspace{1cm} (39)

Eq. (39) represents the triangular shape passing through the origin $O$ in the coordinate system in which $P$ and $Q$ are taken to be orthogonal (see Fig. 2 for the specification of $P-Q$ and $Q_1-Q_2$ diagrams). It is easily understood from Fig. 3 expressing Eq. (39) schematically that the resultant shape clearly indicates the feature of anisotropic functions; i.e., a biased shape with respect to the hydrostatic axis $P$ (Note that the degree of biased shape depends on the relative magnitude of the stress ratio and inherent anisotropy: As the stress ratio increases, the degree of biased shape decreases).

Keeping the hydrostatic pressure $P$ to be constant (denoted as $P_0$), Eq. (38) becomes a quadratic equation expressing an ellipse on $Q_1$ and $Q_2$.

$$CQ_1^2 + 2DQ_1Q_2 + Q_2^2 = EP_0^2$$  \hspace{1cm} (40)

where $C = 1 - (\alpha^*H_0)^2$, $D = H_0H_0(\alpha^*H_0-1)$, $E = \alpha^*H_0^2 - H_0^2$.

Since we do not have a reliable evolution model for the fabric tensor yet, we hereafter confine our attention to the effect of inherent anisotropy on the strength of granular materials. The strength criterion of granular materials in the two dimension may be given by

$$\sin\phi' = \frac{\sqrt{Q_1^2 + Q_2^2}}{P} = \frac{\bar{Q}}{P_0}$$  \hspace{1cm} (41)

Using the standard transformation rule with respect to the second order tensor, we have

$$Q_1 = \bar{Q}\cos(2\phi')$$
$$Q_2 = \bar{Q}\sin(2\phi')$$

$$\bar{Q} = \sqrt{Q_1^2 + Q_2^2}$$  \hspace{1cm} (42)

where $\phi'$ denotes the angle defined by $\cos\phi' = \frac{Q_1}{\bar{Q}}$.
Fig. 4(a). Anisotropic strength of granular materials in the relations maximum stress ratio and the deviation angle \( \Phi \) (with \( h_1 = 1.20 \) and \( h_2 = 0.80 \))

\[ (e^*_1, e_1) \] where \( e^*_1 \) is a unit vector in the direction of \( \sigma_1 \); and \( e_1 \) is that in \( h_1 \), dot denotes an inner product. Under the hydrostatic pressure constant condition, the strength anisotropy is expressed by the dependency of \( Q \) on \( \Phi \). Substitution from Eq. (42) into Eq. (40) results in

\[ N_1 \bar{Q}^2 + N_2 \bar{Q} + N_3 = 0 \]  

(43)

where \( N_1 = C \cos^2 (2\Phi) + \sin^2 (2\Phi) \), \( N_2 = 2D \cos (2\Phi) \), \( N_3 = -EP_3^3 \). Solving Eq. (43), we can obtain the strength anisotropy as a function of \( \Phi \). Fig. 4(a) shows the calculation results for the case that \( h_1 = 1.20, h_2 = 0.80 \). These values are taken for the purpose of illustration without strict microscopic observations despite of the fact that \( H_i \) can have a close connection with microfabrics. Further studies will be required to determine the fabric tensor to be used for the determination of the modified stress tensor in both aspects from macroscopic stress strain behavior and microscopic observations.

\( \alpha^* \) was obtained as 0.854 so as to give \( \sin \Phi = 0.9 \) at \( \Phi = 0 \) degree in order to coincide with the result by Miura (1985) and Miura et al. (1986). It should be noted that the calculation corresponds to the case where the unit normal to the bedding plane is in \( e_2 \) direction.

The qualitative agreement can be considered as satisfactory (Note that closer agreements can be obtained for another sets of \( H_1 \) and \( H_2 \); but the drop of strength at around \( \Phi = 60 \) degrees is not represented by the calculation. However, the drop of strength is still a controversial feature depending on the test condition and materials (see, e.g., Gutierrez, 1989). It is believed that the discontinuous drop of strength, if any, must not be treated by employing an elaborate function in the transformation rule, since they may be a different micromechanism leading to the discontinuous drop of strength at around \( \Phi = 60 \) degrees from other angles.

Referring to Fig. 4(b), we can easily understand that the strength anisotropy is embodied mainly by the shift of the center of the ellipse from the origin to \( Q_1 > 0 \) direction. By the shift of the center of the yield surface to \( Q_1 > 0 \), the value of \( \bar{Q} \) becomes large when \( \Phi \) is small.

Fig. 5 shows various yield surfaces under the condition that \( \sigma_1 \) and \( \sigma_2 \) are fixed to be equal to \( \bar{Q} = 0.6 \) (marked as A) and \( \Phi \) has different values. When a fixed set of stresses is applied, the elastic region enclosed by the yield surface becomes larger as \( \Phi \) increases. The increase of \( \Phi \) indicates the situation that a fixed set of stresses is applied to weaker directions; and we hence obtain wider elastic regions since other stronger directions in
fabrics can sustain higher sets of stresses. This result also indicates the effect of anisotropy on the yield behavior of granular materials. We also note that when principal axes of the fabric tensor are not coincident with those of the stress tensor ($\Psi = 30$ and 60 degrees), the center of the yield surfaces also shifted in $Q_2$ direction as is found from close inspections on Fig. 5.

**ANISOTROPIC STRESS STRAIN BEHAVIOR OF GRANULAR MATERIALS WITH INHERENT ANISOTROPY**

As an other example, we simulate the anisotropic stress strain behavior of granular materials with inherent anisotropy. The relative directions of the principal axes of the stress tensor and the fabric tensor are defined again by $\Psi$. The situation discussed here is similar to the last section. We consider the anisotropic behavior when the specimen is subjected to monotonous loadings up to the specified stress ratio while the relative directions of the two tensors are fixed; i.e., $\Psi$ is fixed and no rotation of principal stress axes occurs during loadings. The mathematical formulation of the constitutive equation was carried out based on the non-associative flow rule and the extended stress dilatancy equation with the aid of the modified stress method.

**Non Associated Flow Model with Anisotropic Property**

In order to take into account the effect of inherent anisotropy, we consider the original modified stress tensor (Eq. (32)). The plane strain condition is here treated as a two dimensional problem by supposing that the effects of the plane strain condition are properly taken into account in the definitions of material parameters. In this section we follow the tensorial formulation instead of $P, Q_1, Q_2$. We, however, use these measures to discuss specific properties of the developed constitutive model in what follows.

The yield surface is taken to be given by the Drucker-Prager yield condition (Eq. (31)) in terms of the original modified stress tensor: i.e.,

$$f = \sqrt{T_{i j} T_{i j}} - \alpha^* P^* = 0$$

where $T_{i j}$ denotes the deviatoric part of $T_{i j}$ and $P^* = - \text{tr}(T)/2$ (positive in compression). The assumption of deviatoric normality flow rule (i.e., the deviatoric plastic strain rate is proportional to the deviatoric part of the gradient of the yield surface $(\partial f/\partial \sigma_{i j})$) is expressed as

$$\dot{\varepsilon}_{i j} = \dot{\varepsilon}_{i j} = N_{i j}$$

$$\left\{ \begin{array}{l}
\dot{\varepsilon}_{i j} = (\partial f/\partial \sigma_{i j}) \dot{\sigma}_{i j} > 0 \\
\dot{\varepsilon}_{i j} = 0 \text{ in other cases}
\end{array} \right. \quad (45)$$

where $\dot{\varepsilon}_{i j} = (1/2) \text{tr}(\dot{N}) \dot{\sigma}_{i j}$ with $N_{i j} = (\partial f/\partial \sigma_{i j})$. Using the chain rule of differentiation, we have

$$N_{i j} = (\partial f/\partial T_{k l}) (\partial T_{k l}/\partial \sigma_{i j})$$

$$= (\partial f/\partial T_{k l}) L_{k l i j} \quad (46)$$

where $L_{k l i j}$ and $(\partial f/\partial T_{k l})$ are expressed as follows:

$$L_{k l i j} = (\partial \sigma_{i j} H_{k l} + H_{k l} \partial \sigma_{i j})/2$$

$$\frac{\partial f}{\partial T_{k l}} = \frac{T_{k l}^{\alpha}}{\tau_s} - \frac{\alpha^*}{2} \partial T_{k l} \quad (48)$$

with the definition of $\tau_s = (T_{i j} T_{i j}/2)^{1/2}$. As may be understood from Eqs. (46), (47) and (48), the required mathematical operation for the anisotropic description is embodied in the fourth order tensor $L_{k l i j}$.

In order to give a dilatancy component in
terms of the deviatoric plastic component, we follow the formulation obtained from experimental observations (Gutierrez, 1989)

\[ s_{1j} \dot{\gamma}_{1j} - P \dot{\gamma}_{kk} = PM \dot{\gamma} \]  \hspace{1cm} (49)

where \( \dot{\gamma} = (\dot{\gamma}_{1j} \dot{\gamma}_{1j})^{1/2} \). \( M \) denotes a material constant relating to the frictional property. Eq. (49) was found to be applicable even to the case in which the non-coaxiality between the plastic strain rate and stress tensors is involved. In Eq. (49), it is also assumed that there is no effect of inherent anisotropy on the stress dilatation equation, which has been supported experimentally. The use of Eqs. (45) and (49) yields

\[ tr(\dot{\gamma}) = \dot{\theta} \tan \nu \]

\[ = \lambda \left\{ M(N_{1j} N_{1j})^{1/2} - \frac{1}{P} (N_{1j} N_{1j}) \right\} \]  \hspace{1cm} (50)

\( \lambda \) involved in Eqs. (45) and (50) can be defined through the consistency condition of yield functions; i.e.,

\[ f = N_{1j} \dot{\sigma}_{1j} + (\partial f / \partial \alpha^*) (\partial \alpha^* / \partial \gamma) \chi = 0 \]  \hspace{1cm} (51)

Using Eq. (45) and the definition of \( \tau^* \), we obtain

\[ \dot{\sigma}_{1j} = (N_{1j} \dot{\sigma}_{1j}) / \left( P (\partial \alpha^* / \partial \gamma) (N_{1j} N_{1j})^{1/2} \right) \]  \hspace{1cm} (52)

Combining Eqs. (45), (50), and (52) and the definition of \( \dot{\varepsilon}_{1j} = \dot{\varepsilon}_{1j} + (1/2) tr(\dot{\gamma}) \delta_{ij} \), we obtain

\[ \dot{\varepsilon}_{1j} = \frac{1}{H} P_{ij} N_{kk} \]  \hspace{1cm} (53)

with \( P_{ij} = N_{1j}^{1/2} + (\tan \nu / 2) \delta_{ij} \), \( H = P^* (\partial \alpha^* / \partial \gamma^*) (N_{1j} N_{1j})^{1/2} \)

The remaining formulation is the definition of hardening equation between \( \alpha^* \) and the equivalent shear strain \( \gamma^* \). In the constitutive model, we employ the hyperbolic relation in the form

\[ a^* = \gamma^*/(a + b \gamma^*) \text{ with } \]  \hspace{1cm} (54)

\[ (\partial \alpha^* / \partial \gamma^*) = a/(1 - b \alpha^*)^2 \]

where \( a \) and \( b \) are material parameters.

**Anisotropic Elastic Relation**

In order to define the elastic behavior, we first consider the isotropic elastic relation between the rate of modified stress tensor and the elastic strain rate

\[ \dot{\varepsilon}_{ij} = \frac{1}{4K} \dot{T}_{kk} \delta_{ij} + \frac{1}{2G} \dot{T}_{ij} \]  \hspace{1cm} (55)

where \( K \) and \( G \) are elastic bulk and shear moduli, and \( \dot{T}_{ij} \) is defined as

\[ \dot{T}_{ij} = (\partial T_{ij} / \partial \sigma_{kl}) \partial_{kl} = L_{ijkl} \partial_{kl} \]  \hspace{1cm} (56)

Using the equations derived above, the final elasto-plastic constitutive equation for two dimensional problems becomes of the from

\[ \dot{\varepsilon}_{ij} = C_{ijkl} \partial_{kl} \]  \hspace{1cm} (57)

**Numerical Simulation**

The material parameters used for the numerical simulations are listed in Table 1. No attempt was made to show best correspondences with experimental results, but we will discuss qualitative correspondences only. The inherent anisotropy was selected again to be \( h_1 = 1.20 \) and \( h_2 = 0.80 \). The calculation was carried out in a simple incremental manner without iterations. The calculation results of the elasto-plastic stress strain behavior are shown in Fig. 6 (a) and (b). Fig. 6 (a) is the relationship between the stress ratio and the effective shear strain \( \tau = |\varepsilon_1 - \varepsilon_2| \), where \( \varepsilon_1 \) and \( \varepsilon_2 \) are principal values of the strain rate. Fig. 6 (b) is the relationship between dilatancy and the stress ratio. As is found from Fig. 6, the anisotropic stress strain behavior of granular materials with inherent anisotropy, showing

<table>
<thead>
<tr>
<th>Table 1. Parameters for numerical calculations</th>
</tr>
</thead>
<tbody>
<tr>
<td>parameters</td>
</tr>
<tr>
<td>( K ) (elastic bulk modulus) in Eq. (55)</td>
</tr>
<tr>
<td>( G ) (elastic shear modulus) in Eq. (55)</td>
</tr>
<tr>
<td>( a ) (parameter for hardening) in Eq. (54)</td>
</tr>
<tr>
<td>( b ) (parameter for hardening) in Eq. (54)</td>
</tr>
<tr>
<td>( M ) (parameter for dilatancy) in Eq. (49)</td>
</tr>
<tr>
<td>( H_1 ) and ( H_2 ) for fabric anisotropy</td>
</tr>
<tr>
<td>( P ) (( \sigma_1 + \sigma_2 ))/2</td>
</tr>
<tr>
<td>is specified; and by changing the value of ( P ), the stress strain and dilatancy behavior in Fig. 6 was calculated</td>
</tr>
</tbody>
</table>
coordinate system to be coincident with principal axes of the fabric tensor again.

(a) elastic region

There occur elastic regions in the modified Drucker–Prager yield condition in the sense that \( N_{ij}\delta_{ij} < 0 \) even for a virgin loading. The condition of \( N_{ij}\delta_{ij} < 0 \) becomes by direct calculation using Eqs. (46), (47), and (48)

\[
\tilde{\tau} < (1 + (\tau_s)^{2}/P^*) H_a \cos 2\psi = Q_{cr} \tag{58}
\]

\( \tau_s \) and \( P^* \) include \( \tilde{\tau} \) in their definitions, we can, however, note that \( \tau_s \) and \( P^* \) are all positive quantities. We therefore conclude that when \( 0 < \psi < \pi/4 \), there occur elastic regions satisfying \( \tilde{\tau} < Q_{cr} \). As is easily understood from Eq. (58), the elastic region becomes wider as \( H_a \) increases and \( \psi \) is close to 0. When \( \pi/4 < \psi < \pi/2 \), \( Q_{cr} \) becomes negative; and we therefore expect no elastic region in these directions. Further study is required on the pseudo–elastic regions which is a natural consequence of the modified elasto–plastic model developed here.

(b) Degree of non-coaxiality

The degree of non-coaxiality of the model can be measured by the definition

\[
\tan 2\psi = (2\tilde{\tau}/\tilde{\tau}) = (\tilde{\tau} / \tilde{\tau}) = N_{ij}/N'_{ij} \tag{59}
\]

By direct calculation, we have the following form:

\[
\tan 2\psi = \frac{\tilde{\tau} \sin 2\psi}{PH_a + \tilde{\tau} \cos 2\psi - \frac{1}{2} (\tau_s)^2 H_a / P^*} \tag{60}
\]

It is easily understood by inspection that Eq. (60) yields the following features:

(a) When the principal axes of the fabric tensor coincides with those of the stress tensor \( \psi = 0 \) at present), the coaxiality \( \psi = \delta = 0 \) holds;

(b) When the isotropic property \( H_a = 0 \) holds, \( \psi = \delta \) always holds. With the help of simple numerical calculation, we also note that

(c) As the stress ratio (beyond the fabric region discussed above) increases, the degree of non-coaxiality measured by \( \psi - \delta \) decreases.

These features on the degree of non-coaxiality

satisfactory correspondences with experimental observations (Miura et al., 1986), can be described by the elasto–plastic model based on the modified stress method.

It seems worth discussing some specific features of the developed model: (a) the existence of elastic region in stress space; (b) the degree of non-coaxiality; and (c) the degree of shape change. Here we use the stress and fabric measures in the discussion of strength anisotropy in which we select the

NII-Electronic Library Service
Fig. 7. Evolution of yield surface $Q_1$-$Q_2$ diagram under a monotonous loading with $\Psi = 0$

show satisfactory qualitative correspondences with Miura et al. (1986) and Okada (1986).

The properties from (a) to (c) can be easily understood from Fig. 7 in which the evolutions of the yield surface in $Q_1$ and $Q_2$ diagram are drawn for the case that inherent anisotropy is fixed. When $\Psi = 0$, the normal vector to the yield surface is of course in the direction of $Q_1$ and the coaxiality holds (the property (a)). When $H_4 = 0$, the yield surface becomes a complete circular; and $\Psi = \delta$ always holds for any $\Psi$ (the property (b)). As $\bar{Q}$ increases from the origin to A with a fixed $\Psi$, the ellipse becomes larger with the effect that the normal vector to the yield surface takes similar values to that of circular yield surface.

(c) Shape change of yield surface

The shape change in the proposed model may be measured by the ratio of the length of major axis (in $Q_1$ direction) to the minor axis (in $Q_2$ direction) of the ellipse in $Q_1$ and $Q_2$ diagram. From Eq. (40), we have

$$\left( Q_1 - \frac{D^2}{C} \right)^2 + \frac{1}{C} (Q_2)^2 = EP \frac{1}{C} + \frac{D^2}{C}$$ (61)

The ratio of the length of major axis to that of minor axis becomes $1/\sqrt{C}$. Since $C$ is defined as $C = 1 - (\alpha^* H_4)^2$ and $\alpha^*$ is a monotonic increasing function of $\bar{Q}$. The elliptic property of the yield surface is increasing under the condition that $\bar{Q}$ is increasing and $H_4$ constant. However, $\sqrt{C}$ is always close to 1 (note $\sqrt{C} = 0.92$ when $\alpha^* = 0.8$ corresponding to $\sin \phi = 0.83$ and $H_4 = 0.2$), the shape change of the proposed model is not so eminent. The anisotropic description of the model discussed here is mainly given through the evolution of the center of the ellipse in the $Q_1$ and $Q_2$ diagram.

CONCLUSIONS

The development of simple and reasonable methods for the introduction of the fabric anisotropy into the constitutive formulation of granular materials is of immediate importance. The modified stress tensor as a fabric dependent linear transformation from the stress tensor was used for this objective. The convenient method called the modified stress method was developed and illustrative examples of the anisotropic elasto-plastic behavior of granular materials were studied.

Based on the representation theorem of isotropic tensor functions, the general modified stress tensor was derived as a fabric dependent linear transformation from the stress tensor. It was found that the classical hardening models can be obtained as special cases of the modified stress method.

In order to demonstrate the effectiveness of the modified stress method, two illustrative examples in two dimensional problems were studied: (a) the strength anisotropy based on the Modified Drucker–Prager failure condition; and (b) the anisotropic stress strain behavior of granular materials under the condition that inherent anisotropy is kept unchanged during deformation, which was simulated based on the non-associated flow rule using the Drucker–Prager yield condition and the generalized stress dilatancy equation as well as on the modified stress method. The agreement of the prediction by the developed model with experimental observations was found to be satisfactory at least from a qualitative viewpoint.

In the modified stress method for the description of anisotropic behavior of granular
materials, the additional mathematical operation to the classical formulation of isotropic hardening models is the calculation of the fourth order tensor \( L_{ijkl} = \left( \frac{\partial T_{ij}}{\partial y_{kl}} \right) \). This remarkable simplicity can be considered as an essential and promising feature of the modified stress method.

ACKNOWLEDGEMENTS

This study was partly supported by the grant in aids from the Ministry of Education, Science and Culture through the grant numbers: 63750507 and 01750485. The authors would like to express their tanks to Prof. Boehler (Institut de Mecanique) who kindly sent his relevant works to them.

REFERENCES

APPENDIX: NOTATION

In this appendix, we summarize the main symbols used in this paper. Ordinary symbols are not listed but specific symbols peculiar to this paper are listed.

\( a_k \): scalar functions of basic invariants defined by a set of selected tensorial variables

\( G_k \): the irreducible generators in the representation theorem of isotropic functions.

\( H_{ij} \): the fabric tensor in general, which includes the inverse tensor of physically derived fabric tensor

\( L_{ijkl} \): the transformation tensor from the stress tensor to the modified stress tensor defined as \( (\partial T_{ij}/\partial \sigma_{kl}) \)

\( P = (\sigma_{11} + \sigma_{22})/2 \)

\( Q_1 = (\sigma_{11} - \sigma_{22})/2 \)

\( Q_2 = \sigma_{22} \)

\( \bar{Q} = \sqrt{Q_1^2 + Q_2^2} = \sqrt{1/2 (\epsilon_{ijkl} \delta_{ij})} \)

The corresponding stress measures in modified stress space are marked with superscript of asterisk

\( T_{ij} \): the modified stress tensor obtained as linear transformation from the stress tensor defined as \( T_{ij} = L_{ijkl}(H) \sigma_{kl} \)

\( \alpha \): hardening parameters being a function of plastic history

\( \alpha^* \): hardening parameter in modified stress space

\( \delta_{ij} \): the Kronecker's delta (=1 when \( i = j \), =0 in other cases)

\( \Psi \): the deviation angle between the principal directions of the fabric tensor \( (e_i) \) and that of the stress tensor \( (e^*)_j \). It is defined by \( \cos(\Psi) = (e^* \cdot e_i) \), where dot denotes the inner product

\( \tau_* = \sqrt{1/2 (T_{ij} T^*_{ij})} \); \( T^*_{ij} \) is the deviatoric part of the modified stress tensor

19, pp. 899–916.


